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Algebras satisfying a polynomial identity of degree six that are principal train

Daouda Kabre[1], André Conseibo[1,*]

[1] *Départment de Mathématiques, Université Norbert ZONGO, Koudougou, Burkina Faso*

Abstract. In this paper we study the class of algebras satisfying a polynomial identity of degree six that are principal train algebras of rank 3 or 4, for which we give the explicit form of the train equation. If the rank of A is $n \geq 5$ in general, we provide the form of the train equation in some cases.

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1. Introduction

In 1923, Serge Bernstein gave a mathematical proof of the principle of stationarity of Hardy-Weinberg ([3]). From 1939 onwards, Etherington introduced the notion of weighted algebra and principal train algebra for an algebraic model of genetics. However, it was not until 1975 ([5])that Philip Holgate defined algebraically the so-called Bernstein ([6]). Following him, several authors studied various classes of algebras satisfying polynomial identities, in order to model the process of genetic transmission. (see, for instances, [9],[1], [2]). The aim of this paper is to study the algebras verifying the polynomial identity $2x^2x^4 = \omega(x)^2x^4 + \omega(x)^4x^2$ that are principal train algebras. In ([8], the authors prove that such an algebra,assuming the existence of nonzero idempotent, admits a Peirce decomposition. The use of the Peirce decomposition will allow us to finally establish links between this class of algebras and principal train algebras.

2. Preliminaries

Let K be a commutative field and A a commutative K-algebra, not necessarily associative. For any element x of A we define the principal powers of x by $x^1 = x$ and $x^{k+1} = xx^k$ for any integer $k \geq 1$. An idempotent is any element e of A such that $e^2 = e$. In this paper the idempotents considered are all non-zero.

*Corresponding author.
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Email addresses: daoudakabre@yahoo.fr (D. Kabre), andreconsebo@yahoo.fr (A. Conseibo)

Definition 1. *We will say that the algebra A is a baric if there exists a non-zero morphism of algebras $\omega : A \to K$. The morphism ω is then called the weight function of the algebra A. The weight of an element x of A is the scalar $\omega(x)$.*

Definition 2. *A baric K-algebra (A, ω) is a principal train algebra of rank $n \geq 2$ if there are scalars $\gamma_1, \ldots, \gamma_{n-1} \in K$ such that $x^n + \gamma_1\omega(x)x^{n-1} + \cdots + \gamma_{n-1}\omega(x)^{n-1}x = 0$, where the integer $n \geq 2$ is the smallest having this property.*

Definition 3. *A baric K-algebra (A, ω) is a Bernstein algebra if $(x^2)^2 = \omega(x)^2x^2$ for any x in A.*

In the rest of the document, K denotes an algebraically closed infinite commutative field with characteristic different from 2.

In [11], it is shown that if A denotes a Bernstein algebra, then for any x in A, $2x^ix^j = \omega(x)^ix^j + \omega(x)^jx^i$, $\forall i, j \geq 2$; in particular, for $i = 2$ and $j = 4$, $2x^2x^4 = \omega(x)^2x^4 + \omega(x)^4x^2, \forall x \in A$. In this paper, our attention will be focused on the structure of algebras satisfying the latter polynomial identity and that are principal train. Let us consider the identity

$$2x^2x^4 = \omega(x)^2x^4 + \omega(x)^4x^2 \tag{1}$$

In the rest of the paper, $K = \mathbb{C}$, i.e the field of complex numbers. In [8], the authors obtained the following two theorems. .

Theorem 1. *[8] Let (A, ω) be a K-algebra verifying (1) and e be a non-zero idempotent of A. Then A admits a Peirce decomposition relative to e: $A = Ke \oplus A_0 \oplus A_{\frac{1}{2}} \oplus A_\lambda \oplus A_{\bar{\lambda}}$ where $A_\alpha = \{x \in Ker\omega, ex = \alpha x\}$, with $\alpha \in \{0; \frac{1}{2}; \lambda = \frac{-1-i\sqrt{23}}{4}; \bar{\lambda} = \frac{-1+i\sqrt{23}}{4}\}$.*

Theorem 2. *[8] Let $A = K_e \oplus A_0 \oplus A_{\frac{1}{2}} \oplus A_\lambda \oplus A_{\bar{\lambda}}$ be the Peirce decomposition of an algebra verifying (1), then:*

i) $A_0A_0 \subset A_{\frac{1}{2}}$;

ii) $A_{\frac{1}{2}}A_{\frac{1}{2}} \subset A_0 \oplus A_\lambda \oplus A_{\bar{\lambda}}$;

iii) $A_\lambda A_{\bar{\lambda}} = 0$;

iV) $A_\lambda A_\lambda = 0$;

$\dot{V}$) $A_{\bar{\lambda}}A_{\bar{\lambda}} = 0$:

Vi) $A_0A_{\frac{1}{2}} \subset A_{\frac{1}{2}} \oplus A_\lambda \oplus A_{\bar{\lambda}}$;

Vii) $A_\lambda A_{\frac{1}{2}} \subset A_{\frac{1}{2}} \oplus A_0 \oplus A_{\bar{\lambda}}$;

Viii) $A_{\bar{\lambda}}A_{\frac{1}{2}} \subset A_{\frac{1}{2}} \oplus A_0 \oplus A_\lambda$;

iX) $A_0A_\lambda \subset A_{\frac{1}{2}}$;

X) $A_0A_{\bar{\lambda}} \subset A_{\frac{1}{2}}$.

Let (A,ω) be a baric commutative K-algebra not necessarily associative verifying the identity (1). The partial linearisation of this identity gives the following result:

Proposition 1. *Let (A,ω) be a K-algebra verifying (1). For all x,y,z,t in A we have:*
$4x^2[z(t(xy))+z(x(ty))+t(z(xy))+x(z(ty))+t(x(yz))+x(t(yz))+z(y(tx))+t(y(xz))+x(y(tz))+y(z(tx))+y(t(xz))+y(x(tz))]+4xt[2z(x(xy))+2x(z(xy))+2x(x(yz))+2x(y(xz))+2y(x(xz))+z(x^2y)+y(x^2z)]+4ty[zx^3+x(zx^2)+2x(x(xz))]+4xy[z(tx^2)+2z(x(xt))+t(zx^2)+2x(z(xt))+2t(x(xz))+2x(t(xz))+2x(x(tz))]+4tz[2x(x(xy))+x(yx^2)+yx^3]+4yz[tx^3+x(tx^2)+2x(x(xt))]+4xz[2t(x(xy))+2x(t(xy))+2x(x(ty))+t(yx^2)+2x(y(xt))+y(tx^2)+2y(x(xt))] = 2\omega(tz)[2x(x(xy)+x(x^2y)+x^3y]+2\omega(xz)[2t(x(xy)+2x(t(xy)+2x(x(ty)+t(x^2y)+2x(y(xt)+y(tx^2)+2y(x(tx)]+2\omega(xt)[2z(x(xy)+2x(z(xy)+2x(x(zy)+z(x^2y)+2x(y(xz)+y(zx^2)+2y(x(zx)]+2\omega(x^2)[z(t(xy))+z(x(ty))+t(z(xy))+x(z(ty))+t(x(yz))+x(t(yz))+z(y(xt))+t(y(xz))+x(y(tz))+y(z(xt))+y(t(xz))+y(x(tz))]+24[\omega(xyzt)x^2+\omega(yzx^2)xt+\omega(ytx^2)xz+\omega(ztx^2)xy]+8[\omega(x^3y)zt+\omega(x^3z)yt+\omega(x^3t)zy]+2\omega(yz)[tx^3+x(tx^2)+2x(x(xt))]+2\omega(ty)[zx^3+x(zx^2)+2x(x(xz))]+2\omega(xy)[z(tx^2)+t(zx^2)+2z(x(xt))+2x(z(xt))+2t(x(xz))+2x(t(xz))+2x(x(zt))]$

