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An Iterative Approach to Solve Volterra Nonlinear Integral Equations

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Abstract. In this study, we provide the Aboodh decomposition method, a novel analytical technique. The fundamental definitions and theorems of the suggested approach are provided and analyzed. This new method is a novel mixture of the Aboodh transform and the Adomian decomposition method. The new method is used to solve nonlinear integro-differential equations (IDEs), and the solutions are given as quickly expanding series of terms. We compute the maximum absolute error and provide some figures to compare the resulting approximative solutions with the exact ones in order to demonstrate the method's applicability and efficiency.

2020 Mathematics Subject Classifications: 44A05,45D05,49M27

Key Words and Phrases: Aboodh transform, Decomposition method, Integral equations, Nonlinear integro-differential equations

1. Introduction

Several scientific and engineering problems include integral equations. Volterra or Fredholm integral equations can be used to solve a wide variety of initial and boundary value problems. The potential theory made the greatest contribution to the development of integral equations. The development of integral equations was further facilitated by mathematical physics models of diffraction issues, astrophysics, quantum mechanics scattering, conformal mapping, and water waves [4, 5, 7, 9, 26]. Moreover, the study of nonlinear IDEs has appeared in many fields of science, because of the great number of applications that could describe, such as chemical kinetics, queuing theory, and others [12, 21, 27, 28, 34]. Thus researchers have developed many techniques to handle these problems such as He's homotopy perturbation method [37], variation iteration method [14], least square method [8], decomposition method [13] and others.
Decomposition method is one of the most powerful methods to solve nonlinear differential and integral equations, it presents approximate analytical series solutions of the target problems. Adomian decomposition method was presented by Adomian in [6, 7] to solve integral equations, then it was developed by Wazwaz to solve Volterra IDEs [39].

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Then the method was used by many researchers to solve various kinds of problems [15–19, 22, 31, 32].
Integral transforms have played important roles in solving integral equations, such as Laplace transform [38], ARA transform [36], formable transform [35] and others [10, 20]. One of the most important transforms in literature is Aboodh transform, which was introduced in 2013 [1], and it has great applications in mathematics. Aboodh transform is defined by the following improper integral:

$$A\left[\varphi\left(\tau\right)\right]=\frac{1}{v}\int_{0}^{\infty}e^{-v\tau}\varphi\left(\tau\right)d\tau,\quad v>0.$$

This transform has a great attention from mathematicians, because of its applicability to solve different types of problems, also it could be combined easily with other iteration methods to solve nonlinear problems [2, 3, 11].
The main goal of this article, is to introduce a new combination between the Aboodh transform and the Adomian decomposition method, namely the Aboodh -decomposition method (ADM). The proposed method is utilized to establish analytical series solutions of nonlinear Volterra IDEs, these approximate solutions converge rapidly to the exact ones. for the nonlinear VIE. The novelty of this approach is the powerful combination between the decomposition method and Aboodh transform for the first time. Moreover, the high speed of convergence of the approximate analytical solutions obtained by ADM to the exact solutions, make it an effective method to solve nonlinear IDEs in comparison to other numerical methods.
This research investigates the solution of the nonlinear Volterra IDE of the form

$$\varphi^{(n)}\left(\tau\right)=\psi\left(\tau\right)+\int_{0}^{\tau}k(\tau-u)H(\varphi\left(u\right))du,$$

where the kernel $k(\tau-v)$ and $\psi(\tau)$ are real-valued functions, and $H(\varphi\left(u\right))$ is a nonlinear function of $\varphi\left(v\right)$, such as $\varphi^{3}\left(v\right)$, $\cosh\varphi\left(v\right)$, $\sinh\varphi\left(v\right)$. The main contribution of this work is to present a new analytical approach for solving nonlinear IDEs with simple and easy steps, the method basically depends on applying the Aboodh transform the using the technique of ADM to handle the nonlinear terms. The method is new and simple with less computatios than other methods.
This paper is structured as follows, in Section 2, we introduce the definition of Aboodh transform and some basic properties of it, and we illustrate the main idea of the Adomian decomposition method. In Section 3, the ADM is presented to handle nonlinear Volterra IDEs. To show the applicability of the method, we solve some numerical examples on IDEs. Finally, the conclusion of this article is introduced in Section 5.

2. Basic preliminaries of ADM

2.1. Aboodh integral transform

In this section, we present the basic definitions and properties of Aboodh transform.

