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Generalization of Jensen Mercer inequality on Delta integral with Applications

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Abstract. The aim of this article is to give generalization of Jensen Mercer inequality on delta integral along with applications to Ky Fan inequality and related results.

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1. Introduction

One of the most well-known inequality in mathematics and statistics is Jensen's inequality for convex functions. Because of their importance, Jensen's inequality has received numerous variants, generalizations, and refinements (for reference see [1, 3, 7, 9–12, 14–18, 21, 22]). In 2003, Mercer established a variant of Jensen's inequality known as the Jensen-Mercer inequality [19], which as follows:

Proposition 1. *Let $\zeta : [\mu, \nu] \subset I \to \mathbb{R}$ be a convex function and $x_i \in [\mu, \nu]$, s. t. $\sum_{i=1}^{n} \omega_i = 1$, for $1 \leq i \leq n$, then*

$$\zeta\left(\mu + \nu - \sum_{i=1}^{n} \omega_i x_i\right) \leq \zeta(\mu) + \zeta(v) - \sum_{i=1}^{n} \omega_i \zeta(x_i).$$

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In 1988, Stefan Hilger introduced the idea of theory of time scale calculus in order to unify discrete and continuous analysis and also extend the traditional differential and difference equations in [13]. There have been over a thousand publications in this field, with numerous applications [5] in all branches of science. For interest, readers can see Bohner and Peterson's monograph [6] for an introduction to single-variable time scale calculus and its applications.

Definition 1. *A time scale is an arbitrary nonempty closed subset of the real numbers. Examples of time scales are* $\mathbb{R}, \mathbb{Z}$ *and* $q^{\mathbb{N}_0} := \{q^k | k \in \mathbb{N}_0\}$. *The complex number are not time scale.*

Definition 2. *If* $\mathbb{T}$ *is a time scale, then we define forward jump operator* $\sigma : \mathbb{T} \to \mathbb{R}$ *by* $\sigma(\hat{\theta}) := \inf\{\tau \in \mathbb{T} | \tau > \hat{\theta}\}$ *for all* $\hat{\theta} \in \mathbb{T}$, *the backward jump operator* $\rho : \mathbb{T} \to \mathbb{R}$ *by* $\rho(\hat{\theta}) := \sup\{\tau \in \mathbb{T} | \tau < \hat{\theta}\}$ *for all* $\hat{\theta} \in \mathbb{T}$, *and the graininess function* $\mu : \mathbb{T} \to [0, \infty)$ *by* $\mu(\hat{\theta}) := \sigma(\hat{\theta}) - \hat{\theta}$ *for all* $\hat{\theta} \in \mathbb{T}$. *Furthermore for a function* $g : \mathbb{T} \to \mathbb{R}$, *we define* $g^{\sigma}(\hat{\theta}) = g(\sigma(\hat{\theta}))$ *for all* $\hat{\theta} \in \mathbb{T}$ *and* $g^{\rho}(\hat{\theta}) = g(\rho(\hat{\theta}))$ *for all* $\hat{\theta} \in \mathbb{T}$. *In this definition we use* $\inf \emptyset = \sup \mathbb{T}$ *(i.e.,* $\rho(\hat{\theta}) = \hat{\theta}$ *if* $\hat{\theta}$ *is the maximum of* $\mathbb{T}$*) and* $\sup \emptyset = \inf \mathbb{T}$ *(i.e.,* $\rho(\hat{\theta}) = \hat{\theta}$ *if* $\hat{\theta}$ *is the minimum of* $\mathbb{T}$*).*

These definitions allow us to characterize every point in a time scale as following classification of points:

(i) $\hat{\theta}$ right-scattered $\Longrightarrow \hat{\theta} < \sigma(\hat{\theta})$,

(ii) $\hat{\theta}$ right-dense $\Longrightarrow \hat{\theta} = \sigma(\hat{\theta})$,

(iii) $\hat{\theta}$ left-scattered $\Longrightarrow \rho(\hat{\theta}) < \hat{\theta}$,

(iv) $\hat{\theta}$ left-dense $\Longrightarrow \rho(\hat{\theta}) = \hat{\theta}$,

(v) $\hat{\theta}$ isolated $\Longrightarrow \rho(\hat{\theta}) < \hat{\theta} < \sigma(\hat{\theta})$,

(vi) $\hat{\theta}$ dense $\Longrightarrow \rho(\hat{\theta}) = \hat{\theta} = \sigma(\hat{\theta})$.

We define,
If $\mathbb{T}$ has a left scattered maximum M_1, then we define $\mathbb{T}^k = \mathbb{T}/M_1$; otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right scattered maximum M_2, then we define $\mathbb{T}^k = \mathbb{T}/M_2$; otherwise $\mathbb{T}^k = \mathbb{T}$. Finally we define $\mathbb{T}^* = \mathbb{T}^k \bigcap \mathbb{T}_k$. The mapping $\mu, \nu : \mathbb{T} \to [0, \infty)$ defined by

$$\mu(t) = \sigma(t) - t$$

and

$$\nu(t) = t - \rho(t)$$

are called the forward and backward graininess functions, respectively. In the following consideration, $I_{\mathbb{T}} = I \bigcap \mathbb{T}$ will denote a time scale interval.

Definition 3. *Let $f : \mathbb{T} \to \mathbb{R}$ be a real function on time scale $\mathbb{T}$. Then for $t \in \mathbb{T}^k$, we define a $f^{\Delta}(t)$ to be a number with the property that given any $\varepsilon > 0$, there is a neighbour hood $\mathbf{U}_{\mathbb{T}}$ of t such that*

$$|(f(\rho(t)) - f(s)) - f^{\Delta}(t)[\rho(t) - s]| \leq \varepsilon|\rho(t) - s|$$

for all $s \in \mathbf{U}_{\mathbb{T}}$ we call $f^{\Delta}(t)$ the delta derivative of f at t.