The previous proposition allows us to establish the following lemma.

Lemma 1. *For all $x_0,y_0,z_0 \in A_0$; $x_{\frac{1}{2}},y_{\frac{1}{2}},z_{\frac{1}{2}} \in A_{\frac{1}{2}}$; $x_\lambda,y_\lambda,z_\lambda \in A_\lambda$; $x_{\bar{\lambda}},y_{\bar{\lambda}},z_{\bar{\lambda}} \in A_{\bar{\lambda}}$ the following identities are verified:*

1) $[z_0(x_0y_0)+x_0(y_0z_0)+y_0(x_0z_0)]_\lambda = [z_0(x_0y_0)+x_0(y_0z_0)+y_0(x_0z_0)]_{\bar{\lambda}} = 0$;

2) $[4(x_{\frac{1}{2}}(y_{\frac{1}{2}}z_{\frac{1}{2}})_0+y_{\frac{1}{2}}(x_{\frac{1}{2}}z_{\frac{1}{2}})_0+z_{\frac{1}{2}}(y_{\frac{1}{2}}x_{\frac{1}{2}})_0)+(\lambda+1)(x_{\frac{1}{2}}(y_{\frac{1}{2}}z_{\frac{1}{2}})_\lambda+y_{\frac{1}{2}}(x_{\frac{1}{2}}z_{\frac{1}{2}})_\lambda+z_{\frac{1}{2}}(y_{\frac{1}{2}}x_{\frac{1}{2}})_\lambda)+(\bar{\lambda}+1)(x_{\frac{1}{2}}(y_{\frac{1}{2}}z_{\frac{1}{2}})_{\bar{\lambda}}+y_{\frac{1}{2}}(x_{\frac{1}{2}}z_{\frac{1}{2}})_{\bar{\lambda}}+z_{\frac{1}{2}}(y_{\frac{1}{2}}x_{\frac{1}{2}})_{\bar{\lambda}})]_{\frac{1}{2}} = 0$;

3) $(x_{\frac{1}{2}}(y_0z_0))_0 = 0$;

4) $(\lambda+1)(x_{\frac{1}{2}}(y_0z_0))_\lambda = -4(y_0(x_{\frac{1}{2}}z_0)_{\frac{1}{2}}+z_0(y_0x_{\frac{1}{2}})_{\frac{1}{2}})_\lambda$;

5) $(\bar{\lambda}+1)(x_{\frac{1}{2}}(y_0z_0))_{\bar{\lambda}} = -4(y_0(x_{\frac{1}{2}}z_0)_{\frac{1}{2}}+z_0(y_0x_{\frac{1}{2}})_{\frac{1}{2}})_{\bar{\lambda}}$;

6) $[x_{\bar{\lambda}}(y_0z_0)]_0 = [x_\lambda(y_0z_0)]_0 = 0$;

7) $[y_\lambda(x_0z_\lambda)_{\frac{1}{2}}]_0 = [y_{\bar{\lambda}}(x_0z_{\bar{\lambda}})_{\frac{1}{2}}]_0 = 0$;

8) $[x_\lambda(y_0z_0)]_{\frac{1}{2}} = [x_{\bar{\lambda}}(y_0z_0)]_{\frac{1}{2}} = 0$;

9) $[y_\lambda(x_0z_\lambda)_{\frac{1}{2}}]_{\frac{1}{2}} = [y_{\bar{\lambda}}(x_0z_{\bar{\lambda}})_{\frac{1}{2}}]_{\frac{1}{2}} = 0$;

10) $[x_\lambda(y_0z_0)_{\frac{1}{2}}]_{\bar{\lambda}} = [x_{\bar{\lambda}}(y_0z_0)_{\frac{1}{2}}]_\lambda = 0$;

11) $[y_\lambda(x_0z_\lambda)_{\frac{1}{2}}]_{\bar{\lambda}} = [y_{\bar{\lambda}}(x_0z_{\bar{\lambda}})_{\frac{1}{2}}]_{\bar{\lambda}} = 0$;

12) $[y_0(x_{\bar{\lambda}}z_0)_{\frac{1}{2}} + z_0(x_{\bar{\lambda}}y_0)_{\frac{1}{2}}]_{\bar{\lambda}} = 0;$

13) $[y_0(x_\lambda z_0)_{\frac{1}{2}} + z_0(x_\lambda y_0)_{\frac{1}{2}}]_\lambda = 0;$

14) $(-2\lambda-1)((y_{\frac{1}{2}}(x_0z_{\frac{1}{2}})_\lambda)_0 + (z_{\frac{1}{2}}(x_0y_{\frac{1}{2}})_\lambda)_0) + (-2\bar{\lambda}-1)((y_{\frac{1}{2}}(x_0z_{\frac{1}{2}})_{\bar{\lambda}})_0 + (z_{\frac{1}{2}}(x_0y_{\frac{1}{2}})_{\bar{\lambda}})_0) + 6(y_{\frac{1}{2}}(x_0z_{\frac{1}{2}})_{\frac{1}{2}})_0 + (z_{\frac{1}{2}}(x_0y_{\frac{1}{2}})_{\frac{1}{2}})_0 = 0;$

15) $(x_\lambda(y_{\frac{1}{2}}z_{\frac{1}{2}})_0)_{\frac{1}{2}} = (x_{\bar{\lambda}}(y_{\frac{1}{2}}z_{\frac{1}{2}})_0)_{\frac{1}{2}};$