Definition 1. *[1] Let $\varphi(\tau)$ be a piecewise continuous function defined on $(0,\infty)$. Then Aboodh transform for $\varphi(\tau)$ is denoted and defined by*

$$A[\varphi(\tau)] = \Phi(v) = \frac{1}{v}\int_0^\infty e^{-v\tau}\varphi(\tau)\,d\tau, \quad \tau > 0.$$

The inverse Aboodh transform for $\Phi(v)$ denoted and defined by

$$A^{-1}[\Phi(v)] = \varphi(\tau) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ve^{v\tau}\Phi(v)\,dv.$$

Theorem 1. *(Existence Condition)[1]*
If$\varphi(\tau)$ is a piecewise continuous function on $[0,\infty)$ and satisfies the condition

$$|\varphi(\tau)| \le Ne^{\alpha\tau},$$

for some $N > 0$.
Then, Aboodh transform $A[\varphi(\tau)]$ exists for $Re(v) > \alpha$.

Proof. Using the definition of Aboodh transform, we obtain

$$|\Phi(v)| = \left|\frac{1}{v}\int_0^\infty e^{-v\tau}\varphi(\tau)\,d\tau\right| \le \frac{1}{v}\int_0^\infty e^{-v\tau}|\varphi(\tau)|\,d\tau \le \frac{1}{v}\int_0^\infty e^{-v\tau}Ne^{\alpha\tau}d\tau$$
$$= \frac{1}{v}N\int_0^\infty e^{-\tau(v-\alpha)}d\tau = \frac{N}{v(v-\alpha)}, \quad Re(v) > \alpha > 0.$$

Hence, Aboodh integral transform exists for $Re(v) > \alpha > 0$.

Now, we mention some properties of Aboodh transform to the basic functions. Suppose that $\Phi_1(v) = A[\varphi_1(\tau)]$ and $\Phi_2(v) = A[\varphi_2(\tau)]$ and $\alpha, \beta \in \mathbb{R}$, then

- $A[\alpha\varphi_1(\tau) + \beta\varphi_2(\tau)] = \alpha\Phi_1(v) + \beta\Phi_2(v)$.
- $A^{-1}[\alpha\Phi_1(v) + \beta\Phi_2(v)] = \alpha\varphi_1(\tau) + \beta\varphi_2(\tau)$.

Now the following table (Table 1) introduces some values of Aboodh transform to some elementary functions, for more details, see [1].

Table 1: Aboodh transform for some functions.

$\varphi(\tau)$	$A[\varphi(\tau)]$
1	$1/v^2$
τ^a	$\frac{\Gamma(a+1)}{v^{a+2}}, a > -1$
$e^{a\tau}$	$\frac{1}{v(v-a)}, s > a$
$\sin(a\tau)$	$\frac{a}{v(v^2+a^2}$
$\cos(a\tau)$	$\frac{1}{v^2+a^2}$
$\sinh(a\tau)$	$\frac{a}{v(v^2-a^2}$
$\cosh(a\tau)$	$\frac{1}{v^2-a^2}$
$\varphi'(\tau)$	$u\Phi(v) - \frac{1}{v}\varphi(0)$
$\varphi^{(n)}(\tau)$	$\Phi(v) - \sum_{j=0}^{n-1} v^{n-j-2}\varphi^{(j)}(0)$
$(\varphi * \psi)(\tau)$	$vA[\varphi(\tau)]\,A[\psi(\tau)]$

2.2. Adomian Decomposition Method

The Adomian decomposition method [7], which has many applications in engineering, physics, and applied mathematics, is a very effective technique for solving many classes of nonlinear partial and ordinary differential equations. The core idea behind the Adomian decomposition technique is to decompose the nonlinear term in the equation into a sum of component. These parts add up to a highly accurate representation of the solution. We explain the steps of the method as:

- Suppose that the solution of the target problem has the following series representation
$$\varphi(\tau) = \sum_{n=0}^{\infty} \varphi_n(\tau) = \varphi_0(\tau) + \varphi_1(\tau) + \dots.$$
- Establish a recursive relation of the nonlinear term of the discussed equation, then substitute the value of the series solution in the equation.
- Simplify the resulting equation and solve it for the series components recursively.