For $f : \mathbb{T} \to \mathbb{R}$, then we define $f^{\sigma} : \mathbb{T} \to \mathbb{R}$ by $f^{\sigma}(t) = f(\sigma(t))$ for $t \in \mathbb{T}$. We define $f^{\rho} : \mathbb{T} \to \mathbb{R}$ by $f^{\rho}(t) = f(\rho(t))$ for $t \in \mathbb{T}$.
Following properties holds for $t \in \mathbb{T}^k$.

(*i*) If f is Δ differentiable at t, then f is continuous at t.

(*ii*) If f is continuous at t and t is right-scattered, then f is delta differentiable at t with $f^{\Delta}(t) = (f(t) - f(\sigma(t))/\upsilon(t))$

(*iii*) If f is right-dense, then f is delta differentiable at t if and only if the $\lim_{s\to t}(f(t) - f(s))/(t - s)$ exit, then $f^{\Delta}(t) = \lim_{s\to t}(f(t) - f(s))/(t - s)$.

(*iv*) If f is Δ differentiable at t, then $f^{\rho}(t) = f(t) + f^{\Delta}(t)\upsilon(t))$.

For more details on time scale, we refer the reader to [20].
In [23], Jensen inequality on delta integral is given as follow:

Proposition 2. *Let $a, b \in \mathbb{T}, a < b$ and $I \subset \mathbb{R}, g \in C([a, b]_{\mathbb{T}}, I)$ and $\omega \in C([a, b]_{\mathbb{T}}, \mathbb{R})$ is a probability density function and also $\zeta \in C(I, \mathbb{R})$ is convex, then*

$$\zeta\left(\int_a^b \omega(\dagger)g(\eta)\Delta\eta\right) \leq \int_a^b \omega(\dagger)\zeta\left(g(\eta)\right)\Delta\eta. \tag{1}$$

In this article we generalize Jensen Mercer inequality for $\Delta-$integral . Also give generalization and refinement of Ky Fan inequality and its related results for $\Delta-$integral.

We establish a Jensen Mercer $\Delta-$integral inequality. Before we further proceed we recall here a lemma from [19] stated as under:

Lemma 1. *Let $\zeta : [\mu, \nu] \subset I \to \mathbb{R}$ be a convex function and $x_i \in [\mu, \nu]$. Then*

$$\zeta\left(\mu + \nu - x_i\right) \leq \zeta\left(\mu\right) + \zeta\left(v\right) - \zeta\left(x_i\right), \quad 1 \leq i \leq n.$$

Theorem 1. *If $g \in C([a, b]_{\mathbb{T}}, [\mu, \nu])$ and $\omega \in C([a, b]_{\mathbb{T}}, [\mu, \nu])$ is a probability density function and also $\zeta \in C([\mu, \nu], \mathbb{R})$ is convex, then*

$$\zeta\left(\mu + \nu - \int_a^b \omega(\dagger)g(\eta)\Delta\eta\right) \leq \zeta\left(\mu\right) + \zeta\left(v\right) - \int_a^b \omega(\dagger)\zeta\left(g(\eta)\right)\Delta\eta. \tag{2}$$

Proof.

Since ζ is convex function and $\left(\mu+\nu-\int_a^b \omega(\dagger)g(\eta)\Delta\eta\right) \in [\mu,\nu]$, therefore by inequality (1) and Lemma 1, we have

$$\begin{aligned}
&\zeta\left(\mu+\nu-\int_a^b \omega(\dagger)g(\eta)\Delta\eta\right)\\
&=\zeta\left(\int_a^b \omega(\dagger)(\mu+\nu-g(\eta))\Delta\eta\right)\\
&\leq \int_a^b \omega(\dagger)\left[\zeta\left(\mu+\nu-g(\eta)\right)\Delta\eta\right]\\
&=\zeta\left(\mu\right)+\zeta\left(v\right)-\int_a^b \omega(\dagger)\zeta\left(g(\eta)\right)\Delta\eta.
\end{aligned}$$

Corollary 1. *Let $\mathbb{T}=\mathbb{R}$ and by considering assumptions of Theorem 1. Then*

$$\zeta\left(\mu+\nu-\int_a^b \omega(\dagger)g(\eta)d\eta\right) \leq \zeta\left(\mu\right)+\zeta\left(v\right)-\int_a^b \omega(\dagger)\zeta\left(g(\eta)\right)d\eta. \tag{3}$$

1.1. Cases

(i) Let $g(\eta)>0$ on $[a,b]_{\mathbb{T}}$ and $\zeta(\dagger)=t^{\beta}$ is convex and concave on $(0,+\infty)$ for $\beta<0$ or $\beta>1$ and for $\beta\in(0,1)$ respectively. Then

$$\zeta\left(\mu+\nu-\int_a^b \omega(\dagger)g(\eta)\Delta\eta\right)^{\beta} \leq \zeta\left(\mu\right)^{\beta}+\zeta\left(v\right)^{\beta}-\int_a^b \omega(\dagger)\zeta\left(g(\eta)\right)^{\beta}\Delta\eta, \quad \beta<0 \text{ or } \beta>1, \tag{4}$$

$$\zeta\left(\mu+\nu-\int_a^b \omega(\dagger)g(\eta)\Delta\eta\right)^{\beta} \geq \zeta\left(\mu\right)^{\beta}+\zeta\left(v\right)^{\beta}-\int_a^b \omega(\dagger)\zeta\left(g(\eta)\right)^{\beta}\Delta\eta, \quad \beta\in(0,1). \tag{5}$$