16) $(-6\lambda-13)((y_{\frac{1}{2}}(x_\lambda z_{\frac{1}{2}})_0)_\lambda + (z_{\frac{1}{2}}(x_\lambda y_{\frac{1}{2}})_0)_\lambda) + (-6\lambda-2\bar{\lambda}-7)((y_{\frac{1}{2}}(x_\lambda z_{\frac{1}{2}})_{\bar{\lambda}})_\lambda + (z_{\frac{1}{2}}(x_\lambda y_{\frac{1}{2}})_{\bar{\lambda}})_\lambda) + (-2\lambda-12)((y_{\frac{1}{2}}(x_\lambda z_{\frac{1}{2}})_{\frac{1}{2}})_\lambda + (z_{\frac{1}{2}}(x_\lambda y_{\frac{1}{2}})_{\frac{1}{2}})_\lambda) = 0;$

17) $(-6\bar{\lambda}-13)((y_{\frac{1}{2}}(x_{\bar{\lambda}}z_{\frac{1}{2}})_0)_{\bar{\lambda}} + (z_{\frac{1}{2}}(x_{\bar{\lambda}}y_{\frac{1}{2}})_0)_{\bar{\lambda}}) + (-6\bar{\lambda}-2\lambda-7)((y_{\frac{1}{2}}(x_{\bar{\lambda}}z_{\frac{1}{2}})_\lambda)_{\bar{\lambda}} + (z_{\frac{1}{2}}(x_{\bar{\lambda}}y_{\frac{1}{2}})_\lambda)_{\bar{\lambda}}) + (-2\bar{\lambda}-12)((y_{\frac{1}{2}}(x_{\bar{\lambda}}z_{\frac{1}{2}})_{\frac{1}{2}})_{\bar{\lambda}} + (z_{\frac{1}{2}}(x_{\bar{\lambda}}y_{\frac{1}{2}})_{\frac{1}{2}})_{\bar{\lambda}}) = 0;$

18) $(y_\lambda(x_{\frac{1}{2}}z_\lambda)_{\frac{1}{2}})_0 + (z_\lambda(x_{\frac{1}{2}}y_\lambda)_{\frac{1}{2}})_0 = (y_{\bar{\lambda}}(x_{\frac{1}{2}}z_{\bar{\lambda}})_{\frac{1}{2}})_0 + (z_{\bar{\lambda}}(x_{\frac{1}{2}}y_{\bar{\lambda}})_{\frac{1}{2}})_0 = (y_\lambda(x_{\frac{1}{2}}z_\lambda)_{\frac{1}{2}})_{\bar{\lambda}} + (z_\lambda(x_{\frac{1}{2}}y_\lambda)_{\frac{1}{2}})_{\bar{\lambda}} = (y_{\bar{\lambda}}(x_{\frac{1}{2}}z_{\bar{\lambda}})_{\frac{1}{2}})_\lambda + (z_{\bar{\lambda}}(x_{\frac{1}{2}}y_{\bar{\lambda}})_{\frac{1}{2}})_\lambda = 0;$

19) $(-14\lambda-6)((y_\lambda(x_{\frac{1}{2}}z_\lambda)_{\frac{1}{2}})_{\frac{1}{2}} + (z_\lambda(x_{\frac{1}{2}}y_\lambda)_{\frac{1}{2}})_{\frac{1}{2}}) - 16\lambda((y_\lambda(x_{\frac{1}{2}}z_\lambda)_0)_{\frac{1}{2}} + (z_\lambda(x_{\frac{1}{2}}y_\lambda)_0)_{\frac{1}{2}} = 0;$

20) $(-14\bar{\lambda}-6)((y_{\bar{\lambda}}(x_{\frac{1}{2}}z_{\bar{\lambda}})_{\frac{1}{2}})_{\frac{1}{2}} + (z_{\bar{\lambda}}(x_{\frac{1}{2}}y_{\bar{\lambda}})_{\frac{1}{2}})_{\frac{1}{2}}) - 16\bar{\lambda}((y_{\bar{\lambda}}(x_{\frac{1}{2}}z_{\bar{\lambda}})_0)_{\frac{1}{2}} + (z_{\bar{\lambda}}(x_{\frac{1}{2}}y_{\bar{\lambda}})_0)_{\frac{1}{2}}) = 0;$

21) $(x_\lambda(y_0z_{\frac{1}{2}})_{\frac{1}{2}})_0 + (z_{\frac{1}{2}}(x_\lambda y_0)_{\frac{1}{2}})_0 = (x_{\bar{\lambda}}(y_0z_{\frac{1}{2}})_{\frac{1}{2}})_0 + (z_{\frac{1}{2}}(x_{\bar{\lambda}}y_0)_{\frac{1}{2}})_0 = 0;$

22) $(z_{\frac{1}{2}}(x_\lambda y_0)_{\frac{1}{2}})_\lambda + (y_0(x_\lambda z_{\frac{1}{2}})_{\frac{1}{2}})_\lambda = (z_{\frac{1}{2}}(x_{\bar{\lambda}}y_0)_{\frac{1}{2}})_{\bar{\lambda}} + (y_0(x_{\bar{\lambda}}z_{\frac{1}{2}})_{\frac{1}{2}})_{\bar{\lambda}} = 0;$

23) $(z_{\frac{1}{2}}(x_\lambda y_0)_{\frac{1}{2}})_{\bar{\lambda}} + (y_0(x_\lambda z_{\frac{1}{2}})_{\frac{1}{2}})_{\bar{\lambda}} + (x_\lambda(z_{\frac{1}{2}}y_0)_{\frac{1}{2}})_{\bar{\lambda}} = 0;$

24) $(z_{\frac{1}{2}}(x_{\bar{\lambda}}y_0)_{\frac{1}{2}})_\lambda + (y_0(x_{\bar{\lambda}}z_{\frac{1}{2}})_{\frac{1}{2}})_\lambda + (x_{\bar{\lambda}}(z_{\frac{1}{2}}y_0)_{\frac{1}{2}})_\lambda = 0;$

25) $(-18\lambda-2\bar{\lambda}+5)(z_\lambda(x_{\bar{\lambda}}y_0)_{\frac{1}{2}})_0 + (-18\bar{\lambda}-2\lambda+5)(x_{\bar{\lambda}}(z_\lambda y_0)_{\frac{1}{2}})_0 = 0;$

26) $(-3\lambda+1)(z_\lambda(x_{\bar{\lambda}}y_0)_{\frac{1}{2}})_{\frac{1}{2}} + (-3\bar{\lambda}+1)(x_{\bar{\lambda}}(z_\lambda y_0)_{\frac{1}{2}})_{\frac{1}{2}} = 0;$

27) $(z_\lambda(x_{\bar{\lambda}}y_0)_{\frac{1}{2}})_{\bar{\lambda}} = (x_{\bar{\lambda}}(z_\lambda y_0)_{\frac{1}{2}})_\lambda = 0;$

28) $(-12\lambda-2\bar{\lambda}+4)(z_\lambda(x_{\bar{\lambda}}y_{\frac{1}{2}})_{\frac{1}{2}})_0 + (-12\bar{\lambda}-2\lambda+4)(x_{\bar{\lambda}}(z_\lambda y_{\frac{1}{2}})_{\frac{1}{2}})_0 = 0;$

