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Schultz and Modified Schultz Polynomials of Edges Induce Chain and Ring for Hexagonal Graphs

Asmaa S. Aziz[1,*], Haitham N. Mohammed[2], Ahmed M. Ali [1]

[1] *Department of Mathematics, College of Computer Sciences and Mathematics, Mosul University, Mosul, Iraq*
[2] *Nineveh Education Directorate, Mosul, Iraq*

Abstract. Schultz polynomial is one of the must significant formulas that represent a relationship between the degree's of vertices in a simple connected graph G and the distances between these vertices. In this work, Schultz and modified Schultz polynomials, as well as their topological indices of chain and ring hexagonal graphs, have been successfully identified.

2020 Mathematics Subject Classifications: 05C09, 05C31

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1. Introduction

Mathematical chemistry is a field of theoretical chemistry that examines and predicts molecule structure using mathematical approaches rather than quantum mechanics. Chemical graph theory is a powerful method for determining molecule structures that has made significant contributions to the advancement of chemical science. In the molecular graph G, atoms and bonds are represented by vertices and edges, respectively. The order of the graph G in graph theory is $p = p(G) = |V(G)|$, and the size of G is $q = q(G) = |E(G)|$, while the degree of $\eta \in V(G)$ is the number of vertices joining to η and denoted by δ_η. Furthermore, the distance $d(\mu, \eta) = d(\mu, \eta \mid G)$ between any two vertices μ and η is the length of the shortest path connecting them in G. The greatest distance in G is the diameter denoted by $diam(G)$, [7]. Let $d(G, \xi)$ express the number of random pair of vertices in G with ξ distance. Let $a_\xi(G)$ is the set of all these pairs such that $|a_\xi(G)| = |a_\xi| = d(G, \xi)$ and $\sum_{\xi=1}^{diam(G)} d(G, \xi) = \binom{p}{2}$, where $\binom{p}{2}$ represents the number of unordered pairs of different vertices in G,[12].

*Corresponding author.
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Email addresses: asmaas982@uomosul.edu.iq(A. S. Aziz), haytham.nashwan@yahoo.com(H. N. Mohammed), ahmedgraph@uomosul.edu.iq(A. M. Ali)

Topological indices were first used in Biology and Chemistry in 1947 when scientist Harold Wiener [20] created the Wiener index to show connections between the physico-chemical features of organic molecules in molecular graphs.

The Wiener index, abbreviated W(G), is the sum of all distances between all unordered (μ, η) pairs in a connected graph G :

$$W(G) = \sum_{\{\mu,\eta\}\subseteq V(G)} d(\mu, \eta);$$

where, $d(\mu, \eta)$ indicates the distance between μ and η.

Based on the Wiener index, Hosoya in 1988 [14] invented the new Hosoya polynomial $H(G; x)$ which is defined as:

$$H(G; x) = \sum_{\{\mu,\eta\}\subseteq V(G)} x^{d(\mu,\eta)};$$

Recently, many other polynomials have been obtainted such as detour polynomial [3], m-polynomial [16],[17].

The Schultz index (Sc) is another based structure which was first introduced by Harry Schultz [18] the molecular topological index and is characterized by:

$$Sc(G) = \sum_{\{\mu,\eta\}\subseteq V(G)} (\delta_\mu + \delta_\eta) d(\mu, \eta),$$

where δ_μ is the degree of the vertex μ and δ_η is the degree of η .

The Schultz index is based on this. In 1997, Klavžar and Gutman [15] proposed the Mod. Sch. index, which is defined as:

$$Sc^*(G) = \sum_{\{\mu,\eta\}\subseteq V(G)} (\delta_\mu \delta_\eta) d(\mu, \eta),$$

There are two significant polynomials for these structural descriptors in chemical graph theory, "Schultz polynomial(ScP)" and "Modified Schultz polynomial(MScP)" of G are respectively defined as:

$$Sc(G; x) = \sum_{\mu,\eta\subseteq V(G)} (\delta_\mu + \delta_\eta) x^{d(\mu,\eta)};$$

$$Sc^*(G; x) = \sum_{\mu,\eta\subseteq V(G)} (\delta_\mu \delta_\eta) x^{d(\mu,\eta)},$$

In 2009, Hassani et al. [13] computed the (ScP) and (MScP) of C100 fullerene isomers using the GAP program, while Behmaram et al. [6] obtained (ScP) of some graph operations. Farahani [8] discovered hosoya, (ScP) and (MScP) and their topological indices for benzene, followed by (ScP) and (MScP)of coronene polycyclic aromatic hydrocarbons in a subsequent study [9]. Many researchers have worked over the last decade to determine (ScP) and (MScP) and their indices for graphs consisting of chains and rings of special graphs with chemical applications [10],[19],[5],[11], [4], [1], [2].

2. Results

2.1. The Edges Induce Chain For Hexagonal Graphs $C_e(C_6)_\gamma$

The edges induce chain for hexagonal graphs which is denoted by $C_e(C_6)_\gamma$ is a graph consisting of m hexagonal rings, $m \geq 3, \gamma = 4m-1$ every two successive rings have a common edge induce, forming a chain as shown in Fig. 1.