3. Solving nonlinear Volterra IDEs by ADM

In this part of the study, we operate Aboodh transform to the target IDE, then apply the decomposition method, which is the main idea of the ADM. Moreover, we suppose that the given kernel in the equation has a difference form, that could be presented as: $k(x-\tau)$, for examples, $\cos(x-\tau)$, $(x-\tau)^2$, $e^{x-\tau}$. Now, consider the following nonlinear Volterra IDE:

$$\varphi^{(n)}(\tau) = \psi(\tau) + \int_0^{\tau} k(\tau-u) H(\varphi(u)) du, \tag{1}$$

Subject to the initial conditions (ICs)

$$\varphi^{(i)}(0) = \delta_i, \quad i = 0, 1, \dots, n-1. \tag{2}$$

To get the solution of equation (1) by ADM, we operate Aboodh transform to equation (1)

$$A\left[\varphi^{(n)}(\tau)\right] = A[\psi(\tau)] + A\left[\int_0^{\tau} k(\tau-v) H(\varphi(v)) dv\right].$$

The differential property and the convolution property stated in Table 1 of Aboodh transform imply that equation (1) can be simplified to

$$\begin{aligned} v^{n-1} A[\varphi(\tau)] - v^{n-2}\delta_0 - v^{n-3}\delta_1 - \dots - \frac{1}{v}\delta_{n-1} \\ = A[\psi(\tau)] + vA[k(\tau-v)] A[H(\varphi(\tau))]. \end{aligned} \tag{3}$$

Hence, substituting the ICs (2) in (3) and simplifying equation (3), we obtain

$$\begin{aligned} A\left[\varphi\left(\tau\right)\right] &= \frac{1}{v}\delta_0 + \frac{1}{v^2}\delta_1 + \ldots + \frac{1}{v^n}\delta_{n-1} + \frac{1}{v^{n-1}}A\left[\psi\left(\tau\right)\right] \\ &+ \frac{1}{v^{n-2}}A\left[k\left(\tau - v\right)\right]A\left[H\left(\varphi\left(\tau\right)\right)\right]. \end{aligned} \tag{4}$$

Now, utilizing the Adomian decomposition method to treat the nonlinear function $H\left(\varphi\left(\tau\right)\right)$, we have to present $\varphi\left(\tau\right)$ as an infinite series with the components:

$$\varphi\left(\tau\right) = \sum_{i=0}^{\infty}\varphi_i(\tau) = \varphi_0\left(\tau\right) + \varphi_1\left(\tau\right) + \ldots, \tag{5}$$

where the components $\varphi_i(\tau)$, $\tau = 0, 1, \ldots$, are determined by the recurrence relation and the nonlinear term $H\left(\varphi\left(\tau\right)\right)$ can be expressed as

$$H\left(\varphi\left(\tau\right)\right) = \sum_{i=0}^{\infty}A_i(\tau), \tag{6}$$

where $A_i\left(\tau\right)$, $i = 0, 1, 2, \ldots$ are defined as

$$A_i = \frac{1}{i!}\frac{d^i}{d\lambda^i}\left(H\left[\sum_{j=0}^{i}\lambda^j\varphi_j\right]\Bigg|_{\lambda=0}\right), \; i = 0, 1, 2, \cdots. \tag{7}$$

The A_i's are called the Adomian polynomials for the nonlinear function $H(\varphi\left(\tau\right))$, that can be determined by

$$\begin{aligned} A_0 &= H\left(\varphi_0\right), \\ A_1 &= \varphi_1 H'\left(\varphi_0\right), \\ A_2 &= \varphi_2 H'\left(\varphi_0\right) + \frac{1}{2!}\varphi_1^2 H''\left(\varphi_0\right), \\ A_3 &= \varphi_3 H'\left(\varphi_0\right) + \varphi_1\varphi_2 H''\left(\varphi_0\right) + \frac{1}{3!}\varphi_1^3 H'''\left(\varphi_0\right), \\ A_4 &= \varphi_4 H'\left(\varphi_0\right) + \left(\frac{1}{2!}\varphi_2^2 + \varphi_1\varphi_3\right)H''\left(\varphi_0\right) + \frac{1}{2!}\varphi_1^2\varphi_2 H'''\left(\varphi_0\right) + \frac{1}{4!}\varphi_1^4 H^{(4)}\left(\varphi_0\right). \end{aligned} \tag{8}$$

