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Using G_α-transform to study higher-order differential equations with polynomial coefficients

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Abstract. In this study, solutions of higher-order differential equations with polynomial coefficients (HODEPCs) were obtained by applying the G_α-transform. Based on some characterizations, the solutions of HODEPCs were investigated. With the general solution of the HODEPCs, the curves of the general solution can be shown in several examples.

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1. Introduction

Differential equations can be used to express a wide range of physical laws and relationships. Consequently, differential equations play a vital role in a variety of complicated events that occur all over the world. Mathematics can be used to model any physical phenomenon. Modeling is a generic method used in engineering, science, and other professions to convert physical situations or other data into mathematical models. Subsequently, the differential equations in the models must be solved.

In recent years, many authors have studied solutions to differential equations using various methods. In general, it is still very difficult to obtain closed-form solutions for differential equations for most models of real-life problems, but several techniques have been developed to make it easier to find these solutions. Integral transforms have been widely applied to solve several different types of differential equations. There are many publications in the literature on the theory and application of integral transform for solving differential equations, including contributions by Laplace [3, 4, 6, 14, 20, 21, 27], Sumudu [1, 5, 7, 15, 28–30], and Elzaki [8–13, 26].

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Recently, an extended Laplace transform, called the Laplace-typed integral transform, or the G_α-transform, or the generalized Laplace-typed integral transforms, has been introduced in [16] and some of its properties have been investigated. The G_α-transform is defined by the formula

$$G_\alpha\{f(\tau)\} = w^\alpha \int_0^\infty e^{-\tau/w} f(\tau) d\tau,$$

where $\alpha \in \mathbb{Z}$ and w is a complex variable. By selecting the appropriate α, the G_α-transform can be applied immediately to any situations. Table 1 lists some of the transforms along with their definitions, and we use α to convert the G_α-transform as appropriate.

Table 1: Some integral transform definitions

Transform	**Definition**	**G_α-Transform**
Laplace	$\int_0^\infty e^{-\tau/w} f(\tau) d\tau$	$\alpha = 0$ and $1/s = w$
Sumudu	$\frac{1}{w} \int_0^\infty e^{-\tau/w} f(\tau) d\tau$	$\alpha = -1$
Elzaki	$w \int_0^\infty e^{-\tau/w} f(\tau) d\tau$	$\alpha = 1$

Furthermore, the Laplace transform is well-known with a strong application in derivative transforms. To select a transform that provides a simple tool for integral transforms, one has the option to choose $\alpha = -2$ and obtain

$$G_{-2}\{f(\tau)\} = \frac{1}{w^2} \int_0^\infty e^{-\tau/w} f(\tau) d\tau;$$

see [17] for more details. Kim Hj. [18] used the G_{-2}-transform to solve Laguerre's equation.

Recently, Sattaso S. et al. [24] explored the properties of the G_α-transform and offered various examples to demonstrate its usefulness. Some examples can be easily solved with the G_α-transform but not with the Sumudu or Elzaki transforms.

Furthermore, the application of the n-th partial derivatives to the G_α-transform in partial differential equations was presented by Kim Hj. et al. [22]. Kim Hj. [2] investigated the existence and uniqueness of theorems for a variant of a generalized Laplace transform represented by a logarithmic function. In addition, Kim Hj. [19] considered an argument based on the rigor of mathematical induction for the Laplace transform of the n-th derivative of any order. The results can be extended to the generalized Laplace transform.

Geum Y. H. et al. [25] showed the matrix representation of convolution and related the mathematical notion of convolution to the concept of convolution in a convolutional neural network.

Most recently, the range of G_α-transforms that can be used to solve second-order and third-order ordinary differential equations with variable coefficients was addressed by Prasertsang P. et al. [23]. Motivated by this discussion, the current paper will extend the variable coefficients in general form and identify some characterizations among them.

The remaining sections of the paper are organized as follows. Section 2 introduces a definition and lemmas to prove the theorem. Section 3 derives the solutions of HODEPCs via the G_α-transform and obtains some theorem and corollary. Some applications and conclusions are given in sections 4 and 5, respectively.