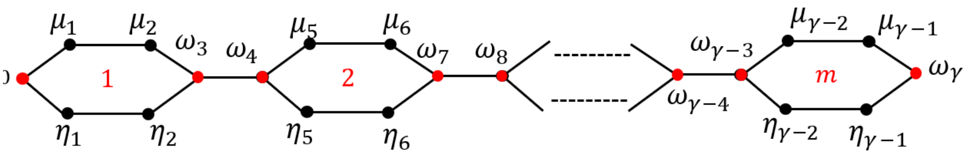


Figure 1: Edges Induce Chain For Hexagonal Graphs $C_e(C_6)_\gamma$.

From Fig.1 we note, $p(C_e(C_6)_\gamma) = 6m$, $q(C_e(C_6)_\gamma) = 7m-1$ and $diam(C_e(C_6)_\gamma) = 4m-1$.

Table 1: Degree Matrix of $C_e(C_6)_\gamma$ For $1 \leq i,j \leq \gamma-2$,and $1 \leq r,s \leq m-1, r \neq s$ and $i,j \neq r,s$.

+ ×	$\delta\omega_0 = 2$	$\delta\mu_i = 2$	$\delta\eta_i = 2$	$\delta\omega_{4r,(4r-1)} = 3$	$\delta\omega_\gamma = 2$
$\delta\omega_0 = 2$		4 / 4	4 / 4	5 / 6	4 / 4
$\delta\mu_j = 2$	4 / 4	4 / 4	4 / 4	5 / 6	4 / 4
$\delta\eta_j = 2$	4 / 4	4 / 4	4 / 4	5 / 6	4 / 4
$\delta\omega_{4s,(4s-1)} = 3$	5 / 6	5 / 6	5 / 6	6 / 9	5 / 6
$\delta\omega_\gamma = 2$	4 / 4	4 / 4	4 / 4	5 / 6	

Theorem 1. *For $m \geq 3, \gamma = 4m-1$, then:*

(i) $Sc(C_e(C_6)_\gamma; x) = \frac{1}{2}(17\gamma-3)x + 12(\gamma-1)x^2 + \frac{1}{2}(25\gamma-51)x^3 + \sum_{\xi=4,8,\dots,\gamma-3}(11\gamma-11\xi +9)x^\xi + \frac{1}{2}\sum_{\xi=5,9,\dots,\gamma-2}(21\gamma-21\xi+22)x^\xi + 2\sum_{\xi=6,10,\dots,\gamma-1}(5\gamma-5\xi+3)x^\xi + \frac{1}{2}\sum_{\xi=7,11,\dots,\gamma}(21\gamma-21\xi+8)x^\xi$.

(ii) $Sc^*(C_e(C_6)_\gamma; x) = \frac{1}{4}(41\gamma-27)x + 2(7\gamma-9)x^2 + \frac{1}{4}(57\gamma-127)x^3 + \frac{1}{2}\sum_{\xi=4,8,\dots,\gamma-3}(25\gamma -25\xi+13)x^\xi + \frac{1}{4}\sum_{\xi=5,9,\dots,\gamma-2}(49\gamma-49\xi+30)x^\xi + 4\sum_{\xi=6,10,\dots,\gamma-1}(3\gamma-3\xi+1)x^\xi + \frac{1}{4}\sum_{\xi=7,11,\dots,\gamma-4}(49\gamma-49\xi+12)x^\xi + 4x^\gamma$.

Proof. For vertex $y,z \in V(C_e(C_6)_\gamma)$ there is $d(y,z) = \xi, 1 \leq \xi \leq \gamma$. And obviously: $\sum_{i=1}^{\gamma}|a_i| = (18\gamma^2+7\gamma+193)/16$.
The proof is consist of the following twelve cases:

(i) If $d(y,z) = \xi = 1$, then $|a_1| = (7\gamma+3)/4$, also is equal to $q(C_e(C_6)_\gamma)$, we have four subsets of it:

(a) $\left|\left\{(\mu_{1(\gamma-1)}, \omega_{0(\gamma)}), (\eta_{1(\gamma-1)}, \omega_{0(\gamma)})\right\}\right| = 4$.

(b) $|\{(\mu_{4i+1}, \mu_{4i+2}), (\eta_{4i+1}, \eta_{4i+2}) : 0 \leq i \leq (\gamma-3)/4\}| = (\gamma+1)/2$.

(c) $|\{(\mu_{4i+2}, \omega_{4i+3}), (\eta_{4i+2}, \omega_{4i+3}) : 0 \leq i \leq (\gamma - 7)/4\}| = (\gamma - 3)/2.$

(d) $|\{(\mu_{4i+1}, \omega_{4i}), (\eta_{4i+1}, \omega_{4i}), (\omega_{4i-1}, \omega_{4i}) : 1 \leq i \leq (\gamma - 3)/4\}| = 3(\gamma - 3)/4.$

(ii) If $d(y, z) = \xi = 2$, then $|a_2| = (5\gamma - 3)/2$, we have six subsets of it:

(a) $|\{(\mu_\xi(\eta_\xi), \omega_0)\}| = 2.$

(b) $|\{(\mu_{\gamma-\xi}(\eta_{\gamma-\xi}), \omega_\gamma)\}| = 2.$

(c) $|\{(\mu_{4i+1}(\eta_{4i+1}), \omega_{4i+\xi+1}) : 0 \leq i \leq (\gamma - \xi - 5)/4\}| = (\gamma - \xi - 1)/2.$