29) $(-12\lambda+6)(z_\lambda(x_{\bar{\lambda}}y_{\frac{1}{2}})_{\frac{1}{2}})_{\frac{1}{2}} + (-12\bar{\lambda}+6)(x_{\bar{\lambda}}(z_\lambda y_{\frac{1}{2}})_{\frac{1}{2}})_{\frac{1}{2}} - 16\lambda(z_\lambda(x_{\bar{\lambda}}y_{\frac{1}{2}})_0)_{\frac{1}{2}} - 16\bar{\lambda}(x_{\bar{\lambda}}(z_\lambda y_{\frac{1}{2}})_0)_{\frac{1}{2}} = 0;$

30) $(z_\lambda(x_{\bar{\lambda}}y_{\frac{1}{2}})_{\frac{1}{2}})_{\bar{\lambda}} = (x_{\bar{\lambda}}(z_\lambda y_{\frac{1}{2}})_{\frac{1}{2}})_\lambda = 0.$

Proof. Consider the identity of Proposition 1. Setting $x = e$, $y \in A_\alpha$, $z \in A_\beta$, $t \in A_\gamma$, we have respectively $ey = \alpha y$, $ez = \beta z$, $et = \gamma t$ and:

$$\begin{gathered}4\alpha e(z(ty) + 4e(z(e(ty))) + 4\alpha e(t(zy)) + 4e(e(z(ty))) + 4e(t(e(yz))) + 4e(e(t(yz)))+\\ 4\gamma e(z(yt)) + 4\beta e(t(yz)) + 4e(e(y(tz))) + 4\gamma e(y(zt)) + 4\beta ey(tz)) + 4e(y(e(tz))+\\ 8\gamma\alpha^2 t(zy) + 8\gamma\alpha t(e(zy)) + 8\gamma t(e(e(yz))) + 8\gamma\beta t(e(yz)) + 8\gamma\beta^2 t(yz) + 4\gamma\alpha t(zy)+\\ 4\gamma\beta t(yz) + 4\beta z(ty) + 4\beta^2 z(ty) + 8\beta^3 z(ty) + 4\alpha\gamma y(zt) + 8\alpha\gamma^2 y(zt) + 4\alpha\beta y(tz)+\\ 8\alpha\gamma y(e(zt)) + 8\alpha\beta^2 y(tz) + 8\alpha\beta y(e(tz)) + 8\alpha e(e(tz)) + 8\beta^3 y(tz) + 4\beta^2 y(tz) + 4\beta y(tz)+\\ 4\gamma t(yz) + 4\gamma^2 t(yz) + 8\gamma^3 t(yz) + 8\beta^3 z(ty) + 8\beta^2 z(e(ty)) + 8\beta z(e(e(ty))) + 4\beta^2 z(ty)+\\ 8\beta\gamma z(e(yt)) + 4\beta\gamma z(yt) + 8\beta\gamma^2 z(yt) = 2\beta z(ty) + 2z(e(ty)) + 2\alpha t(zy) + 2e(z(ty))+\\ 2t(e(yz) + 2e(t(yz)) + 2\gamma z(yt) + 2\beta t(yz) + 2e(y(tz)) + 2\gamma y(zt) + 2\beta y(tz) + 2y(e(tz))\end{gathered} \tag{2}$$

By setting $\alpha = \beta = \gamma = 0$, the relation (2) becomes

$$\begin{aligned}4e(z(e(ty))) + 4e(e(z(ty))) + 4e(t(e(yz))) + 4e(e(t(yz))) + 4e(e(y(tz))) + 4e(y(e(tz))\\ = 2z(e(ty)) + 2e(z(ty)) + 2e(t(yz)) + 2e(y(tz)) + 2y(e(tz)) + 2t(e(yz)\end{aligned} \tag{3}$$

Since $A_0^2 \subset A_{\frac{1}{2}}$ according to Theorem 2, therefore the relation (3) becomes

$$4e(e(z(ty))) + 4e(e(t(yz))) + 4e(e(y(tz))) = z(ty) + y(tz) + t(yz) \tag{4}$$

Using the relations $i)$ and $vi)$ of Theorem 2, relation (4) gives $[z(ty)]_{\frac{1}{2}} + 4\lambda^2[z(ty)]_\lambda + 4\bar{\lambda}^2[z(ty)]_{\bar{\lambda}} + [t(yz)]_{\frac{1}{2}} + 4\lambda^2[t(yz)]_\lambda + 4\bar{\lambda}^2[t(yz)]_{\bar{\lambda}} + [y(tz)]_{\frac{1}{2}} + 4\lambda^2[y(tz)]_\lambda + 4\bar{\lambda}^2[y(tz)]_{\bar{\lambda}} = [z(ty)]_{\frac{1}{2}} + [z(ty)]_\lambda + [z(ty)]_{\bar{\lambda}} + [t(yz)]_{\frac{1}{2}} + [t(yz)]_\lambda + [t(yz)]_{\bar{\lambda}} + [y(tz)]_{\frac{1}{2}} + [y(tz)]_\lambda + [y(tz)]_{\bar{\lambda}}$
which implies that

$$\begin{cases}(4\lambda^2 - 1)([z(ty)]_\lambda + [t(yz)]_\lambda + [y(tz)]_\lambda) = 0\\ (4\bar{\lambda}^2 - 1)([z(ty)]_{\bar{\lambda}} + [t(yz)]_{\bar{\lambda}} + [y(tz)]_{\bar{\lambda}}) = 0\end{cases}$$

As $4\lambda^2 - 1 \neq 0$ and $4\bar{\lambda}^2 - 1 \neq 0$, then $[z(ty)]_\lambda + [t(yz)]_\lambda + [y(tz)]_\lambda = [z(ty)]_{\bar{\lambda}} + [t(yz)]_{\bar{\lambda}} + [y(tz)]_{\bar{\lambda}} = 0$; posing $t = x_0; y = y_0; z = z_0$, we have (1).
By proceeding in a similar way, we find the other identities.

the following two results are immediate consequences of Lemma 1 and Theorem2.