(ii) Let $g(\eta)>0$ on $[a,b]_{\mathbb{T}}$ and $\zeta(\dagger)=\ln(\dagger)$ is concave on $(0,+\infty)$. Then

$$\ln\left(\mu+\nu-\int_a^b \omega(\dagger)g(\eta)\Delta\eta\right) \leq \ln\left(\mu\right)+\ln\left(v\right)-\int_a^b \omega(\dagger)\ln\left(g(\eta)\right)\Delta\eta.$$

(iii) Let $\mathbb{T} = \mathbb{Z}$ and $m \in \mathbb{N}$. Fix $a = 1$ and $b = m + 1$, let $g : \{1, \ldots, m+1\} \to (0, \infty)$, $\zeta = -\ln x$ is convex on $(0, +\infty)$ and by using Theorem 1, we get

$$\begin{aligned}
&\ln\left(\mu+\nu-\sum_{\dagger=1}^{m} g(\dagger)\right)\\
&=\ln\left(\mu+\nu-\int_{1}^{m+1} g(\eta)\Delta\eta\right)\\
&\geq \ln(\mu)+\ln(v)-\int_{1}^{m+1}\ln(g(\eta))\,\Delta\eta\\
&=\ln(\mu\nu)-\left[\sum_{\dagger=1}^{m}\ln(g(\dagger))\right]\\
&=\ln(\mu\nu)-\left[\ln\left(\prod_{\dagger=1}^{m} g(\dagger)\right)\right],
\end{aligned}$$

and hence

$$\ln\left(\mu+\nu-\sum_{\dagger=1}^{m} g(\dagger)\right)\geq \ln\frac{(\mu\nu)}{\left(\prod_{\dagger=1}^{m} g(\dagger)\right)}.$$

(iv) Let $\mathbb{T} = 2^{\mathbb{M}_{\not{v}}}$ and $M \in \mathbb{N}$. Fix $a = 1$ and $b = 2^M$ and consider a function $g : \{2^l : 0 \leq l \leq N\} \to (0, \infty)$ by using Theorem 1, we get

$$\begin{aligned}
&\ln\left(\mu+\nu-\int_{1}^{2M} g(\dagger)\Delta\right)\\
&=\ln\left(\mu+\nu-\sum_{l=0}^{M-1} 2^l g(2^l)\right)\\
&=\ln\left(\mu+\nu-\int_{1}^{2M} g(\eta)\Delta\eta\right)\\
&\geq \ln(\mu)+\ln(v)-\int_{1}^{2M}\ln(g(\eta))\,\Delta\eta.\\
&=\ln(\mu\nu)-\int_{1}^{2M}\ln(g(\dagger))\,\Delta\dagger\\
&=\ln(\mu\nu)-\sum_{l=0}^{M-1} 2^l\ln\left(g(2^l)\right)
\end{aligned}$$

$$= \ln(\mu\nu) - \left[\ln \prod_{l=0}^{M-1} (g(2^l))^{2^l}\right]$$

and hence

$$\mu + \nu - \left(\sum_{l=0}^{M-1} 2^l g(2^l)\right) \geq \frac{\mu\nu}{\prod\limits_{l=0}^{M-1} \left(g(2^l)\right)^{2^l}}.$$

2. Ky Fan inequality and related results

In 1961, the Ky Fan inequality was given in the famous monograph 'Inequalities' in [4] as follow:

$$\frac{\check{G}_n}{\check{G}'_n} \leq \frac{\check{A}_n}{\check{A}'_n}, \quad x_j \in \left(0, \frac{1}{2}\right] \tag{6}$$

equality holds iff $x_1 = \cdots = x_n$, which magnetize the attention of several mathematician. For generalization and refinement of the Ky Fan inequality see papers [2, 8]. (and references therein)

In this section, we are improving Ky Fan inequality and related results for time scale calculus.

By considering assumptions of Theorem 1, we define the generalized weighted arithmetic mean of $g \in C([a,b]_{\mathbb{T}}, [\mu,\nu])$ with weight ω :

$$\check{A}_{[\mu,\nu]}(g,\omega) = \mu + \nu - \int_a^b \omega(\dagger) g(\eta) \Delta\eta, \tag{7}$$

the generalized weighted geometric mean of the $g \in C([a,b]_{\mathbb{T}}, [\mu,\nu])$ of weight ω:

$$\check{G}_{[\mu,\nu]}(g,\omega) = exp\left[\ln(\mu\nu) - \int_a^b \omega(\dagger) \ln(g(\eta))\, \Delta\eta\right], \tag{8}$$

the generalized weighted harmonic mean of the $g \in C([a,b]_{\mathbb{T}}, [\mu,\nu])$ of weight ω:

$$\check{H}_{[\mu,\nu]}(g,\omega) = \left(\frac{1}{\mu} + \frac{1}{\nu} - \int_a^b \omega(\dagger) \frac{1}{(g(\eta))\, \Delta\eta}\right)^{-1}. \tag{9}$$

Examples

(i) Let $\mathbb{T} = \mathbb{R}$. Then

$$\check{A}_{[\mu,\nu]}(g,\omega) = \mu + \nu - \int_a^b \omega(\dagger) g(\eta) d\eta, \tag{10}$$

$$G_{[\mu,\nu]}(g,\omega) = exp\left[\ln(\mu\nu) - \int_a^b \omega(\dagger)\ln(g(\eta))\,d\eta\right], \tag{11}$$

and

$$\breve{H}_{[\mu,\nu]}(g,\omega) = \left(\frac{1}{\mu} + \frac{1}{\nu} - \int_a^b \omega(\dagger)\frac{1}{(g(\eta))\,d\eta}\right)^{-1}. \tag{12}$$