(d) $|\{(\omega_{4i-1}, \mu_{4i+\xi-1}(\eta_{4i+\xi-1})) : 1 \leq i \leq (\gamma - \xi - 1)/4\}| = (\gamma - \xi - 1)/2.$

(e) $|\{(\mu_{4i+2}(\eta_{4i+2}), \omega_{4i+\xi+2}) : 0 \leq i \leq (\gamma - \xi - 5)/4\}| = (\gamma - \xi - 1)/2.$

(f) $|\{(\omega_{4i}, \mu_{4i+\xi}(\eta_{4i+\xi})) : 1 \leq i \leq (\gamma - \xi - 1)/4\}| = (\gamma - \xi - 1)/2.$

(iii) If $d(y, z) = 3$, then $|a_3| = (11\gamma - 21)/4$, we have eight subsets of it:

(a) $\left|\left\{(\omega_{0(\gamma-\xi)}, \omega_{\xi(\gamma)})\right\}\right| = 2.$

(b) $|\{(\omega_{4i}, \omega_{4i+\xi}) : 1 \leq i \leq (\gamma - \xi - 4)/4\}| = (\gamma - \xi - 4)/4.$

(c) $|\{(\mu_{4i-2}, \mu_{4i-2+\xi}(\eta_{4i-2+\xi})) : 1 \leq i \leq (\gamma - \xi)/4\}| = (\gamma - \xi)/2.$

(d) $|\{(\eta_{4i-2}, \mu_{4i-2+\xi}(\eta_{4i-2+\xi})) : 1 \leq i \leq (\gamma - \xi)/4\}| = (\gamma - \xi)/2.$

(e) $|\{(\mu_{4i+1}, \eta_{4i+2}) : 0 \leq i \leq (\gamma - \xi)/4\}| = (\gamma + 1)/4.$

(f) $|\{(\mu_{4i+2}, \eta_{4i+1}) : 0 \leq i \leq (\gamma - \xi)/4\}| = (\gamma + 1)/4.$

(g) $|\{(\mu_{4i-3}(\eta_{4i-3}), \omega_{4i-3+\xi}) : 1 \leq i \leq (\gamma - \xi)/4\}| = (\gamma - \xi)/2.$

(h) $\left|\left\{(\omega_{4i-1}, \mu_{4i+\xi-1}(\eta_{4i+\xi-1})) : 1 \leq i \leq (\gamma - \xi)/4\right\}\right| = (\gamma - \xi)/2.$

(iv) If $d(y, z) = \xi, \xi = 4, 8, \ldots, \gamma - 7$, then $\sum_{\xi=4,8,\ldots,\gamma-7} |a_\xi| = 5(\gamma - 7)(\gamma + 5)/16$, we have seven subsets of it:

(a) $|\{(\mu_{4i+1}, \mu_{4i+1+\xi}(\eta_{4i+1+\xi})) : 0 \leq i \leq (\gamma - \xi - 3)/4\}| = (\gamma + 1 - \xi)/2.$

(b) $|\{(\mu_{4i+2}, \mu_{4i+2+\xi}(\eta_{4i+2+\xi})) : 0 \leq i \leq (\gamma - \xi - 3)/4\}| = (\gamma + 1 - \xi)/2.$

(c) $|\{(\eta_{4i+1}, \eta_{4i+1+\xi}(\mu_{4i+1+\xi})) : 0 \leq i \leq (\gamma - \xi - 3)/4\}| = (\gamma + 1 - \xi)/2.$

(d) $|\{(\eta_{4i+2}, \eta_{4i+2+\xi}(\mu_{4i+2+\xi})) : 0 \leq i \leq (\gamma - \xi - 3)/4\}| = (\gamma + 1 - \xi)/2.$

(e) $\left|\left\{(\omega_{0(\gamma)}, \omega_{\xi(\gamma-\xi)})\right\}\right| = 2.$

(f) $|\{(\omega_{4i-1}, \omega_{4i-1+\xi}) : 1 \leq i \leq (\gamma - \xi - 3)/4\}| = (\gamma - 3 - \xi)/4.$

(g) $|\{(\omega_{4i}, \omega_{4i+\xi}) : 1 \leq i \leq (\gamma - \xi - 3)/4\}| = (\gamma - 3 - \xi)/4.$

(v) If $d(y, z) = \xi, \xi = 5, 9, \ldots, \gamma - 6$, then $\sum_{\xi=5,9,\ldots,\gamma-6} |a_\xi| = (\gamma - 7)(9\gamma + 37)/32$, we have seven subsets of it:

(a) $|\{(\mu_{4i+1}, \mu_{4i+1+\xi}(\eta_{4i+1+\xi})) : 0 \leq i \leq (\gamma - \xi - 2)/4\}| = (\gamma + 2 - \xi)/2.$

(b) $|\{(\eta_{4i+1}, \eta_{4i+1+\xi}(\mu_{4i+1+\xi})) : 0 \leq i \leq (\gamma - \xi - 2)/4\}| = (\gamma + 2 - \xi)/2.$

(c) $|\{(\mu_\xi(\eta_\xi), \omega_0)\}| = 2.$

(d) $|\{(\mu_{\gamma-\xi}(\eta_{\gamma-\xi}), \omega_\gamma)\}| = 2$.