Corollary 1. *If* $A_0 = 0$, *then* $A_{\frac{1}{2}}^2 \subset A_\lambda \oplus A_{\bar{\lambda}}$, $A_{\frac{1}{2}}A_\lambda \subset A_{\frac{1}{2}} \oplus A_{\bar{\lambda}}$, $A_{\frac{1}{2}}A_{\bar{\lambda}} \subset A_{\frac{1}{2}} \oplus A_\lambda$, $A_\lambda^2 = A_{\bar{\lambda}}^2 = A_\lambda A_{\bar{\lambda}} = 0$ *and for all* $x_{\frac{1}{2}} \in A_{\frac{1}{2}}$; $x_\lambda \in A_\lambda$; $x_{\bar{\lambda}} \in A_{\bar{\lambda}}$ *the following identities are verified:*

i) $[(\lambda+1)x_{\frac{1}{2}}(x_{\frac{1}{2}}^2)_\lambda + (\bar{\lambda}+1)x_{\frac{1}{2}}(x_{\frac{1}{2}}^2)_{\bar{\lambda}}]_{1/2} = 0$;

ii) $(2\lambda+3)(x_{\frac{1}{2}}(x_{\frac{1}{2}}x_\lambda)_{\bar{\lambda}})_\lambda + (\lambda+6)(x_{\frac{1}{2}}(x_{\frac{1}{2}}x_\lambda)_{\frac{1}{2}})_\lambda = 0$;

iii) $(2\bar{\lambda}+3)(x_{\frac{1}{2}}(x_{\frac{1}{2}}x_{\bar{\lambda}})_{\lambda})_{\bar{\lambda}}+(\bar{\lambda}+6)(x_{\frac{1}{2}}(x_{\frac{1}{2}}x_{\bar{\lambda}})_{\frac{1}{2}})_{\bar{\lambda}}=0$;

iv) $x_{\lambda}(x_{\lambda}x_{\frac{1}{2}})=0$;

v) $x_{\bar{\lambda}}(x_{\bar{\lambda}}x_{\frac{1}{2}})=0$;

vi) $(x_{\lambda}(x_{\bar{\lambda}}x_{\frac{1}{2}}))_{\bar{\lambda}}=0$;

vii) $(x_{\bar{\lambda}}(x_{\lambda}x_{\frac{1}{2}}))_{\lambda}=0$;

viii) $(2\lambda-1)x_{\lambda}(x_{\bar{\lambda}}x_{\frac{1}{2}})+(2\bar{\lambda}-1)x_{\bar{\lambda}}(x_{\lambda}x_{\frac{1}{2}})=0$.

Corollary 2. *If $A_{\bar{\alpha}}=0$ with $\alpha\in\{\lambda,\bar{\lambda}\}$, then the following identities are verified:*

i) $2ex_0^3=x_0^3$;

ii) $[12(x_{\frac{1}{2}}(x_{\frac{1}{2}}^2)_0)+3(\alpha+1)(x_{\frac{1}{2}}(x_{\frac{1}{2}}^2)_{\alpha})]_{\frac{1}{2}}=0$;

iii) $[(\alpha+1)(x_{\frac{1}{2}}(x_0^2)_{\frac{1}{2}})+8(x_0(x_0x_{\frac{1}{2}})_{\frac{1}{2}})]_{\alpha}=0$;

iV) $[(-4\alpha-2)(x_{\frac{1}{2}}(x_0x_{\frac{1}{2}})_{\alpha})+12(x_{\frac{1}{2}}(x_0x_{\frac{1}{2}})_{\frac{1}{2}})]_0=0$;

V) $[(6\alpha+13)(x_{\frac{1}{2}}(x_{\alpha}x_{\frac{1}{2}})_0)+(2\alpha+12)(x_{\frac{1}{2}}(x_{\alpha}x_{\frac{1}{2}})_{\frac{1}{2}})]_{\alpha}=0$;

Vi) $[(7\alpha+3)(x_{\alpha}(x_{\alpha}x_{\frac{1}{2}})_{\frac{1}{2}})+8\alpha(x_{\alpha}(x_{\alpha}x_{\frac{1}{2}})_0)]_{\frac{1}{2}}=0$;

Vii) $[x_{\frac{1}{2}}(x_0^2)_{\frac{1}{2}}]_0=[x_{\alpha}(x_0^2)_{\frac{1}{2}}]_0=[x_{\alpha}(x_0x_{\alpha})_{\frac{1}{2}}]_0=[x_{\alpha}(x_{\frac{1}{2}}x_{\alpha})_{\frac{1}{2}}]_0=[x_{\alpha}(x_{\frac{1}{2}}x_0)_{\frac{1}{2}}]_0=0$;

Viii) $[x_{\alpha}(x_0^2)_{\frac{1}{2}}]_{\frac{1}{2}}=[x_{\alpha}(x_0x_{\alpha})_{\frac{1}{2}}]_{\frac{1}{2}}=[x_{\alpha}(x_{\frac{1}{2}}^2)_0]_{\frac{1}{2}}=0$;

iX) $[x_0(x_0x_{\alpha})_{\frac{1}{2}}]_{\alpha}=[x_{\frac{1}{2}}(x_0x_{\alpha})_{\frac{1}{2}}]_{\alpha}=0$.

In [4], the author give Peirce decomposition of a principal train algebra

Theorem 3. *(see Theorem 1, [4]) Let (A,ω) be a principal train algebra with an idempotent e and with principal train polynomial $P(X)=(X-1)(X-\lambda_1)\cdots(X-\lambda_{r-1})$ (the λ_i are two by two distinct). Then A splits into the direct sum $A=Ke\oplus V_1\oplus V_2\oplus\cdots\oplus V_s$ where $V_i=N\cap(L_e-\gamma_i i_d)$, $L_e:A\to A$, $x\mapsto ex$, $i_d:A\to A$, $x\mapsto x$.*

In [7] the authors gave a characterization of principal train algebras of rank 4.

Theorem 4. *(Theorem 5, [7]) Let (A,ω) be a baric algebra. The algebra A is a principal train algebra of rank 4, with principal train polynomial $X(X-1)(X-\lambda_1)(X-\lambda_2)$ where λ_1 and λ_2 different and different from $\frac{1}{2}$, if and only if:*

1) *A possesses an idempotent e and, with respect to e, it has the Peirce decomposition $A=Ke\oplus U_{1/2}\oplus U_{\lambda_1}\oplus U_{\lambda_2}$ where $U_i=\{x\in ker\omega, ex=ix\}$ $(i\in\{1/2,\lambda_1,\lambda_2\})$*

2) $U_{1/2}^2 \subset U_{\lambda_1} \oplus U_{\lambda_2}$, $U_{1/2}U_{\lambda_1} \subset U_{1/2} \oplus U_{\lambda_2}$, $U_{1/2}U_{\lambda_2} \subset U_{1/2} \oplus U_{\lambda_1}$, $U_{\lambda_1}^2 \subset U_{\lambda_2}$, $U_{\lambda_2}^2 \subset U_{\lambda_1}$, $U_{\lambda_1}U_{\lambda_2} = 0$,$(U_{\lambda_1} \oplus U_{\lambda_2})^3 = 0$.