(ii) Let $\mathbb{T} = \mathbb{Z}, a = 1$ and $b = n+1$, we define $\omega(i) = \omega_i$ and $g(i) = g_i$. The condition for weight for ω means that $\sum_{i=1}^{n} \omega_i > 0$. Then, we have

$$\breve{A}_{[\mu,\nu]}(g,\omega) = \breve{A}_n(g,\omega) = \mu + \nu - \sum_{i=1}^{n} \omega_i g_i, \tag{13}$$

$$\breve{G}_{[\mu,\nu]}(g,\omega) = \bar{G}_n(g,\omega) = \frac{(\mu\nu)}{\prod_{i=1}^{n} g_i^{\omega_i}}, \tag{14}$$

and

$$\breve{H}_{[\mu,\nu]}(g,\omega) = \breve{H}_n(g,\omega) = \left(\frac{1}{\mu} + \frac{1}{\nu} - \sum_{i=1}^{n} \omega_i \frac{1}{(g_i)}\right)^{-1}. \tag{15}$$

Now we establish generalized Ky Fan inequality for time scale.

Theorem 2. *By considering assumptions of Theorem* 1*, and* $g(\eta) \in (0, \frac{\gamma}{2}]$*, where* $0 < r < \gamma < 1$*, then*

$$\frac{\breve{A}_{[\mu,\nu]}(g,\omega)}{\breve{G}_{[\mu,\nu]}(g,\omega)} \geq \frac{\breve{A}_{[\mu,\nu]}(\gamma - g,\omega)}{\breve{G}_{[\mu,\nu]}(\gamma - g,\omega)}.$$

Proof. By applying $\zeta(x) = \ln\frac{\gamma - x}{x}, x \in (0, \frac{\gamma}{2}]$ to the Theorem 1, we obtain required results.

In next theorem we provide refinement of the Ky Fan inequality as follow:

Theorem 3. *By considering assumptions of Theorem* 1 *and* $0 < n \leq g(\eta) \leq N, \gamma > 0$*, then*

$$\frac{\breve{A}_{[\mu,\nu]}(g,\omega)}{\breve{G}_{[\mu,\nu]}(g,\omega)} \geq \left[\frac{\breve{A}_{[\mu,\nu]}(g,\omega)}{\breve{G}_{[\mu,\nu]}(g,\omega)}\right]^{\frac{N^2}{(\gamma - N)^2}} \geq \frac{\breve{A}_{[\mu,\nu]}(\gamma - x,\omega)}{\breve{G}_{[\mu,\nu]}(\gamma - x,\omega)}$$

$$\geq \left[\frac{\breve{A}_{[\mu,\nu]}(g,\omega)}{\breve{G}_{[\mu,\nu]}(g,\omega)}\right]^{\dfrac{n^2}{(\gamma-n)^2}} \geq 1. \tag{16}$$

Proof. From the inequality $\frac{\breve{A}_{[\mu,\nu]}(g,\omega)}{\breve{G}_{[\mu,\nu]}(g,\omega)} \geq 1$ and $n, N \in (0, \frac{\gamma}{2}]$, the first and last inequalities deduced directly.

Let $\phi : (0,\gamma) \to \mathbb{R}$, $\phi(r) = \ln[(\frac{\gamma-r}{r})] + \alpha \ln(r)$ with $\alpha \in \mathbb{R}$, we have

$$\begin{aligned} \phi'(r) &= -\frac{1}{r(\gamma-r)} + \frac{\alpha}{r}, \quad r \in (0,\gamma), \\ \phi''(r) &= \frac{1}{r^2}\left[\frac{\gamma(\gamma-2r)}{(\gamma-r)^2} - \alpha\right], \quad r \in (0,\gamma). \end{aligned}$$

If $\varphi : (0,\gamma) \to \mathbb{R}$, defined as $\varphi(r) = \frac{\gamma(\gamma-2r)}{(\gamma-r)^2}$, then $\varphi'(r) = \frac{2r(r-1)}{(1-r)^4}$, indicating φ is monotonically strictly decreasing on $(0,\gamma)$. Consequently for $r \in (n,N)$, we have

$$\frac{1-2N}{(1-N)^2} = \varphi(N) \leq \varphi(r) \leq \varphi(n) = \frac{1-2n}{(1-n)^2}. \tag{17}$$

If $\alpha \leq \frac{\gamma(\gamma-2N)}{(\gamma-N)^2}$, we conclude from (17) that the function φ is strictly convex on (n,N). Applying Theorem 1 to the function

$$\begin{aligned} &\phi : (n,N) \to \mathbb{R}, \\ &\phi(r) = \ln\left[\left(\frac{\gamma-r}{r}\right)\right] + \alpha \ln(r), \end{aligned}$$

with $\alpha \leq \frac{\gamma(\gamma-2N)}{(\gamma-N)^2}$, we conclude that

$$\begin{aligned} \ln\left(\frac{\gamma-u}{u}\right) + \alpha \ln u + \ln\left(\frac{\gamma-\nu}{\nu}\right) + \alpha \ln v - \int_a^b \omega(\dagger)\left[\ln\left(\frac{\gamma-g(\eta)}{g(\eta)}\right) + \alpha \ln(g(\eta))\right]\Delta\eta \\ \geq \ln\left(\frac{\gamma-u-\nu+\displaystyle\int_a^b \omega(\dagger)g(\eta)\Delta\eta}{\mu+\nu-\displaystyle\int_a^b \omega(\dagger)g(\eta)\Delta\eta}\right) + \alpha \ln\left(\mu+\nu-\int_a^b \omega(\dagger)g(\eta)\Delta\eta\right) \end{aligned}$$