(e) $|\{(\mu_{4i+2}(\eta_{4i+2}), \omega_{4i+2+\xi}) : 0 \leq i \leq (\gamma-\xi-6)/4\}| = (\gamma-2-\xi)/2$.

(f) $|\{(\omega_{4i}, \mu_{4i+\xi}(\eta_{4i+\xi})) : 0 \leq i \leq (\gamma-\xi-2)/4\}| = (\gamma-2-\xi)/2$.

(g) $|\{(\omega_{4i-1}, \omega_{4i-1+\xi}) : 1 \leq i \leq (\gamma-\xi-2)/4\}| = (\gamma-2-\xi)/4$.

(vi) If $d(y,z) = \xi, \xi = 6, 10, ..., \gamma-5$, then $\sum_{\xi=7,11,...,\gamma-5} |a_\xi| = (\gamma-7)(\gamma+1)/4$, we have six subsets of it: Similar to $(ii(a-f))$, put $\xi = 6, 10, ..., \gamma-5$.

(vii) If $d(y,z) = \xi, \xi = 7, 11, ..., \gamma-8$, then $\sum_{\xi=7,11,...,\gamma-8} |a_\xi| = (\gamma-11)(9\gamma+17)/32$, we have six subsets of it: Similar to $(iii(a-d), (g)$ and $(h))$, put $\xi = 7, 11, ..., \gamma-9$.

(viii) If $d(y,z) = \gamma-4$, $|a_{\gamma-4}| = 10$ then we have six subsets of it: Similar to $(iii(a-d), (f)$ and $(h))$, put $\xi = \gamma-4$.

(ix) If $d(y,z) = \gamma-3$, $|a_{\gamma-3}| = 10$, there are five subsets of it: Similar to $(iv(a-e))$, put $\xi = \gamma-3$.

(x) If $d(y,z) = \gamma-2$, $|a_{\gamma-2}| = 8$, the four subsets of it: Similar to $(iv(a-d))$, put $\xi = \gamma-2$.

(xi) If $d(y,z) = \gamma-1$, $|a_{\gamma-1}| = 4$ then two subsets of it: Similar to $(ii(a-b))$, put $\xi = \gamma-1$.

(xii) If $d(\mu,\eta) = \gamma$ then $|a_\gamma| = 1$, we have: $|\{(\omega_0, \omega_\gamma)\}| = 1$.

Corollary 1. *For $m \geq 3, \gamma = 4m-1$, then:*

(i) $Sc(C_e(C_6)_\gamma) = (7\gamma^3 + 15\gamma^2 + 29\gamma + 21)/4$.

(ii) $Sc^*(C_e(C_6)_\gamma) = (49\gamma^3 + 63\gamma^2 + 185\gamma + 147)/24$.

2.2. The Edges Induce Ring For Hexagonal Graphs $R_e(C_6)_\gamma$

This graph is said to be a hexagonal bracelet graphs $R_e(C_6)_\gamma$ which is a connected graph consisting of $m \geq 3$, hexagonal rings such that two hexagons are joined by exactly one added edge as shown in Fig. 2.

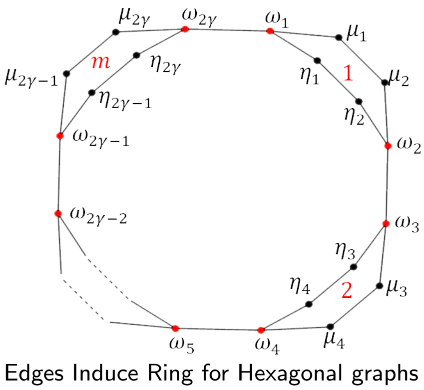


Figure 2: Edges Induce Ring for Hexagonal graphs $R_e(C_6)_\gamma$.

Table 2: Degree Matrix of $R_e(C_6)_\gamma$ For $1 \le i,j \le 2\gamma-1$, and $1 \le r,s \le m-1, r \ne s$, and $i,j \ne r,s$.

$+$ \ $\times$	$\delta\omega_1 = 3$	$\delta\mu_j = 2$	$\delta\eta_i = 2$	$\delta\omega_{4r,(4r-1)} = 3$	$\delta\omega_{2\gamma} = 3$
$\delta\omega_1 = 3$		5 \ 6	5 \ 6	6 \ 9	6 \ 9
$\delta\mu_j = 2$	5 \ 6	4 \ 4	4 \ 4	5 \ 6	4 \ 4
$\delta\eta_j = 2$	5 \ 6	4 \ 4	4 \ 4	5 \ 6	4 \ 4
$\delta\omega_{4s,(4s-1)} = 3$	6 \ 9	5 \ 6	5 \ 6	6 \ 9	5 \ 6
$\delta\omega_{2\gamma} = 3$	6 \ 9	5 \ 6	5 \ 6	6 \ 9	