Thus, substituting equations (5) and (6) in equation (4), we get

$$\begin{aligned} A\left[\sum_{i=0}^{\infty}\varphi_i(\tau)\right] &= \frac{1}{v}\delta_0 + \frac{1}{v^2}\delta_1 + \ldots + \frac{1}{v^n}\delta_{n-1}\left(0\right) + \frac{1}{v^{n-1}}A\left[\psi\left(\tau\right)\right] \\ &+ \frac{1}{v^{n-2}}A\left[k\left(\tau - v\right)\right]A\left[\sum_{i=0}^{\infty}A_i(\tau)\right]. \end{aligned} \tag{9}$$

The recursive relation from Adomian decomposition method implies

$$A\left[\varphi_0\left(\tau\right)\right]=\frac{1}{v}\delta_0+\frac{1}{v^2}\delta_1+\ldots+\frac{1}{v^n}\delta_{n-1}\left(0\right)+\frac{1}{v^{n-1}}A\left[\psi\left(\tau\right)\right]. \tag{10}$$

From equation (9), one can get

$$A\left[\varphi_{n+1}\ \left(\tau\right)\right]=\frac{1}{v^{n-2}}A\left[k\left(\tau-v\right)\right]\ \ A\left[A_n(\tau)\right]. \tag{11}$$

Operating the inverse Aboodh transform to the equations in (10) and (11) recursively, one can obtain the values of the components $\varphi_0\left(\tau\right),\varphi_1\left(\tau\right),\cdots$.
The solution of the Volterra IDE (1) is

$$\varphi\left(\tau\right)=\varphi_0\left(\tau\right)+\varphi_1\left(\tau\right)+\ldots.$$

Remark 1. *A necessary condition for equation (11) to be well defined is that*

$$\lim_{v\to\infty}\frac{1}{v^{n-2}}A\left[k\left(\tau\right)\right]=0.$$

The presented method is effective in establishing approximate solutions of nonlinear Volterra IDEs. To test the validity of the method, we discuss some applications and compute the maximum absolute error, defined as

$$AbsErr=\max|\varphi_{exact}-\varphi_{app}|,$$

which is given in some interval.

4. Applications

In this section, we apply ADM to solve some applications of Volterra IDEs, and compute the maximum absolute error to show the efficiency of our results.

Problem 1. *Consider the following nonlinear Volterra integral equation:*

$$\varphi\left(\tau\right)=2\tau-\frac{\tau^4}{12}+\frac{1}{4}\int_0^\tau(\tau-u)\varphi^2\left(u\right)du. \tag{12}$$

Solution The exact solution of equation (12) is $\varphi\left(\tau\right)=2\tau$.
To get the solution by ADM, we apply Aboodh transform to equation (12) to get

$$\Phi\left(v\right)=A\left[2\tau-\frac{\tau^4}{12}\right]+\frac{1}{4}vA\left[\tau\right]A\left[\varphi^2\left(\tau\right)\right]=\frac{2}{v^3}-\frac{5!}{12\ v^6}+\frac{1}{4}v\frac{1}{v^3}A\left[\varphi^2\left(\tau\right)\right]. \tag{13}$$

Substituting the value of the series solution $\Phi\left(v\right)$ and the Adomian components for $\varphi^2\left(u\right)$, we obtain

$$A\left[\varphi_0\left(\tau\right)\right]=\frac{2}{v^3}-\frac{5!}{12v^6},$$

$$A\left[\varphi_{n+1}(\tau)\right]=\frac{1}{4v^2}A\left[A_n(\tau)\right],\ n\geq 0.$$

For the nonlinear term $\varphi^2(v)$, it can be decomposed using the formula in equation (7), one can obtain the following components

$$\begin{aligned}A_0&=\varphi_0^2,\\A_1&=2\varphi_0\varphi_1,\\A_2&=\varphi_1^2+2\varphi_0\varphi_2,\\A_3&=2\varphi_1\varphi_2+2\varphi_0\varphi_3,\\A_4&=\varphi_2^2+2\varphi_1\varphi_3+2\varphi_0\varphi_4.\end{aligned}\tag{14}$$