$$\begin{aligned} \ln\frac{\breve{G}_{[\mu,\nu]}(\gamma-g,\omega)}{\breve{G}_{[\mu,\nu]}(g,\omega)} + \alpha \ln \breve{G}_{[\mu,\nu]}(g,\omega) &\geq \ln\frac{\breve{A}_{[\mu,\nu]}(\gamma-g,\omega)}{\breve{A}_{[\mu,\nu]}(g,\omega)} + \alpha \ln \breve{A}_{[\mu,\nu]}(g,\omega) \\ \left(\frac{\breve{G}_{[\mu,\nu]}(g,\omega)}{\breve{A}_{[\mu,\nu]}(g,\omega)}\right)^{\alpha} &\geq \frac{\breve{A}_{[\mu,\nu]}(\gamma-g,\omega)}{\breve{A}_{[\mu,\nu]}(g,\omega)}\frac{\breve{G}_{[\mu,\nu]}(g,\omega)}{\breve{G}_{[\mu,\nu]}(\gamma-g,\omega)} \\ \left(\frac{\breve{G}_{[\mu,\nu]}(g,\omega)}{\breve{A}_{[\mu,\nu]}(g,\omega)}\right)^{\alpha-1} &\geq \left(\frac{\breve{A}_{[\mu,\nu]}(\gamma-x,\omega)}{\breve{G}_{[\mu,\nu]}(\gamma-x,\omega)}\right) \end{aligned} \tag{18}$$

from (18) we observe that this inequality is best possible if we have α is maximal, i.e, $\alpha = \frac{\gamma-2N}{(\gamma-N)2}$, that leads

$$\left(\frac{\breve{G}_{[\mu,\nu]}(g,\omega)}{\breve{A}_{[\mu,\nu]}(g,\omega)}\right)^{\frac{\gamma-2N}{(\gamma-N)^2}-1} \geq \frac{\breve{A}_{[\mu,\nu]}(\gamma-x,\omega)}{\breve{G}_{[\mu,\nu]}(\gamma-x,\omega)}$$

which yield to the second inequality in (16).

We established the third inequality by using the function $F(r) = \beta \ln r - \ln\left[\frac{(\gamma-r)}{r}\right]$ and the same technique. If $\beta \geq \frac{\gamma-2n}{(\gamma-n)^2}$ is true, then the function is strictly convex on (n, N).

Remark 1. *Since the Ky Fan inequality is also equivalent to*

$$\frac{\breve{A}_{[\mu,\nu]}(g,\omega)}{\breve{G}_{[\mu,\nu]}(g,\omega)} \geq \frac{\breve{A}_{[\mu,\nu]}(\gamma-g,\omega)}{\breve{G}_{[\mu,\nu]}(\gamma-g,\omega)},$$

then the first part of the inequality may be seen as refinement of the Ky Fan inequality while the second part

$$\frac{\breve{A}_{[\mu,\nu]}(\gamma-g,\omega)}{\breve{G}_{[\mu,\nu]}(\gamma-g,\omega)} \geq \left(\frac{\breve{A}_{[\mu,\nu]}(g,\omega)}{\breve{G}_{[\mu,\nu]}(g,\omega)}\right)^{\frac{n^2}{(\gamma-n)^2}}$$

can be considered as a counter part of the Ky Fan inequality.

Remark 2. *(i) Let $\mathbb{T} = \mathbb{Z}, a = 1$ and $b = n+1$, we define $\omega(i) = \omega_i$ and $g(i) = g_i$. The condition for ω means that $\sum_{i=1}^{n} \omega_i > 0$. Then, we have*

$$\frac{\breve{A}_n(g,\omega)}{\breve{G}_n(g,\omega)} \geq \left[\frac{\breve{A}_n(g,\omega)}{\breve{G}_n(g,\omega)}\right]^{\frac{N^2}{(\gamma-N)^2}} \geq \frac{\breve{A}_n(\gamma-g,\omega)}{\breve{G}_n(\gamma-g,\omega)} \geq \left[\frac{\breve{A}_n(g,\omega)}{\breve{G}_n(g,\omega)}\right]^{\frac{n^2}{(\gamma-n)^2}} \geq 1.$$

(ii) Let $\mathbb{T} = \mathbb{R}$. Then for weight $\omega : \mathbb{R} \to \mathbb{R}$ and for continuous function $g : \mathbb{R} \to \mathbb{R}$ with $g([\mu,\nu]) \subset [n, N] \subset (0, \frac{\gamma}{2}]$, we have

$$\frac{\breve{A}_{[\mu,\nu]}(g,\omega)}{\breve{G}_{[\mu,\nu]}(g,\omega)} \geq \left[\frac{\breve{A}_{[\mu,\nu]}(g,\omega)}{\breve{G}_{[\mu,\nu]}(g,\omega)}\right]^{\frac{N^2}{(\gamma-N)^2}} \geq \frac{\breve{A}_{[\mu,\nu]}(\gamma-g,\omega)}{\breve{G}_{[\mu,\nu]}(\gamma-g,\omega)} \geq \left[\frac{\breve{A}_{[\mu,\nu]}(g,\omega)}{\breve{G}_{[\mu,\nu]}(g,\omega)}\right]^{\frac{n^2}{(\gamma-n)^2}} \geq 1.$$

Now, we will prove a result related to the inequality $\breve{A}_{[\mu,\nu]}(\gamma-x,\omega) \geq \breve{G}_{[\mu,\nu]}(\gamma-x,\omega)$.

Theorem 4. *By considering the assumptions of Theorem* 1 *and also* $\zeta \in C([\mu,\nu],\mathbb{R})$ *is convex and* $\gamma > 0$, *then*

$$\breve{A}_{[\mu,\nu]}(\gamma-g,\omega) \geq \breve{G}_{[\mu,\nu]}(\gamma-g,\omega).$$

Proof. By applying $\zeta(x) = x - \ln(\gamma - x)$ for all $x \in (0, \frac{\gamma}{2}]$ to the Theorem 1, we get required result.

Now, we present refinement of Ky Fan inequality via convexity.