2. Preliminaries

To analyze the study for HODEPCs via the G_α-transform, a definition and lemmas are given, as follows

Definition 1. [24] Let $f(\tau)$ be a piecewise continuous function on $\tau \geq 0$ and has an exponential order k. The G_α-transform of $f(\tau)$, briefly $G_\alpha\{f(\tau)\}$, is characterized with the formula

$$G_\alpha\{f(\tau)\} = w^\alpha \int_0^\infty e^{-\tau/w} f(\tau) d\tau,$$

where $\alpha \in \mathbb{Z}$ and $w > 0$ is a complex variable and $G_\alpha\{f(\tau)\}$ exists for $w < 1/k$.

Lemma 1. [24] If $y^{(m)}(\tau)$ is a piecewise continuous function on $[0, \infty)$ for $m \in \mathbb{N} \cup \{0\}$ and has an exponential order k for $w < \frac{1}{k}$, then

$$\begin{aligned} G_\alpha\{\tau^n y^{(m)}(\tau)\} &= w^{2n} \frac{d^n G_\alpha\{y^{(m)}(\tau)\}}{dw^n} - \binom{n}{1} [\alpha - (n-1)]\, w^{2n-1} \frac{d^{n-1} G_\alpha\{y^{(m)}(\tau)\}}{dw^{n-1}} \\ &+ \cdots - \binom{n}{n-1} [\alpha - (n-1)] [\alpha - (n-2)] \cdots (\alpha - 1) w^{n+1} \frac{dG_\alpha\{y^{(m)}(\tau)\}}{dw} \\ &+ [\alpha - (n-1)] [\alpha - (n-2)] \cdots \alpha w^n G_\alpha\{y^{(m)}(\tau)\}. \end{aligned} \tag{1}$$

From Lemma 1, Eq. (1) can be rewritten as the following,

$$\begin{aligned} G_\alpha\{\tau^n y^{(m)}(\tau)\} &= \sum_{l=0}^{n} (-1)^{n-l} \binom{n}{l} \prod_{s=l}^{n-1} (m + \alpha - s) \frac{Y^{(l)}(w)}{w^{m-n-l}} \\ &- \sum_{k=0}^{m-1} \prod_{L=1}^{n} (-m + k + L) w^{\alpha - m + k + L + 1} y^{(k)}(0). \end{aligned} \tag{2}$$

If $m = 0$ in Eq. (2),

$$G_\alpha\{\tau^n y(\tau)\} = \sum_{l=0}^{n} (-1)^{n-l} \binom{n}{l} \prod_{s=l}^{n-1} (\alpha - s) \frac{Y^{(l)}(w)}{w^{-n-l}}.$$

If $n = 0$ in Eq. (2),

$$G_\alpha\{y^{(m)}(\tau)\} = \frac{Y(w)}{w^m} - \sum_{k=0}^{m-1} w^{\alpha - m + k + 1} y^{(k)}(0),$$

where $Y(w) = G_\alpha\{y(\tau)\}$.

Lemma 2. [24] Assume that $y(\tau) = \sum_{n=0}^{\infty} a_n \tau^n$ is a piecewise continuous function on $[0, \infty)$ and has an exponential order at infinity with the function on $|f(\tau)| \leq Me^{k\tau}$ for $\tau \geq \bar{C}$ where $\bar{C}$ is a constant, then