Theorem 2. *For $m \ge 3, \gamma = m$, then:*

(i) $$Sc(R_e(C_6);x) = 34\gamma x + 48\gamma x^2 + 50\gamma x^3 + \left[\begin{array}{l} \sum_{\xi=4,8}^{2\gamma-4} 44\gamma x^{\xi}, \gamma \text{ is an even} \\ \sum_{\xi=4,8}^{2\gamma-2} 44\gamma x^{\xi}, \gamma \text{ is an odd} \end{array}\right.$$
$$+ \sum_{\xi=5,7}^{2\gamma-1} 42\gamma x^{\xi} \left[\begin{array}{l} \sum_{\xi=4,8}^{2\gamma-2} 40\gamma x^{\xi}, \gamma \text{ is an even} \\ \sum_{\xi=4,8}^{2\gamma-2} 40\gamma x^{\xi}, \gamma \text{ is an odd} \end{array}\right. + \left[\begin{array}{l} 22\gamma x^{2\gamma}, \gamma \text{ is an even,} \\ 20\gamma x^{2\gamma}, \gamma \text{ is an odd.} \end{array}\right.$$

(ii) $$Sc^*(R_e(C_6);x) = 41\gamma x + 56\gamma x^2 + 57\gamma x^3 + \left[\begin{array}{l} \sum_{\xi=4,8}^{2\gamma-4} 50\gamma x^{\xi}, \gamma \text{ is an even} \\ \sum_{\xi=4,8}^{2\gamma-2} 50\gamma x^{\xi}, \gamma \text{ is an odd} \end{array}\right.$$
$$+ \sum_{\xi=5,7}^{2\gamma-1} 49\gamma x^{\xi} + \left[\begin{array}{l} \sum_{\xi=4,8}^{2\gamma-2} 48\gamma x^{\xi}, \gamma \text{ is an even} \\ \sum_{\xi=4,8}^{2\gamma-2} 48\gamma x^{\xi}, \gamma \text{ is an odd} \end{array}\right. + \left[\begin{array}{l} 25\gamma x^{2\gamma}, \gamma \text{ is an even,} \\ 24\gamma x^{2\gamma}, \gamma \text{ is an odd.} \end{array}\right.$$

Proof. For any two vertices $y, z \in V(R_e(C_6)_)$ there is $d(y,z) = \xi, 1 \le \xi \le 2\gamma$. And clearly $\sum_{\xi=1}^{\gamma} d(R_e(C_6)_\gamma, \xi) = \left[\begin{array}{l} 3\gamma(6\gamma-1), \ \gamma \text{ is an even,} \\ (1/2)\gamma(36\gamma+11), \ \gamma \text{ is an odd.} \end{array}\right.$

Proof will be divided into eight cases:

(i) If $d(y,z) = 1$, then $|a_1| = 7\gamma$, there are three subsets of it:

 (a) $|\{(\omega_i, \mu_i), (\omega_i, \eta_i) : 1 \le i \le 2\gamma\}| = 4\gamma$.

 (b) $|\{(\mu_i, \mu_{i+1}), (\eta_i, \eta_{i+1}) : i = 1, 3, 5, \ldots, 2\gamma-1\}| = 2\gamma$.

 (c) $|\{(\omega_i, \omega_{i+1}) : i = 2, 4, 6, \ldots, 2\gamma, (\omega_{2\gamma+1} \equiv \omega_1)\}| = \gamma$.

(ii) If $d(y,z) = 2$, then $|a_2| = 10\gamma$, there are three subsets of it:

 (a) $|\{(\mu_i, \eta_i) : 1 \le i \le 2\gamma\}| = 2\gamma$.

 (b) $|\{(\omega_i, \mu_{i+1}), (\omega_i, \eta_{i+1}) : i = 1, 3, 5, \ldots, 2\gamma-1\}| = 2\gamma$.

 (c) $|\{(\omega_i, \mu_{i-1}), (\omega_i, \eta_{i-1}), (\mu_i, \omega_{i+1}), (\eta_i, \omega_{i+1}), (\omega_i, \mu_{i+1}), (\omega_i, \eta_{i+1}) :$
 $i = 2, 4, 6, \ldots, 2\gamma, (\omega_{2\gamma+1} \equiv \omega_1), (\mu_{2\gamma+1} \equiv u_1), (\eta_{2\gamma+1} \equiv \eta_1)\}| = 6\gamma$.

(iii) If $d(y,z) = 3$, then $|a_3| = 11\gamma$, there are three subsets:

 (a) $|\{\ (\omega_i, \omega_{i+1}) : i = 1, 3, 5, \ldots, 2\gamma-1\ \}| = \gamma$.

(b) $|\{(\mu_i,\omega_{i+2}),(\eta_i,\omega_{i+2}),(\mu_i,\eta_{i+1}),(\eta_i,\mu_{i+1}):$
$i=1,3,5,\ldots,2\gamma-1,\ (\omega_{2\gamma+1}\equiv\omega_1)\}|=4\gamma$.

(c) $|\{\ (\mu_i,\mu_{i+1}),(\mu_i,\eta_{i+1}),(\eta_i,\mu_{i+1}),(\eta_i,\eta_{i+1}),(\omega_i,\mu_{i+2}),(\omega_i,\eta_{i+2}):$
$i=2,4,6,\ldots,2\gamma,(\mu_{2\gamma+1}\equiv\mu_1),(\eta_{2\gamma+1}\equiv\eta_1)\ ,(\mu_{2\gamma+2}\equiv\mu_2),(\eta_{2\gamma+2}\equiv\eta_2)\}|=6\gamma$.