Theorem 5. *By considering the assumptions of Theorem* 1, *we get*

$$\frac{\breve{A}_{[\mu,\nu]}(\gamma-g,\omega)}{\breve{G}_{[\mu,\nu]}(\gamma-x,\omega)} \leq \frac{1}{\breve{G}_{[\mu,\nu]}(g,\omega)+\breve{G}_{[\mu,\nu]}(\gamma-g,\omega)} \leq \frac{\breve{A}_{[\mu,\nu]}(g,\omega)}{\breve{G}_{[\mu,\nu]}(g,\omega)}. \tag{19}$$

Proof. By using $\zeta(x) = \dfrac{1}{1+e^x}$, for strictly convex on $[0,\infty)$ and strictly concave on $(-\infty,0]$. We apply convex function to the inequality (2) and we define $g(\eta) = \ln \dfrac{\gamma - g(\eta)}{g(\eta)} \geq 0, \mu = \ln\left(\dfrac{\gamma-\mu}{\mu}\right), \nu = \ln\left(\dfrac{\gamma-\nu}{\nu}\right)$, by which we get,

$$\frac{1}{1+e\left(\mu+\nu-\displaystyle\int_a^b \omega(\dagger) g(\eta)\Delta\eta\right)} \leq \frac{1}{1+e^u} + \frac{1}{1+e^\nu} - \int_a^b \omega(\dagger)\frac{1}{1+e^{g(\eta)}}$$

$$\begin{aligned}
&\frac{1}{1+exp\left(\ln\left(\frac{\gamma-\mu}{\mu}\right)+\ln\left(\frac{\gamma-\nu}{\nu}\right)-\displaystyle\int_a^b \omega(\dagger)\ln\left(\dfrac{\gamma-g(\eta)}{g(\eta)}\right)\Delta\eta\right)} \\
&\leq \frac{1}{1+exp\left(\ln\left(\dfrac{\gamma-\mu}{u}\right)\right)} + \frac{1}{1+exp\left(\ln\left(\dfrac{\gamma-\nu}{\nu}\right)\right)} \\
&\qquad - \int_a^b \omega(\dagger)\left(\frac{1}{1+exp\left(\ln\left(\frac{\gamma-g(\eta)}{g(\eta)}\right)\right)}\right)\Delta\eta
\end{aligned}$$

which gives,

$$\begin{aligned}
&\frac{1}{1+exp\left(\ln(\gamma-\mu)(\gamma-\nu)-\displaystyle\int_a^b \omega(\dagger)\ln(\gamma-g(\eta))\Delta\eta\right)-\ln(\mu\nu)+\displaystyle\int_a^b \omega(\dagger)\ln(g(\eta))\Delta\eta} \\
&\leq \left(\mu+\nu-\int_a^b \omega(\dagger) g(\eta)\Delta\eta\right)
\end{aligned}$$

consequently, $\dfrac{1}{1 + exp\left(\ln \dfrac{G_{[\mu,\nu]}((\gamma - g), \omega)}{G_{[\mu,\nu]}(g, \omega)}\right)} \leq A_{[\mu,\nu]}(g, \omega)$

or $\dfrac{1}{\breve{G}_{[\mu,\nu]}(g, \omega) + \breve{G}_{[\mu,\nu]}(\gamma - g, \omega)} \leq \dfrac{\breve{A}_{[\mu,\nu]}(g, \omega)}{\breve{G}_{[\mu,\nu]}(g, \omega)}$, this gives the right hand side of (19).

Now by applying the Theorem 1 for the convex function $-\zeta$ on $(-\infty, 0]$ with $g(\eta) = \ln \dfrac{g(\eta)}{\gamma - g(\eta)} \leq 0$, we get left side of the inequality (19).

Now, we will establish $\breve{A}_{[\mu,\nu]}(g, \omega) \geq \breve{H}_{[\mu,\nu]}(g, \omega)$ and $\breve{A}_{[\mu,\nu]}(1 - g, \omega) \geq \breve{H}_{[\mu,\nu]}(1 - g, \omega)$.

Theorem 6. *By considering the assumptions of Theorem* 1 *also by considering* $g(\eta) \in (0, \frac{\gamma}{2}] \subset [\mu, \nu]$*, then*

(i) $\breve{A}_{[\mu,\nu]}(g, \omega) \geq \breve{H}_{[\mu,\nu]}(g, \omega)$.

(ii) $\breve{A}_{[\mu,\nu]}(\gamma - g, \omega) \geq \breve{H}_{[\mu,\nu]}(\gamma - g, \omega)$.

Proof. (i) By using $\zeta(x) = \frac{1}{x}$ for all $x \in (0, \frac{\gamma}{2}]$ to the inequality (2) we get,

$$\begin{aligned}
\frac{1}{\mu + \nu - \displaystyle\int_a^b \omega(\dagger) g(\eta) \Delta\eta} &\leq \frac{1}{\mu} + \frac{1}{\nu} - \int_a^b \omega(\dagger) \left(\frac{1}{g(\eta)}\right) \Delta\eta \\
A_{[\mu,\nu]}(g, \omega) &\geq \frac{1}{\frac{1}{\mu} + \frac{1}{\nu} - \displaystyle\int_a^b \omega(\dagger) \left(\frac{1}{g(\eta)}\right) \Delta\eta} \\
A_{[\mu,\nu]}(g, \omega) &\geq H_{[\mu,\nu]}(g, \omega).
\end{aligned}$$

Proof. (ii) By using $\phi(x) = \frac{1}{\gamma - x}$ for all $x \in (0, \frac{\gamma}{2}]$ to the inequality (2), we get,

$$\begin{aligned}
\frac{1}{\gamma - (\mu + \nu - \displaystyle\int_a^b \omega(\dagger) g(\eta) \Delta\eta)} &\leq \frac{1}{\gamma - \mu} + \frac{1}{\gamma - \nu} - \int_a^b \omega(\dagger) \left(\frac{1}{\gamma - g(\eta)}\right) \Delta\eta \\
\frac{1}{(\gamma - \mu) + (\gamma - \nu) - \displaystyle\int_a^b \omega(\dagger)(\gamma - g(\eta)) \Delta\eta} &\leq \frac{1}{\gamma - \mu} + \frac{1}{\gamma - \nu} - \int_a^b \omega(\dagger) \left(\frac{1}{\gamma - g(\eta)}\right) \Delta\eta \\
\breve{A}_{[\mu,\nu]}(\gamma - g, \omega) &\geq \breve{H}_{[\mu,\nu]}(\gamma - g, \omega).
\end{aligned}$$

Now, we present arithmetic and harmonic mean inequality.