$$G_\alpha\{f(\tau)\} = \sum_{n=0}^{\infty} n! a_n w^{\alpha+n+1}.$$

3. Analytical study for HODEPCs via G_α-transform

Denote $n, m \in \mathbb{N} \cup \{0\}$ and $m \geq n$,

$$\begin{aligned}
\varrho &= 2, 3, 4, \dots, n, \quad \varrho_1 = 0, 1, 2, \dots, n-1, \quad \varrho_2 = 0, 1, 2, \dots, n, \quad \rho = 0, 1, 2, \dots, m, \\
\rho_1 &= m - (n - \varrho_1 - 1), m - (n - \varrho_1 - 2), m - (n - \varrho_1 - 3), \dots, m-2, m-1, m, \\
\rho_2 &= \varrho_1 + 1, \varrho_1 + 2, \varrho_1 + 3, \dots, n-2, n-1, n, \\
\rho_3 &= 0, 1, 2, \dots, m - (n - \varrho_1 + 2), m - (n - \varrho_1 + 1), m - (n - \varrho_1), \\
\upsilon(u) &= a_{\rho,0} \sum_{k=0}^{\rho-1} u^{\alpha-\rho+k+1} y^{(k)}(0) + a_{\rho,1} \sum_{k=0}^{\rho-1} (-\rho + k + 1) u^{\alpha-\rho+k+2} y^{(k)}(0) \\
&\quad + a_{\rho,\varrho} \sum_{k=0}^{\rho-1} \prod_{L=1}^{\varrho} (-\rho + k + L) u^{\alpha-\rho+k+L+1} y^{(k)}(0), \\
\Theta_{\varrho_2}(u) &= \sum_{\rho=0}^{m} \frac{a_{\rho,\varrho_2}}{u^{\rho-2\varrho_2}} - \binom{\varrho_2+1}{\varrho_2} \sum_{\rho=0}^{m} (\rho + \alpha - \varrho_2) \frac{a_{\rho,\varrho_2+1}}{u^{\rho-\varrho_2-(\varrho_2+1)}} \\
&\quad + \binom{\varrho_2+2}{\varrho_2} \sum_{\rho=0}^{m} \prod_{s=\varrho_2}^{\varrho_2+1} (\rho + \alpha - s) \frac{a_{\rho,\varrho_2+2}}{u^{\rho-\varrho_2-(\varrho_1+2)}} \\
&\quad - \binom{\varrho_2+3}{\varrho_2} \sum_{\rho=0}^{m} \prod_{s=\varrho_2}^{\varrho_2+2} (\rho + \alpha - s) \frac{a_{\rho,\varrho_2+3}}{u^{\rho-\varrho_2-(\varrho_2+3)}} + \cdots \\
&\quad + (-1)^{n-2-\varrho_2} \binom{n-2}{\varrho_2} \sum_{\rho=\varrho_2}^{m} \prod_{s=\varrho_2}^{n-3} (\rho + \alpha - s) \frac{a_{\rho,n-2}}{u^{\rho-\varrho_2-n+2}} \\
&\quad + (-1)^{n-1-\varrho_2} \binom{n-1}{\varrho_2} \sum_{\rho=\varrho_2}^{m} \prod_{s=\varrho_2}^{n-2} (\rho + \alpha - s) \frac{a_{\rho,\varrho_2+n-1}}{u^{\rho-\varrho_2-n+1}} \\
&\quad + (-1)^{n-\varrho_2} \binom{n}{\varrho_2} \sum_{\rho=\varrho_1}^{m} \prod_{s=\varrho_2}^{n-1} (\rho + \alpha - s) \frac{a_{\rho,n}}{u^{\rho-\varrho_2-n}}.
\end{aligned}$$

Theorem 1. Consider the higher-order differential equation in the form

$$\sum_{i=0}^{m} (\sum_{j=0}^{n} a_{i,j} \tau^j) y^{(i)}(\tau) = \Phi(\tau), \tag{3}$$

where $\sum_{j=0}^{n} a_{i,j}\tau^j$ are polynomial functions with degree n in terms of τ where $i = 0, 1, 2, \ldots, m$, $j = 0, 1, 2, \ldots, n$, where $m \geq n$ and $a_{i,j}$ are polynomial coefficients with $a_{m,n} \neq 0$ and $\Phi(\tau)$ is an unknown function. If Eq. (3) satisfies the following conditions

$$a_{\rho_1,\varrho_1} = 0, \tag{4}$$

$$\sum_{j=\rho_2}^{n} (-1)^{j-\varrho_1} \binom{j}{\varrho_1} \prod_{s=\varrho_1}^{j-1} (-\rho_2 + j + \alpha - s) a_{-\rho_2+j,j} = 0, \tag{5}$$

$$a_{\rho_3,\varrho_1} + \sum_{j=\varrho_1+1}^{n} (-1)^{j-\varrho_1} \binom{j}{\varrho_1} \prod_{s=\varrho_1}^{j-1} (\rho_3 - \varrho_1 + j + \alpha - s) a_{\rho_3-\varrho_1+j,j} = 0, \tag{6}$$

then, it is appropriately solved using the G_α-transform.