(iv) If $d(y,z)=\xi,\xi=4,8,\ldots,2\gamma-4$, then we have: **A:** If γ **is an even** number, then we have two subsets of it

(a) $|\{\ (\omega_i,\omega_{i+(\xi/2)}),(\mu_i,\mu_{i+(\xi/2)}),(\mu_i,\eta_{i+(\xi/2)}),(\eta_i,\mu_{i+(\xi/2)}),(\eta_i,\eta_{i+(\xi/2)}):$
$i=1,3,5,\ldots,2\gamma-1\ \}|=5\gamma$.
If $i+\frac{\xi}{2}>2\gamma$, then $(\omega_{i+(\xi/2)}\equiv\omega_{i+(\xi/2)-2\gamma}),(\mu_{i+(\xi/2)}\equiv\mu_{i+(\xi/2)-2\gamma}),(\eta_{i+(\xi/2)}\equiv\eta_{i+(\xi/2)-2\gamma})$.

(b) $|\{\ (\omega_i,\omega_{i+(\xi/2)}),(\mu_i,\eta_{i+(\xi/2)}),(\mu_i,\mu_{i+(\xi/2)}),(\eta_i,\mu_{i+(\xi/2)}),(\eta_i,\eta_{i+(\xi/2)}):$
$i=2,4,6,\ldots,2\gamma\ \}|=5\gamma$.
If $i+(\xi/2)>2\gamma$, then $(\omega_{i+(\xi/2)}\equiv\omega_{i+(\xi/2)-2\gamma}),(\mu_{i+(\xi/2)}\equiv\mu_{i+(\xi/2)-2\gamma}),(\eta_{i+(\xi/2)}\equiv\eta_{i+(\xi/2)-2\gamma})$.
Hence $\sum_{\xi=4,8}^{2\gamma-4}|a_\xi|=10\gamma$

B: If γ **is an odd** number, we get $\sum_{\xi=4,8}^{2\gamma-2}|a_\xi|=10\gamma$, note we add the distance of $\xi=2\gamma-2$.

(v) If $d(y,z)=\xi,\xi=5,9,13,\ldots,2\gamma-3$, then we have: **A:** If γ **is an even** number, then we have four subsets of it

(a) $|\{\ (\omega_i,\mu_{i+(\xi-1)/2}),(\omega_i,\eta_{i+(\xi-1)/2}):i=1,3,5,\ldots,2\gamma-1\ \}|=2\gamma$.
If $i+(\xi-1/2)>2\gamma$, then $(\mu_{i+(\xi-1/2)}\equiv\mu_{i+(\xi-1)/2-2\gamma}),(\eta_{i+(\xi-1)/2}\equiv\eta_{i+(\xi-1)/2-2\gamma})$.

(b) $|\{\ (\mu_i,\mu_{i+(\xi-1)/2+1}),(\mu_i,\eta_{i+(\xi-1)/2+1}),(\eta_i,\mu_{i+(\xi-1)/2+1}),(\eta_i,\eta_{i+(\xi-1)/2+1}):$
$i=1,3,5,\ldots,2\gamma-1\ \}|=4\gamma$.
If $i+(\xi-1)/2+1>2\gamma$, then $(\mu_{i+(\xi-1)/2+1}\equiv\mu_{i+(\xi-1)/2+1-2\gamma}),(\eta_{i+(\xi-1)/2+1}\equiv\eta_{i+(\xi-1)/2+1-2\gamma})$.

(c) $|\{\ (\mu_i,\omega_{i+(\xi-1)/2}),(\eta_i,\omega_{i+(\xi-1)/2}):$
$i=2,4,6,\ldots,2\gamma\ \}|=2\gamma$.
If $i+(\xi-1)/2>2\gamma$, then $(\omega_{i+(\xi-1)/2}\equiv\omega_{i+(\xi-1)/2-2\gamma})$.

(d) $|\{\ (\omega_i,\omega_{i+(\xi-1)/2+1}):i=2,4,6,\ldots,2\gamma\ \}|=\gamma$. If $i+(\xi-1)/2+1>2\gamma$, then $(\omega_{i+(\xi-1)/2+1}\equiv\omega_{i+(\xi-1)/2+1-2\gamma})$.

Hence $\sum_{\xi=5,9}^{2\gamma-3}|a_\xi|=9\gamma$, **B:** If γ **is an odd** number, we get $\sum_{\xi=5,9}^{2\gamma-1}|a_\xi|=9\gamma$, note we add the distance of $\xi=2\gamma-1$. If $d(y,z)=\xi,\xi=6,10,14,\ldots,2\gamma-2$, then $\sum_{\xi=6,10}^{2\gamma-2}|a_\xi|=8\gamma$, we have two subsets of it:

(a) $|\{\ (\omega_i,\mu_{i+(\xi/2)}),(\omega_i,\eta_{i+(\xi/2)}),(\mu_i,\omega_{i+(\xi/2)}),(\eta_i,\omega_{i+(\xi/2)}):$
$i=1,3,5,\ldots,2\gamma-1\ \}|=4\gamma$.

If $i+(\xi/2) > 2\gamma$, then $(\mu_{i+(\xi/2)} \equiv \mu_{i+(\xi/2)-2\gamma}), (\eta_{i+(\xi/2)} \equiv \eta_{i+(\xi/2)-2\gamma}), (\omega_{i+(\xi/2)} \equiv \omega_{i+(\xi/2)-2\gamma})$.