Theorem 7. *By considering the assumptions of Theorem* 1*, we get*

$$\frac{1}{\check{A}_{[\mu,\nu]}(g,\omega)}-\frac{1}{\check{A}_{[\mu,\nu]}(\gamma-g,\omega)}\leq\frac{1}{\check{H}_{[\mu,\nu]}(g,\omega)}-\frac{1}{\check{H}_{[\mu,\nu]}(\gamma-g,\omega)}.$$

Proof. We establish it by applying the Theorem 1 to the function,

$$\zeta(z)=\frac{1}{z}-\frac{1}{\gamma-z},\quad \left(0<z\leq\frac{\gamma}{2}\right).$$

By which we get,

$$\left(\frac{1}{\mu+\nu-\int_a^b\omega(\dagger)g(\eta)\Delta\eta}\right)-\left(\frac{1}{(\gamma-\mu)+(\gamma-\nu)-\int_a^b\omega(\dagger)\left(\gamma-g(\eta)\right)\Delta\eta}\right)$$
$$\leq\left(\frac{1}{\mu}-\frac{1}{1-\mu}+\frac{1}{\nu}-\frac{1}{\gamma-\nu}-\int_a^b\omega(\dagger)\left(\frac{1}{g(\eta)}-\frac{1}{\gamma-g(\eta)}\right)\Delta\eta\right)$$

left side of inequality gives,

$$\frac{1}{\check{A}_{[\mu,\nu]}(g,\omega)}-\frac{1}{\check{A}_{[\mu,\nu]}(\gamma-g,\omega)}$$

from the right side of the inequality we obtain,

$$\left[\frac{1}{\mu}+\frac{1}{\nu}-\int_a^b\omega(\dagger)\frac{1}{g(\eta)}\Delta\eta\right]-\left[\frac{1}{\gamma-\mu}+\frac{1}{\gamma-\nu}-\int_a^b\omega(\dagger)\left(\frac{1}{(\gamma-g(\eta))}\right)\Delta\eta\right]$$

$$\frac{1}{\left[\frac{1}{\mu}+\frac{1}{\nu}-\int_a^b\omega(\dagger)\frac{1}{g(\eta)}\Delta\eta\right]^{-1}}-\frac{1}{\left[\frac{1}{\gamma-\mu}+\frac{1}{\gamma-\nu}-\int_a^b\omega(\dagger)\left(\frac{1}{\gamma-g(\eta)}\right)\Delta\eta\right]^{-1}}$$
$$=\frac{1}{\check{H}_{[\mu,\nu]}(g,\omega)}-\frac{1}{\check{H}_{[\mu,\nu]}(\gamma-g,\omega)}$$

that completes the proof.

Now, we establish geometric and harmonic mean inequality for time scale.

Theorem 8. *By considering the assumptions of Theorem* 1*, we get* $\check{H}_{[\mu,\nu]}(g,\omega)\leq\check{G}_{[\mu,\nu]}(g,\omega)$.

Proof. By using $\phi(x)=e^x$ for all $x\in[-\infty,\infty)$ to the inequality (2) we get

$$\exp\left(\mu+\nu-\int_a^b\omega(\dagger)g(\eta)\Delta\eta\right)\leq\exp(\mu)+\exp(\nu)-\int_a^b\omega(\dagger)\exp\left(g(\eta)\right)\Delta\eta.$$

By replacing μ by $\ln\left(\frac{1}{\mu}\right)$, ν by $\ln\left(\frac{1}{\nu}\right)$ and $g(\eta)$ by $\ln\left(\frac{1}{g(\eta)}\right)$, then we obtained result.

Theorem 9. *By considering the assumptions of Theorem* 1*, we get*
$\check{H}_{[\mu,\nu]}(\gamma - g, \omega) \leq \check{G}_{[\mu,\nu]}(\gamma - g, \omega)$.

Proof. By applying $\phi(x) = e^x$ for all $x \in (0, \frac{\gamma}{2}]$ to the inequality (2),

$$\exp\left(\mu + \nu - \int_a^b \omega(\dagger) g(\eta) \Delta\eta\right) \leq \exp(\mu) + \exp(\nu) - \int_a^b \omega(\dagger) \exp(g(\eta)) \Delta\eta.$$

By using $\mu = \ln\left(\frac{1}{\gamma-\mu}\right), \nu = \ln\left(\frac{1}{\gamma-\nu}\right)$ and $g(\eta) = \ln\left(\frac{1}{\gamma-g(\eta)}\right)$, then we get required result.

Theorem 10. *By considering the assumptions of Theorem* 1*, we get*

$$\frac{\check{H}_{[\mu,\nu]}(\gamma - g, \omega)}{\check{H}_{[\mu,\nu]}(g, \omega)} \leq \frac{\check{G}_{[\mu,\nu]}(\gamma - g, \omega)}{\check{G}_{[\mu,\nu]}(g, \omega)}. \tag{20}$$

Proof. Without loss of generality we suppose that $g_j's$ are not equal and by using the strictly convex function $\phi(z) = \ln\left(\frac{\gamma - z}{z}\right)$, for all $z \in (0, \frac{\gamma}{2}]$.