Proof. Taking the G_α-transform along both sides of (3) and applying Eq. (2), it follows that,

$$\begin{aligned} &\sum_{\rho=0}^{m} a_{\rho,\varrho_2} \sum_{l=0}^{\varrho_2} (-1)^{n-l} \binom{\varrho_2}{l} \prod_{s=l}^{\varrho_2-1} (\rho + \alpha - s) \frac{Y^{(l)}(u)}{u^{\rho-\varrho_2-l}} \\ &- \sum_{\rho=0}^{m} a_{\rho,\varrho_2} \sum_{k=0}^{\rho-1} \prod_{L=1}^{\varrho_2} (-\rho + k + L) u^{\alpha-\rho+k+L+1} y^{(k)}(0) = G_\alpha \left[\Phi(\tau)\right], \end{aligned} \tag{7}$$

for $\varrho_2 = 0, 1, 2, \ldots, n$. Therefore, Eq. (7) can be represented in terms of all derivatives of $Y(u)$ as follows

$$\sum_{\varrho_2=0}^{n} Y^{(\varrho_2)}(u)\Theta_{\varrho_2}(u) = G_\alpha \left[\Phi(\tau)\right] + \upsilon(u). \tag{8}$$

Solving the equation (8) using the G_α-transform method, means that the coefficients of $Y(u)$, $Y'(u)$, $Y''(u), \cdots, Y^{(n-3)}(u)$, $Y^{(n-2)}(u)$, $Y^{(n-1)}(u)$ are equal to zero. That is, $\Theta_{\varrho_2}(u) = 0$ for all ϱ_2 except $\varrho_2 = n$.

Let us consider the coefficient of $Y(u)$ is zero, or $\Theta_0(u) = 0$, by setting $\gamma_{1,0} = 1, 2, 3, \ldots, n-1$, $\gamma_{2,0} = n, n+1, n+2, \ldots, m$, and $\gamma_{3,0} = m+1, m+2, m+3, \ldots, m+n$, yields

$$u^m \rightarrow a_{m,0} = 0, \tag{9}$$

$$u^{m-\gamma_{1,0}} \rightarrow a_{m-\gamma_{1,0},0} + \sum_{j=1}^{\gamma_{1,0}} (-1)^{j-0} \binom{j}{0} \prod_{s=0}^{j-1} (m - \gamma_{1,0} + j + \alpha - s) a_{m-\gamma_{1,0}+j,j} = 0, \tag{10}$$

$$u^{m-\gamma_{2,0}} \rightarrow a_{m-\gamma_{2,0},0} + \sum_{j=1}^{n} (-1)^{j-0} \binom{j}{0} \prod_{s=0}^{j-1} (m - \gamma_{2,0} + j + \alpha - s) a_{m-\gamma_{2,0}+j,j} = 0, \tag{11}$$

$$u^{m-\gamma_{3,0}} \quad \rightarrow \quad \sum_{j=\gamma_{3,0}-m}^{n} (-1)^{j-0}\binom{j}{0}\prod_{s=0}^{j-1}(m-\gamma_{3,0}+j+\alpha-s)a_{m-\gamma_{3,0}+j,j}=0. \tag{12}$$