(b) $|\{ (\mu_i, \omega_{i+(\xi/2)}), (\eta_i, \omega_{i+(\xi/2)}), (\omega_i, \mu_{i+(\xi/2)}), (\omega_i, \eta_{i+(\xi/2)}) :$
$i = 2, 4, 6, \ldots, 2\gamma \}| = 4\gamma$.
If $i+(\xi/2) > 2\gamma$, then $(\omega_{i+(\xi/2)} \equiv \omega_{i+(\xi/2)-2\gamma}), (\mu_{i+(\xi/2)} \equiv \mu_{i+(\xi/2)-2\gamma}), (\eta_{i+(\xi/2)} \equiv \eta_{i+(\xi/2)-2\gamma})$.

(vi) If $d(y, z) = \xi, \xi = 7, 11, 15, \ldots, 2\gamma - 1$, then $\sum_{\xi=7,11}^{2\gamma-1} |a_\xi| = 9\gamma$,

we have three subsets of it:

(a) $|\{ (\omega_i, \omega_{i+(\xi-1)/2}), (\mu_i, \omega_{i+(\xi+1)/2}), (\eta_i, \omega_{i+(\xi+1)/2}) :$
$i = 1, 3, 5, \ldots, 2\gamma - 1 \}| = 3\gamma$.
If $i+(\xi-1)/2, i+(\xi+1)/2 > 2\gamma$, then $(\omega_{i+(\xi-1)/2} \equiv \omega_{i+(\xi-1)/2-2\gamma}), (\omega_{i+(\xi+1)/2} \equiv \omega_{i+(\xi+1)/2-2\gamma})$.

(b) $|\{ (\mu_i, \mu_{i+(\xi-1)/2}), (\mu_i, \eta_{i+(\xi-1)/2}), (\eta_i, \mu_{i+(\xi-1)/2}), (\eta_i, \eta_{i+(\xi-1)/2}) :$
$i = 2, 4, 6, \ldots, 2\gamma \}| = 4\gamma$.
If $i+(\xi-1)/2 > 2\gamma$, then $(\mu_{i+(\xi-1)/2} \equiv \mu_{i+(\xi-1)/2-2\gamma}), (\eta_{i+(\xi-1)/2} \equiv \eta_{i+(\xi-1)/2-2\gamma})$.

(c) $|\{ (\omega_i, \mu_{i+(\xi+1)/2}), (\omega_i, \eta_{i+(\xi+1)/2}) : i = 2, 4, 6, \ldots, 2\gamma \}| = 2\gamma$.
If $i+(\xi+1)/2 > 2\gamma$, then $(\mu_{i+(\xi+1)/2} \equiv \mu_{i+(\xi+1)/2-2\gamma}), (\eta_{i+(\xi+1)/2} \equiv \eta_{i+(\xi+1)/2-2\gamma})$.

(vii) If $d(y, z) = 2\gamma$, then we have:

A: If γ is an even number, then we have two subsets of it

(a) $|\{ (\omega_i, \omega_{i+\gamma}), (\mu_i, \mu_{i+\gamma}), (\mu_i, \eta_{i+\gamma}), (\mu_i, \eta_{i+\gamma}), (\eta_i, \eta_{i+\gamma}) :$
$i = 1, 3, 5, \ldots, \gamma - 1 \}| = 5\gamma/2$.

(b) $|\{ (\mu_i, \mu_{i+\gamma}), (\mu_i, \eta_{i+\gamma}), (\eta_i, \mu_{i+\gamma}), (\eta_i, \eta_{i+\gamma}), (\omega_i, \omega_{i+\gamma}) :$
$i = 2, 4, 6, \ldots, \gamma \}| = 5\gamma/2$.

Hence $|a_{2\gamma}| = 5\gamma$.

B: If n is an odd number, then we have: $|\{ (\omega_i, \mu_{i+\gamma}), (\omega_i, \eta_{i+\gamma}), (\mu_i, \omega_{i+\gamma}), (\eta_i, \omega_{i+\gamma}) :$
$i = 1, 3, 5, \ldots, 2\gamma - 1 \}| = 4\gamma$.
If $i + \gamma > 2\gamma$, then $(\mu_{i+\gamma} \equiv \mu_{i-\gamma}), (\eta_{i+\gamma} \equiv \eta_{i-\gamma}), (\omega_{i+\gamma} \equiv \omega_{i-\gamma})$. Hence $|a_{2\gamma}| = 4\gamma$.

Corollary 2. *For $m \geq 3, \gamma = m$, we have:*

$$Sc(R_e(C_6)_\gamma) = \begin{bmatrix} 4\gamma(21\gamma^2 + 8), \gamma \textit{ is an even}, \\ 6\gamma(14\gamma^2 + 5), \gamma \textit{ is an odd}. \end{bmatrix}.$$

$$Sc^*(R_e(C_6)_\gamma) = \begin{bmatrix} 2\gamma(49\gamma^2 + 16), \gamma \textit{ is an even}, \\ \gamma(98\gamma^2 + 31), \gamma \textit{ is an odd}. \end{bmatrix}.$$

3. Examples:

To clarify the previous results, some examples were taken, which were verified programmatically using the Mathematica program.

Table 3: The Edges Induce Chain for Hexagonal Graphs $C_e(C_6)_\gamma, \gamma = 15, 19$.