We set,

$$y = \frac{H_{[\mu,\nu]}(g, \omega)}{H_{[\mu,\nu]}(g, \omega) + H_{[\mu,\nu]}(\gamma - g, \omega)}, \quad y \in \left(0, \frac{\gamma}{2}\right].$$

Then we get,

$$\begin{aligned}
&\ln\left(\frac{1 - \frac{H_{[\mu,\nu]}(g, \omega)}{H_{[\mu,\nu]}(g, \omega) + H_{[\mu,\nu]}(\gamma - g, \omega)}}{\frac{H_{[\mu,\nu]}(g, \omega)}{H_{[\mu,\nu]}(g, \omega) + H_{[\mu,\nu]}(\gamma - g, \omega)}}\right) \\
&= \ln\left(\frac{H_{[\mu,\nu]}(\gamma - g, \omega)}{H_{[\mu,\nu]}(g, \omega)}\right) \\
&= \ln\left(\frac{\frac{1}{\mu} + \frac{1}{\nu} - \int_a^b \omega(\dagger) \frac{1}{g(\eta)} \Delta\eta}{\frac{1}{\gamma-\mu} + \frac{1}{\gamma-\nu} - \int_a^b \omega(\dagger) \frac{1}{\gamma - g(\eta)} \Delta\eta}\right) \\
&= \ln\left(\frac{1}{\mu} + \frac{1}{\nu} - \int_a^b \omega(\dagger) \frac{1}{g(\eta)} \Delta\eta\right) - \ln\left(\frac{1}{\gamma - \mu} + \frac{1}{\gamma - \mu\nu} - \int_a^b \omega(\dagger) \frac{1}{\gamma - g(\eta)} \Delta\eta\right) \\
&\leq \left(\ln \frac{1}{\mu\nu} - \int_a^b \omega(\dagger) \ln(\frac{1}{g(\eta)}) \Delta\eta\right) - \left(\ln \frac{1}{(\gamma - \mu)(\gamma - \nu)} - \int_a^b \omega(\dagger) \ln\left(\frac{1}{\gamma - g(\eta)}\right) \Delta\eta\right)
\end{aligned}$$

right side of the inequality gives,

$$\ln\left(\frac{\check{G}_{[\mu,\nu]}(\gamma-g,\omega)}{\check{G}_{[\mu,\nu]}(g,\omega)}\right).$$

We get required inequality (20) by taking exponential on both sides.

Theorem 11. *By considering the assumptions of Theorem* 1*, we get*

$$\frac{\check{H}_{[\mu,\nu]}(\gamma-g,\omega)}{\check{G}_{[\mu,\nu]}(\gamma-g,\omega)} \leq \frac{1}{\check{G}_{[\mu,\nu]}(g,\omega)+\check{G}_{[\mu,\nu]}(\gamma-g,\omega)} \leq \frac{\check{H}_{[\mu,\nu]}(g,\omega)}{\check{G}_{[\mu,\nu]}(g,\omega)}. \tag{21}$$

Proof. For the left hand inequality we apply function $\zeta(x)=\dfrac{1}{e^x}+1$ *on* $(-\infty,\infty]$ *to the inequality* (2) *that is,*

$$\frac{1}{\exp\left(\mu+\nu-\int_a^b \omega(\dagger)g(\eta)\Delta\eta\right)}+1$$
$$\leq\left(\frac{1}{\exp(\mu)}+1\right)+\left(\frac{1}{\exp(\nu)}+1\right)-\int_a^b\omega(\dagger)\frac{1}{\exp(g(\eta))+1}\Delta\eta$$

by replacing $g(\eta)=\ln(\frac{\gamma-g(\eta)}{g(\eta)})\geq 0$, μ *by* $\ln(\frac{\gamma-\mu}{\mu})$ *and* ν *by* $\ln(\frac{\gamma-\nu}{\nu})$*, we get*

$$\begin{aligned}\frac{1}{\exp\left[\ln\left(\frac{\check{G}_{[\mu,\nu]}(\gamma-g,\omega)}{\check{G}_{[\mu,\nu]}(\gamma-g,\omega)}\right)\right]}+1 &\leq \left(\frac{1}{\gamma-\mu}+\frac{1}{\gamma-\nu}-\int_a^b\omega(\dagger)\frac{1}{(\gamma-g(\eta))}\Delta\eta\right)\\ \frac{\check{G}_{[\mu,\nu]}(g,\omega)+\check{G}_{[\mu,\nu]}(\gamma-g,\omega)}{\check{G}_{[\mu,\nu]}(\gamma-g,\omega)} &\leq \left(\frac{1}{\gamma-\mu}+\frac{1}{\gamma-\nu}-\int_a^b\omega(\dagger)\frac{1}{(\gamma-g(\eta))}\Delta\eta\right)\\ \frac{\check{H}_{[\mu,\nu]}(\gamma-g,\omega)}{\check{G}_{[\mu,\nu]}(\gamma-g,\omega)} &\leq \frac{1}{\check{G}_{[\mu,\nu]}(\gamma-g,\omega)+\check{G}_{[\mu,\nu]}(g,\omega)}.\end{aligned}$$

To prove right-hand of the inequality (21) we apply inequality (2) *to the convex function* $-\zeta$ *on* $(-\infty,\infty]$ *with* $g(\eta)=\ln\left(\frac{g(\eta)}{\gamma-g(\eta)}\right)\leq 0$, $\mu=\ln\left(\frac{\mu}{\gamma-\mu}\right)$, $\nu=\ln\left(\frac{\nu}{\gamma-\nu}\right)$.

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