3) *For all $x \in ker\omega$, $u \in U_{1/2}$, $v \in U_{\lambda_1}$ and $w \in U_{\lambda_2}$, the following relations are verified:*

(i) $(\frac{1}{2} - \lambda_2)(u(u^2)_{\lambda_1})_{1/2} + (\frac{1}{2} - \lambda_1)(u(u^2)_{\lambda_2})_{1/2}$ *;*

(ii) $(\lambda_1 - \lambda_2)(u(uv)_{1/2})_{\lambda_1} + (\lambda_1 - \frac{1}{2})(u(uv)_{\lambda_2})_{\lambda_1}$ *;*

(iii) $(\lambda_2 - \lambda_1)(u(uw)_{1/2})_{\lambda_2} + (\lambda_2 - \frac{1}{2})(u(uw)_{\lambda_1})_{\lambda_2}$ *;*

(iv) $(1 - 2\lambda_2)(v(uv))_{1/2} + (\lambda_1 - \frac{1}{2})uv^2$ *;*

(v) $(1 - 2\lambda_1)(w(uw))_{1/2} + (\lambda_2 - \frac{1}{2})uw^2$ *;*

(vi) $(\frac{1}{2} - \lambda_1)(v(uw)_{1/2})_{1/2} + (\frac{1}{2} - \lambda_2)(w(uv)_{1/2})_{1/2}$ *;*

(vii) $(\lambda_1 - \lambda_2)(w(uv)_{1/2})_{\lambda_1} + (\lambda_1 - \frac{1}{2})w(uv)_{\lambda_2}$ *;*

(viii) $(\lambda_2 - \lambda_1)(w(uv)_{1/2})_{\lambda_2} + (\lambda_2 - \frac{1}{2})w(uv)_{\lambda_1}$ *; where $(x)_i$ denotes the projection of $x \in ker\omega$ onto the subspace U_i, $i \in \{\frac{1}{2}, \lambda_1, \lambda_2\}$;*

(ix) $x^4 = 0$.

3. Relation with principal train algebras

Proposition 2. *Let $A = Ke \oplus A_0 \oplus A_{\frac{1}{2}} \oplus A_\lambda \oplus A_{\bar{\lambda}}$ an algebra satisfying the identity $2x^2x^4 = \omega(x)^2x^4 + \omega(x)^4x^2$. If $A_{\frac{1}{2}} = 0$ then A is a principal train algebra satisfying the equation $x^5 - \frac{1}{2}\omega(x)x^4 + \omega(x)^2x^3 - \frac{3}{2}\omega(x)^3x^2 = 0$.*

Proof. $A_{\frac{1}{2}}$ being zero, we have $A_0^2 = A_\lambda^2 = A_{\bar{\lambda}}^2 = A_\lambda A_0 = A_{\bar{\lambda}}A_0 = A_{\bar{\lambda}}A_\lambda = 0$.
For $x = e + x_0 + x_\lambda + x_{\bar{\lambda}}$, we have $x^2 = e + 2\lambda x_\lambda + 2\bar{\lambda}x_{\bar{\lambda}}$, $x^3 = e - 3x_\lambda - 3x_{\bar{\lambda}}$, $x^4 = e - 2\lambda x_\lambda - 2\bar{\lambda}x_{\bar{\lambda}}$, $x^5 = e + (2\lambda + 3)x_\lambda + (2\bar{\lambda} + 3)x_{\bar{\lambda}}$. So, $x^5 - \frac{1}{2}x^4 + x^3 - \frac{3}{2}x^2 = 0$ and we obtain $x^5 - \frac{1}{2}\omega(x)x^4 + \omega(x)^2x^3 - \frac{3}{2}\omega(x)^3x^2 = 0$ because the set of elements of weight 1 is dense in A according to Zariski's topology.

Proposition 3. *Let $A = Ke \oplus A_0 \oplus A_{\frac{1}{2}} \oplus A_\lambda \oplus A_{\bar{\lambda}}$ be a Peirce decomposition of an algebra satisfying the identity $2x^2x^4 = \omega(x)^2x^4 + \omega(x)^4x^2$. If $A_0 = A_{\bar{\alpha}} = 0$ with $\alpha \in \{\bar{\lambda}, \bar{\bar{\lambda}}\}$, then A checks the train equation*

$$x^3 - (1 + \alpha)\omega(x)x^2 + \alpha\omega(x)^2x = 0.$$

Proof. Let $A = Ke \oplus A_0 \oplus A_{\frac{1}{2}} \oplus A_\lambda \oplus A_{\bar{\lambda}}$ be an algebra satisfying the identity $2x^2x^4 = \omega(x)^2x^4 + \omega(x)^4x^2$. Suppose $A_0 = A_{\bar{\lambda}} = 0$.
Let $x = e + x_{\frac{1}{2}} + x_\lambda$ be an element of weight 1 of A. By exploiting the relations of the Corrolary 2, we have: $x^2 = e + x_{\frac{1}{2}} + 2\lambda x_\lambda + x_{\frac{1}{2}}^2 + 2x_{\frac{1}{2}}x_\lambda$, $x^2 - x = (2\lambda - 1)x_\lambda + x_{\frac{1}{2}}^2 + 2x_{\frac{1}{2}}x_\lambda$, $x(x^2 - x) = \lambda x_{\frac{1}{2}}^2 + (2\lambda^2 - \lambda)x_\lambda + 2\lambda x_\lambda x_{\frac{1}{2}} = \lambda(x^2 - x)$, so $x(x^2 - x) = \lambda(x^2 - x)$ and

$x^3-(1+\lambda)x^2+\lambda x=0$. Since the set of elements of weight 1 is dense in A by the Zariski's topology, then for any x in A, we have $x^3-(1+\lambda)\omega(x)x^2+\lambda\omega(x)^2x=0$. The proof is similar when we assume $A_0=A_\lambda=0$.

Theorem 5. *Let A be an algebra satisfying the identity $2x^2x^4=\omega(x)^2x^4+\omega(x)^4x^2$; then A is a principal train algebra of rank 3 if and only its train equation is of the form $x^3-(1+\gamma)\omega(x)x^2+\gamma\omega(x)^2x=0$, where $\gamma\in\{0,\lambda,\bar{\lambda}\}$*

Proof. Let A be an algebra satisfying the identity $2x^2x^4=\omega(x)^2x^4+\omega(x)^4x^2$.