Making comparisons in the iterative form of equation (7) and applying the inverse Aboodh transform, we obtain

$$\begin{aligned}\varphi_0(\tau)&=2\tau-\frac{\tau^4}{12},\\\varphi_1(\tau)&=\frac{\tau^4}{12}-\frac{\tau^7}{126}+\frac{\tau^{10}}{51840},\\\varphi_2(\tau)&=\frac{\tau^7}{504}-\frac{\tau^{10}}{181440}+\frac{127\,\tau^{13}}{56609280}-\frac{\tau^{16}}{298598400},\\\varphi_3(\tau)&=\frac{\tau^4}{12}-\frac{\tau^7}{504}+\frac{\tau^{10}}{2792}-\frac{19\tau^{13}}{14152320}+\frac{71\tau^{16}}{2264371200}-\frac{7893\,\tau^{19}}{575787643000000}.\end{aligned}$$

Thus, the approximate solution can be expressed as

$$\begin{aligned}\varphi(\tau)&=\varphi_0(\tau)+\varphi_1(\tau)+\varphi_2(\tau)+\varphi_3(\tau)+\dots\\&=2\tau+\frac{\tau^4}{12}-\frac{\tau^7}{126}-\frac{\tau^{10}}{362880}+\frac{51\tau^{13}}{56609280}+\dots.\end{aligned}$$

Table 2, below presents the values of the exact solution and approximate solution of Problem 1, and to test the efficiency we compute the absolute error.

Table 2: The exact and approximate solution of equation (12), and the absolute error.

Nodes	Exact Solution	Approximate Solution	Absolute Error
0.0	0.0	0.0000000000	0.0000000000
0.1	0.2	0.2000083325	0.0000083325
0.2	0.4	0.4001332317	0.0001332317
0.3	0.6	0.6006732643	0.0006732643
0.4	0.8	0.8021203322	0.0021203322
0.5	1.0	1.0051463480	0.0051463480
0.6	1.2	1.2105779450	0.0105779450
0.7	1.4	1.4193552730	0.0193552730
0.8	1.6	1.6328034650	0.0328034650
0.9	1.8	1.8508857190	0.0508857190
1.0	2.0	1.8833526250	0.1166473750

In the following figure below, we sketch the exact and approximate solutions in Figure 1 below. In Figure 2, we sketch the absolute error of the exact and approximate solution of Problem 1.

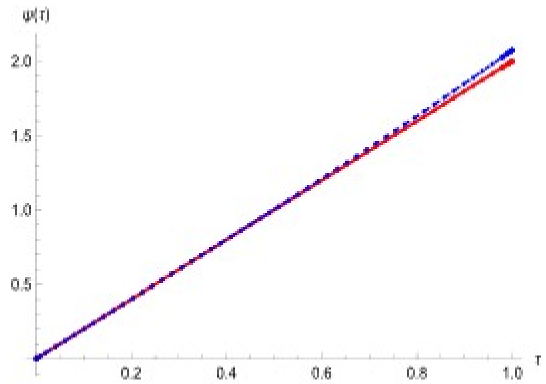

Figure 1: The exact and approximate solutions of the Problem 1.

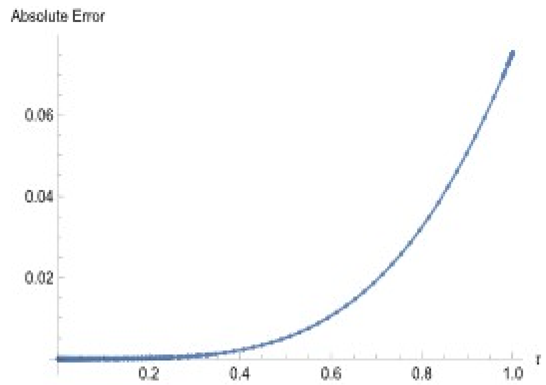

Figure 2: The absolute error of the exact and approximate solutions of equation 12.

Problem 2. *Consider the following nonlinear Volterra integral equation:*

$$\varphi(\tau) = \tau + \int_0^\tau \varphi^2(u)\,du. \tag{15}$$

Solution. The exact solution of equation (15) is $\varphi(\tau) = \tan\tau$.
Applying Aboodh transform to equation (15), we get

$$\Phi(v) = \frac{1}{v^2} + \frac{1}{v} A\left[\varphi^2(\tau)\right]. \tag{16}$$

Thus, by similar arguments to Problem 1, one can obtain

$$\varphi_0(\tau) = \tau,$$
$$\varphi_1(\tau) = \frac{\tau^3}{3},$$

$$\varphi_2(\tau) = \frac{2\tau^5}{15},$$
$$\varphi_3(\tau) = \frac{17\tau^7}{315}.$$

Thus, the approximate solution can be expressed as

$$\varphi(\tau) = \tau + \frac{\tau^3}{3} + \frac{2\tau^5}{15} + \frac{17\tau^7}{315} + \dots.$$

Table 3 below presents the values of the exact solution and approximate solution of Problem 2, and to test the efficiency we compute the absolute error below.