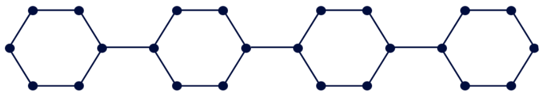
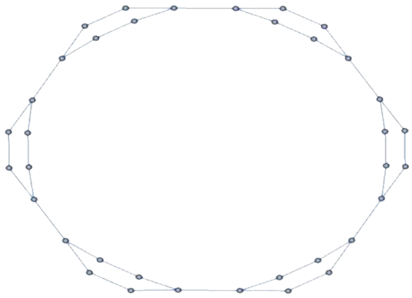
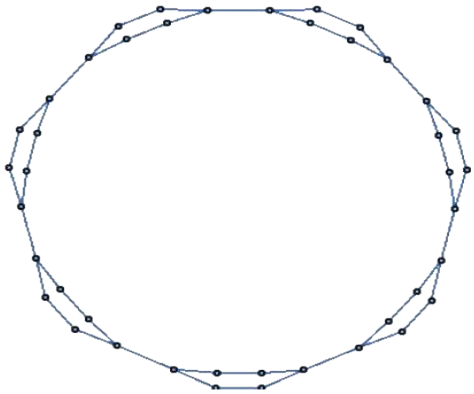
Structure	 $C_e(C_6)_{15}$
(ScP) and (MScP)	$Sc(C_e(C_6)_{15};x) = 126x + 168x^2 + 162x^3 + 130x^4 + 116x^5 + 96x^6 + 88x^7 + 86x^8 + 74x^9 + 56x^{10} + 46x^{11} + 42x^{12} + 32x^{13} + 16x^{14} + 4x^{15}$ $Sc^*(C_e(C_6)_{15};x) = 147x + 192x^2 + 182x^3 + 144x^4 + 130x^5 + 112x^6 + 101x^7 + 94x^8 + 81x^9 + 64x^{10} + 52x^{11} + 44x^{12} + 32x^{13} + 16x^{14} + 4x^{15}$
Indices and diameter	$Sc(C_e(C_6)_{15}) = 6864$. $Sc^*(C_e(C_6)_{15}) = 7603$. $diam(C_e(C_6)_{15}) = 15$.
Structure	 $C_e(C_6)_{19}$
(ScP) and (MScP)	$Sc(C_e(C_6)_{19};x) = 160x + 216x^2 + 212x^3 + 174x^4 + 158x^5 + 136x^6 + 130x^7 + 130x^8 + 116x^9 + 96x^{10} + 88x^{11} + 86x^{12} + 74x^{13} + 56x^{14} + 46x^{15} + 42x^{16} + 32x^{17} + 16x^{18} + 4x^{19}$ $Sc^*(C_e(C_6)_{19};x) = 188x + 248x^2 + 239x^3 + 194x^4 + 179x^5 + 160x^6 + 150x^7 + 144x^8 + 130x^9 + 112x^{10} + 101x^{11} + 94x^{12} + 81x^{13} + 64x^{14} + 52x^{15} + 44x^{16} + 32x^{17} + 16x^{18} + 4x^{19}$
Indices and diameter	$Sc(C_e(C_6)_{19}) = 13500$. $Sc^*(C_e(C_6)_{19}) = 15104$. $diam(C_e(C_6)_{19}) = 19$.

Table 4: The Edges Induce Chain for Hexagonal Graphs $C_e(C_6)_\gamma, \gamma = 15, 19$.

The Structure	 $\boldsymbol{R_e(C_6)_6}$
(ScP) and (MScP)	$Sc(R_e(C_6)_6; x) = 204x + 288x^2 + 300x^3 + 264x^4 + 252x^5 + 240x^6 + 252x^7 + 264x^8 + 252x^9 + 240x^{10} + 252x^{11} + 132x^{12}$ $Sc^*(R_e(C_6)_6; x) = 246x + 336x^2 + 342x^3 + 300x^4 + 294x^5 + 288x^6 + 294x^7 + 300x^8 + 294x^9 + 288x^{10} + 294x^{11} + 150x^{12}$
Indices and diameter	$Sc(R_e(C_6)_6) = 18336$, $Sc^*(R_e(C_6)_6) = 21360$. $diam(R_e(C_6)_6) = 12$.
The Structure	 $\boldsymbol{R_e(C_6)_7}$
(ScP) and (MScP)	$Sc(R_e(C_6)_7; x) = 238x + 336x^2 + 350x^3 + 308x^4 + 294x^5 + 280x^6 + 294x^7 + 308x^8 + 294x^9 + 280x^{10} + 294x^{11} + 308x^{12} + 294x^{13} + 140x^{14}$ $Sc^*(R_e(C_6)_7; x) = 287x + 392x^2 + 399x^3 + 350x^4 + 343x^5 + 336x^6 + 343x^7 + 350x^8 + 343x^9 + 336x^{10} + 343x^{11} + 350x^{12} + 343x^{13} + 168x^{14}$
Indices and diameter	$Sc(R_e(C_6)_7) = 29022$, $Sc^*(R_e(C_6)_7) = 33831$. $diam(R_e(C_6)_7) = 14$.

4. Conclusion

In this paper, we were able to obtain general formulas for the Schultz and modified Schultz polynomials with their indices for both types of hexagonal rings joining to each other by an edge or bridge, and we compared the results using Mathematica program for many examples, and the results were identical

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