Suppose A is a principal train algebra of rank 3, its equation is

$$x^3-(1+\alpha)\omega(x)x^2+\alpha\omega(x)^2x=0 \quad \textit{with} \quad \alpha\in K \tag{5}$$

And a partial linearisation of (5) gives us

$$x^2y+2x(xy)-(1+\alpha)[\omega(y)x^2+2\omega(x)xy]+\alpha[2\omega(xy)x+\omega(x)^2y]=0 \tag{6}$$

setting $y=x^4$ in (6), we have

$$x^2x^4+2x^6-(1+\alpha)\omega(x)^4x^2-2(1+\alpha)\omega(x)x^5+2\alpha\omega(x)^5x+\alpha\omega(x)^2x^4=0 \tag{7}$$

or $2x^6=2(1+\alpha)\omega(x)x^5-2\alpha\omega(x)^2x^4$, we also know that $x^2x^4=\frac{1}{2}\omega(x)^2x^4+\frac{1}{2}\omega(x)^4x^2$; substituting $2x^6$ and x^2x^4 by their expressions in (7), we get

$$(\frac{1}{2}-\alpha)\omega(x)^2x^4-(\frac{1}{2}+\alpha)\omega(x)^4x^2+2\alpha\omega(x)^5x=0 \tag{8}$$

We can notice that $x^3=(1+\alpha)\omega(x)x^2-\alpha\omega(x)^2x$; which implies that $x^4=(1+\alpha)\omega(x)x^3-\alpha\omega(x)^2x^2=(1+\alpha+\alpha^2)\omega(x)^2x^2+(-\alpha^2-\alpha)\omega(x)^3x$. Substituting x^4 by its expression in (8), we get
$-\frac{1}{2}\alpha(2\alpha^2+\alpha+3)\omega(x)^4(x^2-\omega(x)x)=0$. The algebra A being of rank 3, then $x^2-\omega(x)x\neq 0$ which implies that $-\frac{1}{2}\alpha(2\alpha^2+\alpha+3)=0$ hence $\alpha=0$, $\alpha=\lambda$ or $\alpha=\bar{\lambda}$.

Conversely, suppose that A is a principal train algebra of train equation

$$x^3-(1+\alpha)\omega(x)x^2+\alpha\omega(x)^2x=0 \quad \textit{with} \quad \alpha\in\{0,\lambda,\bar{\lambda}\} \tag{9}$$

If $\alpha=0$, A is a Bernstein Jordan algebra (see [10]) and therfore satisfies the identity $2x^2x^4=\omega(x)^2x^4+\omega(x)^4x^2$ (see [11]).

For $\alpha=\lambda$, the partial linearisation of (9) gives us

$$x^2y+2x(xy)-(1+\lambda)[\omega(y)x^2+2\omega(x)xy]+\lambda[2\omega(xy)x+\omega(x)^2y]=0 \tag{10}$$

By setting $y=x^4$, we have $x^2x^4=-2x^6+(1+\lambda)[\omega(x)^4x^2+2\omega(x)x^5]-\lambda[2\omega(x)^5x+\omega(x)^2x^4]$, so $x^2x^4=[-2x^6+2(1+\lambda)\omega(x)x^5-2\lambda\omega(x)^2x^4]+\lambda\omega(x)^2x^4+(1+\lambda)\omega(x)^4x^2-2\lambda\omega(x)^5x=(1+\lambda)\omega(x)^4x^2-2\lambda\omega(x)^5x+\lambda\omega(x)^2x^4$, hence
$x^2x^4-\frac{1}{2}\omega(x)^4x^2-\frac{1}{2}\omega(x)^2x^4=(\lambda^2+\frac{\lambda}{2}-\frac{1}{2})\omega(x)^3x^3+(-\lambda^2+\frac{3\lambda}{2}+\frac{1}{2})\omega(x)^4x^2-2\lambda\omega(x)^5x$, so $x^2x^4-\frac{1}{2}\omega(x)^4x^2-\frac{1}{2}\omega(x)^2x^4=-2\omega(x)^3x^3+(2\lambda+2)\omega(x)^4x^2-2\lambda\omega(x)^5x$. Therefore $x^2x^4-\frac{1}{2}\omega(x)^4x^2-\frac{1}{2}\omega(x)^2x^4=-2\omega(x)^3(x^3-(\lambda+1)\omega(x)x^2+\lambda\omega(x)^2x)=0$ and A satisfies the identity $2x^2x^4=\omega(x)^2x^4+\omega(x)^4x^2$. The proof is similar for $\alpha=\bar{\lambda}$.

Theorem 6. *Let* $A = Ke \oplus A_0 \oplus A_{\frac{1}{2}} \oplus A_\lambda \oplus A_{\bar{\lambda}}$ *be an algebra satisfying the identity* $2x^2x^4 = \omega(x)^2x^4 + \omega(x)^4x^2$*; if* A *is a principal train algebra of rank* 4*, its train equation is one of the following forms:*

i) $x^4 - (1+\gamma)\omega(x)x^3 + \gamma\omega(x)^2x^2 = 0,\ \gamma \in \{0, \lambda, \bar{\lambda}\}$;

ii) $x^4 - \frac{1}{2}\omega(x)x^3 + \omega(x)^2x^2 - \frac{3}{2}\omega(x)^3x = 0$;

iii) $x^4 - (\frac{3}{2}+\gamma)\omega(x)x^3 + (\frac{1}{2}+\frac{3}{2}\gamma)\omega(x)^2x^2 - \frac{1}{2}\gamma\omega(x)^3x = 0\ ;\ \gamma \in \{\frac{1}{2}, \lambda, \bar{\lambda}\}$;

iV) $x^4 - (1+2\gamma)\omega(x)x^3 + \gamma(\gamma+2)\omega(x)^2x^2 - \gamma^2\omega(x)^3x = 0\ ;\ \gamma \in \{\lambda, \bar{\lambda}\}$.

Proof. Let $A = Ke \oplus A_0 \oplus A_{\frac{1}{2}} \oplus A_\lambda \oplus A_{\bar{\lambda}}$ be an algebra satisfying the identity $2x^2x^4 = \omega(x)^2x^4 + \omega(x)^4x^2$. Assuming that A is a principal train algebra of rank 4, its train equation is of the form $x^4 - (1+\alpha+\beta)\omega(x)x^3 + \alpha\omega(x)^2x^2 + \beta\omega(x)^3x = 0$ with $\alpha, \beta \in K$ so its minimal train polynomial is $P(X) = X(X-1)(X-\alpha_1)(X-\alpha_2) = X^4 - (1+\alpha_1+\alpha_2)X^3 + (\alpha_1+\alpha_2+\alpha_1\alpha_2)X^2 - \alpha_1\alpha_2X$; we then notice that $\alpha = \alpha_1+\alpha_2+\alpha_1\alpha_2$ and $\beta = -\alpha_1\alpha_2$ therefore

$$x^4 - (1+\alpha_1+\alpha_2)\omega(x)x^3 + (\alpha_1+\alpha_2+\alpha_1\alpha_2)\omega(x)^2x^2 - \alpha_1\alpha_2\omega(x)^3x = 0 \qquad (11)$$

Now let us look at the different cases related to the train roots α_1 and α_2:

$1st$ **Case**: $\alpha_1 \neq \alpha_2$, $\alpha_1 \neq \frac{1}{2}$ and $\alpha_2 \neq \frac{1}{2}$
By exploiting the theorem 5 of [7] and the theorem 1, we observe that A admits relatively to an idempotent e, the following Peirce decomposition: $A = Ke \oplus A_{\frac{1}{2}} \oplus A_{\alpha_1} \oplus A_{\alpha_2} = Ke \oplus A_0 \oplus A_{\frac{1}{2}} \oplus A_\lambda \oplus A_{\bar{\lambda}}$ then we have by identification $\alpha_1, \alpha_2 \in \{0, \lambda, \bar{\lambda}\}$. Indeed:
for $\alpha_1 = 0$ and $\alpha_2 = \lambda$, (11) becomes $x^4 - (1+\lambda)\omega(x)x^3 + \lambda\omega(x)^2x^2 = 0$,
for $\alpha_1 = 0$ and $\alpha_2 = \bar{\lambda}$, the equation (11) becomes
$x^4 - (1+\bar{\lambda})\omega(x)x^3 + \bar{\lambda}\omega(x)^2x^2 = 0$, and
if $\alpha_1 = \lambda$ and $\alpha_2 = \bar{\lambda}$, (11) becomes $x^4 - \frac{1}{2}\omega(x)x^3 + \omega(x)^2x^2 - \frac{3}{2}\omega(x)^3x = 0$

$2nd$ **Case**: $\alpha_1 \neq \alpha_2$ and $\alpha_1 = \frac{1}{2}$
Considering the theorem 1 of [4] and the theorem (1), it follows that A admits the following Peirce decomposition: $A = Ke \oplus A_{\frac{1}{2}} \oplus A_{\alpha_2}$ with $\alpha_2 \in \{0, \lambda, \bar{\lambda}\}$. The train equation is therefore one of the following forms:
For $\alpha_1 = \frac{1}{2}$ and $\alpha_2 = 0$, (11) becomes $x^4 - \frac{3}{2}\omega(x)x^3 + \frac{1}{2}\omega(x)^2x^2 = 0$;
For $\alpha_1 = \frac{1}{2}$ and $\alpha_2 = \lambda$, the equation (11) becomes
$x^4 - (\frac{3}{2}+\lambda)\omega(x)x^3 + (\frac{1}{2}+\frac{3}{2}\lambda)\omega(x)^2x^2 - \frac{1}{2}\lambda\omega(x)^3x = 0$;
For $\alpha_1 = \frac{1}{2}$ and $\alpha_2 = \bar{\lambda}$, (11) becomes
$x^4 - (\frac{3}{2}+\bar{\lambda})\omega(x)x^3 + (\frac{1}{2}+\frac{3}{2}\bar{\lambda})\omega(x)^2x^2 - \frac{1}{2}\bar{\lambda}\omega(x)^3x = 0$.

$3rd$ **Case**: $\alpha_1 = \alpha_2$, $\alpha_1 \neq \frac{1}{2}$ and $\alpha_2 \neq \frac{1}{2}$
According to the theorem 1 of [4] and as A admits nonzero idempotents, the Peirce decomposition of A with respect to an idempotent e is

$A = Ke \oplus A_{\frac{1}{2}} \oplus B$ with $B = N \cap Ker(\ell_e - \alpha_1 I)^2$. If $B = 0$, we have $x = e + x_{\frac{1}{2}}$ and $x^2 = e + x_{\frac{1}{2}}$ so $x^2 = \omega(x)x$ which is an elementary Bernstein algebra and this contradicts the fact that A is a train algebra of rank 4. Otherwise, there are three possibilities. Indeed:

i) $\alpha_1 = \alpha_2 = 0$ implies that the train equation of A is $x^4 - \omega(x)x^3 = 0$;

ii) $\alpha_1 = \alpha_2 = \lambda$ implies that the train equation of A is
$x^4 - (1 + 2\lambda)\omega(x)x^3 + \lambda(\lambda + 2)\omega(x)^2x^2 - \lambda^2\omega(x)^3x = 0$;

iii) $\alpha_1 = \alpha_2 = \bar{\lambda}$ implies that the train equation of A is
$x^4 - (1 + 2\bar{\lambda})\omega(x)x^3 + \bar{\lambda}(\bar{\lambda} + 2)\omega(x)^2x^2 - \bar{\lambda}^2\omega(x)^3x = 0$.

Definition 4. *For any fixed α in K, we consider the map*
$\varphi_\alpha : K[X] \to K[X]$, $P \mapsto (X - \alpha)P$

We easily establish the following lemma.

Lemma 2. *For $\alpha \in \mathbb{C}$, we have $\varphi_\alpha \circ \varphi_{\bar{\alpha}} = \varphi_{\bar{\alpha}} \circ \varphi_\alpha$*

Theorem 7. *Let $A = Ke \oplus A_0 \oplus A_{\frac{1}{2}} \oplus A_\lambda \oplus A_{\bar{\lambda}}$ be an algebra satisfying the identity $2x^2x^4 = \omega(x)^2x^4 + \omega(x)^4x^2$ such that $A_0 = 0$. Let setting $\mu = X^2 - X$. If A is principal train algebra of rank $n \geq 5$, its train equation is of the following form: $\omega(x)^n((\varphi^t_{\bar{\lambda}} \circ \varphi^s_{\lambda} \circ \varphi^r_{1/2})(\mu)(\frac{x}{\omega(x)}) = 0$, $r \geq 0, s \geq 0, t \geq 0$ are integers and $r + t + s = n - 2$.*

Proof. Let $x = e + x_{\frac{1}{2}} + x_\lambda + x_{\bar{\lambda}}$ an element of weight 1 in A. We have $x^2 - x = x^2_{\frac{1}{2}} + (2\lambda - 1)x_\lambda + (2\bar{\lambda} - 1)x_{\bar{\lambda}} + 2x_{\frac{1}{2}}x_\lambda + 2x_{\frac{1}{2}}x_{\bar{\lambda}}$. By setting $x^2 - x = a_{\frac{1}{2}} + a_\lambda + a_{\bar{\lambda}}$ with $a_\alpha \in A_\alpha$, $\alpha \in \{\lambda, \bar{\lambda}\}$, we show using Corollary 1 that there exists an integer $r \geq 0$ such that $\varphi^r_{1/2}(\mu)(x) = a_{r,\lambda} + a_{r,\bar{\lambda}}$, $a_{r,\lambda} \in A_\lambda$, and $a_{r,\bar{\lambda}} \in A_{\bar{\lambda}}$. Similarly, there exists an integer $s \geq 0$ such that $(\varphi_\lambda \circ \varphi^r_{1/2})(\mu)(x) = b_{\bar{\lambda}}$ with $b_{\bar{\lambda}} \in A_{\bar{\lambda}}$. Finally, for some integer $t \geq 0$, we have $(\varphi_{\bar{\lambda}} \circ \varphi_\lambda \circ \varphi^r_{1/2})(\mu)(x) = 0$. The set of element of weight 1 being dense in A according to Zariski topology, for any x in A, we have $\omega(x)^n((\varphi^t_{\bar{\lambda}} \circ \varphi^s_{\lambda} \circ \varphi^r_{1/2})(\mu)(\frac{x}{\omega(x)}) = 0$.

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