Table 3: The exact and approximate solutions of equation (15) and the absolute error.

Nodes	Exact Solution	Approximate Solution	Absolute Error
0.0	0.00000000000	0.0000000000	0.0000000000
0.1	0.1002940335	0.1003346721	0.0000406386
0.2	0.2026262629	0.2027100241	0.0000837612
0.3	0.3092040035	0.3093358029	0.0001317994
0.4	0.4226035289	0.4227870883	0.0001835594
0.5	0.5460413117	0.5462549603	0.0002136486
0.6	0.6837824776	0.6838787656	0.0000962880
0.7	0.8418070516	0.8411871844	0.0006198672
0.8	1.0289756740	1.0256752970	0.0033003770
0.9	1.2592215210	1.2475448490	0.0116766720
1.0	1.5560303730	1.5206349210	0.03539545198

In the following figure below, we sketch the exact and approximate solutions of Problem 2 in Figure 3 below.

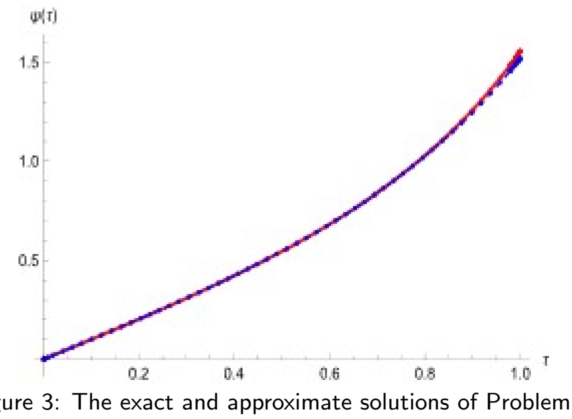

Figure 3: The exact and approximate solutions of Problem 2.

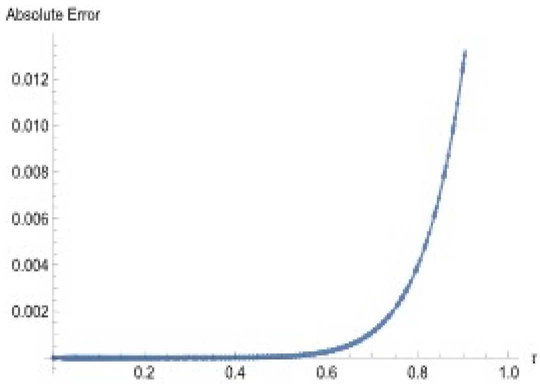

Figure 4: The absolute error of the exact and approximate solutions of equation 15.

Problem 3. *Consider the following nonlinear Volterra IDE of the form*

$$\varphi'(\tau) = \frac{3}{2}e^{\tau} - \frac{1}{2}e^{3\tau} + \int_0^{\tau} e^{u-\tau}\varphi^3(\tau)\,d\tau. \tag{17}$$

$$\varphi(0) = 1. \tag{18}$$

Solution. Applying Aboodh transform to equation (17), we get

$$\Phi(v) = \frac{1}{v} + \frac{3}{2v(v-1)} - \frac{1}{2v(v-3)} + v\frac{1}{v(v-1)}A\left[\varphi^3(\tau)\right].$$

In an equivalent form, we have

$$\Phi(v) = \frac{1}{v} + \frac{3}{2v(v-1)} - \frac{1}{2v(v-3)} + \frac{1}{v-1}A\left[\varphi^3(\tau)\right].$$

Now, we have

$$\begin{aligned} A[\varphi_0(\tau)] &= \frac{1}{v} + \frac{3}{2v(v-1)} - \frac{1}{2v(v-3)}, \\ A[\varphi_{n+1}(\tau)] &= \frac{1}{v-1}A[A_n(\tau)], \quad n \geq 0. \end{aligned} \tag{19}$$