Let us consider the coefficient of $Y'(u)$ is zero, or $\Theta_1(u) = 0$, by setting $\gamma_{1,1} = 1,2,3,\ldots,\ n-2, \gamma_{2,1} = n-1, n, n+1, \ldots, m$, and $\gamma_{3,1} = m+1, m+2, m+3, \ldots, m+n-1$, we obtain

$$u^{m-2} \quad \rightarrow \quad a_{m,1} = 0, \tag{13}$$

$$u^{m-\gamma_{1,1}-2} \quad \rightarrow \quad a_{m-\gamma_{1,1},1} + \sum_{j=2}^{\gamma_{1,1}+1}(-1)^{j-1}\binom{j}{1}\prod_{s=1}^{j-1}(m-1-\gamma_{1,1}+j+\alpha-s)\times \tag{14}$$

$$a_{m-1-\gamma_{1,1}+j,j}=0, \tag{15}$$

$$u^{m-\gamma_{2,1}-2} \quad \rightarrow \quad a_{m-\gamma_{2,1},1} + \sum_{j=2}^{n}(-1)^{j-1}\binom{j}{1}\prod_{s=1}^{j-1}(m-1-\gamma_{2,1}+j+\alpha-s)\times \tag{16}$$

$$a_{m-1-\gamma_{2,1}+j,j}=0, \tag{17}$$

$$u^{m-\gamma_{3,1}-2} \quad \rightarrow \quad \sum_{j=\gamma_{3,1}-m+1}^{n}(-1)^{j-1}\binom{j}{1}\prod_{s=1}^{j-1}(m-1-\gamma_{3,1}+j+\alpha-s)\times \tag{18}$$

$$a_{m-1-\gamma_{3,1}+j,j}=0. \tag{19}$$

Let us consider the coefficient of $Y''(u)$ is zero, or $\Theta_2(u) = 0$, by setting $\gamma_{1,2} = 1,2,3,\ldots,n-3, \gamma_{2,2} = n-2, n-1, n, \ldots, m$, and $\gamma_{3,2} = m+1, m+2, m+3, \ldots, m+n-2$, we have

$$u^{m-4} \quad \rightarrow \quad a_{m,2} = 0, \tag{20}$$

$$u^{m-\gamma_{1,2}-4} \quad \rightarrow \quad a_{m-\gamma_{1,2},2} + \sum_{j=3}^{\gamma_{1,2}+2}(-1)^{j-2}\binom{j}{2}\prod_{s=2}^{j-1}(m-2-\gamma_{1,2}+j+\alpha-s)a_{m-2-\gamma_{1,2}+j,j}=0, \tag{21}$$

$$u^{m-\gamma_{2,2}-4} \quad \rightarrow \quad a_{m-\gamma_{2,2},2} + \sum_{j=3}^{n}(-1)^{j-2}\binom{j}{2}\prod_{s=2}^{j-1}(m-2-\gamma_{2,2}+j+\alpha-s)a_{m-2-\gamma_{2,2}+j,j}=0, \tag{22}$$

$$u^{m-\gamma_{3,2}-4} \quad \rightarrow \quad \sum_{j=\gamma_{3,2}-m+2}^{n}(-1)^{j-2}\binom{j}{2}\prod_{s=2}^{j-1}(m-2-\gamma_{3,2}+j+\alpha-s)a_{m-2-\gamma_{3,2}+j,j}=0. \tag{23}$$

Similarly, the coefficients of $Y'''(u)$, $Y^{(4)}(u)$, $Y^{(5)}(u), \ldots,$ $Y^{(n-3)}(u)$, $Y^{(n-2)}(u)$, $Y^{(n-1)}(u)$ are equal to zero. Next, we will state the following equations from the coefficient of $Y^{(n-3)}(u) = Y^{(n-2)}(u) = Y^{(n-1)}(u) = 0$, by letting $\gamma_{1,n-3} = 1,2,$ $\gamma_{2,n-3} =$