The Adomian polynomials $A_n(\tau)$ of $\varphi^3(\tau)$, can be determined as

$$\begin{aligned} A_0 &= \varphi_0^3, \\ A_1 &= 3\varphi_0^2\varphi_1, \\ A_2 &= 3\varphi_0^2\varphi_2 + 3\varphi_0\varphi_1^3, \\ A_3 &= 3\varphi_0^2\varphi_3 + 6\varphi_0\varphi_1\varphi_2 + \varphi_1^3. \end{aligned}$$

Taking the inverse Aboodh transform to the functions (19) and use the given recursive relation, one can obtain

$$\begin{aligned}\varphi_0(\tau) &= 1+\tau-\frac{1}{2}\tau^3-\frac{\tau^4}{2}-\frac{13}{40}\tau^5+\ldots,\\ \varphi_1(\tau) &= \frac{1}{2}\tau^2+\frac{2}{3}\tau^3+\frac{5}{12}\tau^4+\frac{7}{120}\tau^5+\ldots,\\ \varphi_2(\tau) &= \frac{1}{8}\tau^4+\frac{11}{40}\tau^5+\ldots.\end{aligned}$$

Hence, the approximate series solution of Problem 3 is

$$\varphi(\tau)=1+\tau+\frac{\tau^2}{2!}+\frac{\tau^3}{3!}+\frac{\tau^4}{4!}+\ldots,$$

which converges to the exact solution $\varphi(\tau)=e^{\tau}$. Table 4 below, presents the values of the exact solution and approximate solution of Problem 3, and to test the efficiency we compute the absolute error.

Table 4: The exact and approximate solution of Problem 3, and the absolute error.

Nodes	Exact Solution	Approximate Solution	Absolute Error
0.0	1	1	0
0.1	1.1051709181	1.1051709181	$2.2204460493\times10^{-16}$
0.2	1.2214027582	1.2214027582	0
0.3	1.3498588076	1.3498588076	$2.2204460493\times10^{-16}$
0.4	1.4918246976	1.4918246976	$2.2204460492\times10^{-16}$
0.5	1.6487212707	1.6487212707	$8.8817841970\times10^{-16}$
0.6	1.8221188004	1.8221188004	$9.5479180118\times10^{-15}$
0.7	2.0137527075	2.0137527075	$8.1268325403\times10^{-14}$
0.8	2.2255409285	2.2255409285	$5.3290705182\times10^{-13}$
0.9	2.4596031112	2.4596031112	$2.7911006839\times10^{-12}$
1.0	2.7182818285	2.7182818284	$1.228617207\times10^{-11}$

In the following figure below, we sketch the exact and approximate solutions in Figure 5 below. We also sketch the absolute error of the exact and approximate solutions of Problem 3.

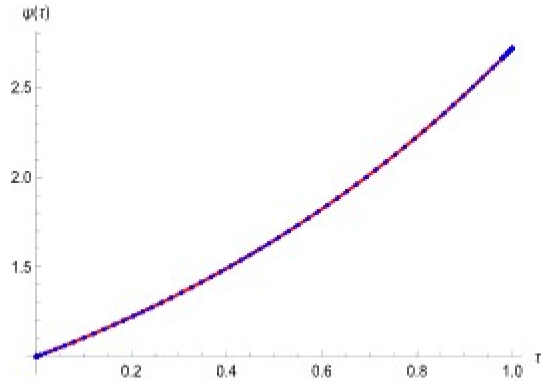

Figure 5: The exact and approximate solutions of Problem 3.

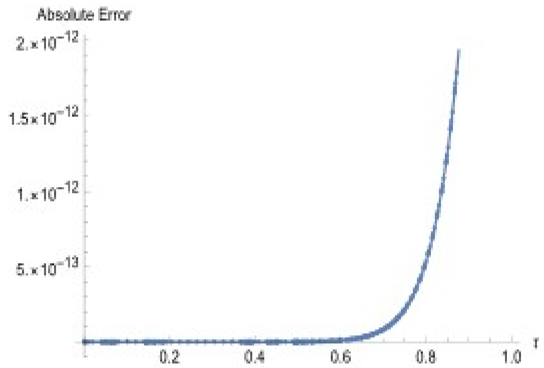

Figure 6: The absolute error of the exact and approximate solutions of Problem 3.

Problem 4. *Consider the following nonlinear Volterra IDE of the form*

$$\varphi'(\tau) = -2\sin\tau - \frac{2\tau}{3}\cos\tau + \int_0^\tau \cos(u-\tau)\varphi^2(\tau)\,d\tau, \tag{20}$$

$$\varphi(0) = 1. \tag{21}$$

Solution. Applying the same procedure from the previous examples, we can obtain

$$\varphi_0(\tau) = 1 - \tau - \tau^2 + \frac{1}{2}\tau^3 + \frac{1}{12}\tau^4 - \frac{11}{120}\tau^5 + \ldots,$$
$$\varphi_1(\tau) = \frac{1}{2}\tau^2 - \frac{1}{3}\tau^3 - \frac{1}{8}\tau^4 + \frac{1}{6}\tau^5 + \ldots,$$
$$\varphi_2(\tau) = \frac{1}{12}\tau^4 - \frac{1}{12}\tau^5 + \ldots.$$

Thus, the approximate solution of (20) and (21) can be expressed as

$$\varphi(\tau) = \left(1 - \frac{\tau^2}{2!} + \frac{\tau^4}{4!} + \ldots\right) - \left(\tau - \frac{\tau^3}{3!} - \frac{\tau^5}{5!} + \ldots\right),$$

which converge to the exact solution

$$\varphi(\tau) = \cos\tau - \sin\tau.$$

Table 5 below presents the values of the exact solution and approximate solution of Problem 4, and to test the efficiency we compute the absolute error.

Table 5: The exact and ARA-DM solutions of Problem 4, and the absolute error.

Nodes	Exact Solution	Approximate Solution	Absolute Error
0.0	1	1	0
0.1	0.8951707486	0.8951709167	0.0000001680
0.2	0.7813972470	0.7814026667	0.0000054196
0.3	0.6598162825	0.65985775	0.0000414675
0.4	0.5316426517	0.5318186667	0.00017601497
0.5	0.3981570233	0.3986979167	0.0005408933
0.6	0.2606931415	0.262048	0.0013548589
0.7	0.12062450	0.1235714167	0.0029469166
0.8	-0.0206493816	-0.0148693333	0.0057800482
0.9	-0.1617169414	-0.151241750	0.0104751914
1.0	-0.3011686789	-0.2833333333	0.0178353456

In the following figure below, we sketch the exact and approximate solutions of Problem 4 in Figure 7 below.

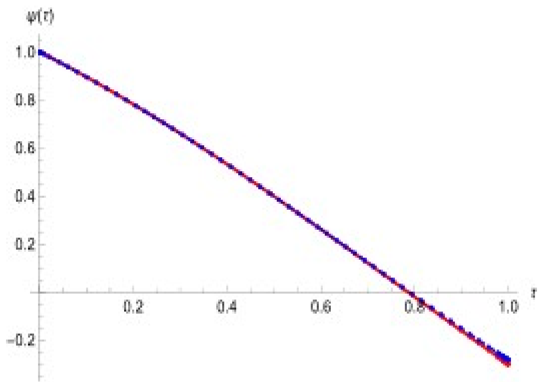

Figure 7: The exact and approximate solutions of the nonlinear Problem 4.

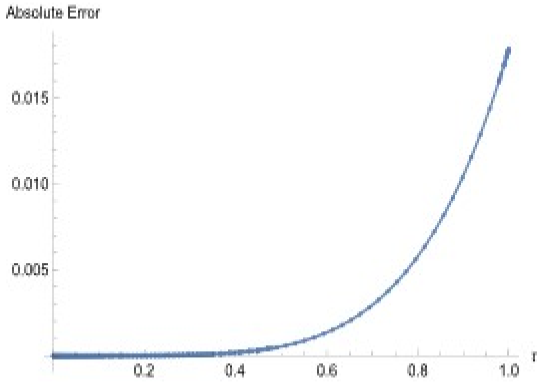

Figure 8: The absolute error of the exact and approximate solutions of Problem 4.

5. Conclusion

The purpose of this study is to provide a new efficient method for solving nonlinear Volterra IDEs. We presented approximate solutions of a family of nonlinear IDE in a form of infinite series solutions using the ADM, that is a combines Aboodh transform with the decomposition technique. Some examples of Volterra IDEs are discussed to verify the validity and applicability of the proposed method. As a result, it turned out that the ADM is an effective and simple method for solving nonlinear IDEs. In the future, we will modify the method [29, 30] and solve nonlinear fractional integral equations of several types [23–25, 33].

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Conflict of interest

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