$3,4,5,\ldots,m,\ \ \gamma_{3,n-3}=m+1,m+2,m+3,\ \ \gamma_{2,n-2}=2,3,4,\ldots,m,\ \ \gamma_{3,n-2}=m+1,m+2,$ and $\gamma_{2,n-1}=1,2,3,\ldots,m$, it follows that

$$u^{m-2(n-3)} \rightarrow a_{m,n-3}=0, \tag{24}$$

$$u^{m-\gamma_{1,n-3}-2(n-3)} \rightarrow a_{m-\gamma_{1,n-3},n-3}+\sum_{j=n-2}^{\gamma_{1,n-3}+n-3}(-1)^{j-(n-3)}\binom{j}{n-3}\times \prod_{s=n-3}^{j-1}(m-(n-3)-\gamma_{1,n-3}+j+\alpha-s)\times a_{m-(n-3)-\gamma_{1,n-3}+j,j}=0, \tag{25}$$

$$u^{m-\gamma_{2,n-3}-2(n-3)} \rightarrow a_{m-\gamma_{2,n-3},n-3}+\sum_{j=n-2}^{n}(-1)^{j-(n-3)}\binom{j}{n-3}\times \prod_{s=n-3}^{j-1}(m-(n-3)-\gamma_{2,n-3}+j+\alpha-s)\times a_{m-(n-3)-\gamma_{2}+j,j}=0, \tag{26}$$

$$u^{m-\gamma_{3,n-3}-2(n-3)} \rightarrow \sum_{j=\gamma_{3,n-3}-m+(n-3)}^{n}(-1)^{j-(n-3)}\binom{j}{n-3}\times \prod_{s=n-3}^{j-1}(m-(n-3)-\gamma_{3,n-3}+j+\alpha-s)\times a_{m-(n-3)-\gamma_{3,n-3}+j,j}=0, \tag{27}$$

$$u^{m-2(n-2)} \rightarrow a_{m,n-2}=0, \tag{28}$$

$$u^{m-2(n-2)-1} \rightarrow a_{m-1,n-2}+\sum_{j=n-1}^{n-1}(-1)^{j-(n-2)}\binom{j}{n-2}\times \prod_{s=n-2}^{j-1}(m-(n-1)+j+\alpha-s)a_{m-(n-1)+j,j}=0, \tag{29}$$

$$u^{m-\gamma_{2,n-2}-2(n-2)} \rightarrow a_{m-\gamma_{2,n-2},n-2}+\sum_{j=n-1}^{n}(-1)^{j-(n-2)}\binom{j}{n-2}\times \prod_{s=n-2}^{j-1}(m-(n-2)-\gamma_{2,n-2}+j+\alpha-s)\times a_{m-(n-2)-\gamma_{2,n-2}+j,j}=0, \tag{30}$$

$$u^{m-\gamma_{3,n-2}-2(n-2)} \rightarrow \sum_{j=\gamma_{3,n-2}-m+(n-2)}^{n}(-1)^{j-(n-2)}\binom{j}{n-2}\times$$

$$\prod_{s=n-2}^{j-1}(m-(n-2)-\gamma_{3,n-2}+j+\alpha-s)\times$$
$$a_{m-(n-2)-\gamma_{3,n-2}+j,j}=0, \quad (31)$$
$$u^{m-2(n-1)} \rightarrow a_{m,n-1}=0, \quad (32)$$
$$u^{m-\gamma_{2,n-1}-2(n-1)} \rightarrow a_{m-\gamma_{2,n-1},n-1}-n\prod_{s=n-1}^{j-1}(m-(n-1)-\gamma_{2,n-1}+j+\alpha-s)\times$$
$$a_{m-(n-1)-\gamma_{2,n-1}+j,j}=0, \quad (33)$$
$$u^{m-(m+1)-2(n-1)} \rightarrow -n\prod_{s=n-1}^{j-1}(-n+j+\alpha-s)a_{-n+j,j}=0. \quad (34)$$

Hence, according to Eqs. (9), (10), (13), (14), (17), (18), (21), (22), (25), (26), and (29), it can be reduced to the following forms,

$$\begin{aligned}
a_{m,\varrho} &= 0; \quad \varrho=0,1,2,\ldots,n-1,\\
a_{m-1,\varrho} &= 0; \quad \varrho=0,1,2,\ldots,n-2,\\
a_{m-2,\varrho} &= 0; \quad \varrho=0,1,2,\ldots,n-3,\\
&\vdots\\
a_{m-(n-3),\varrho} &= 0; \quad \varrho=0,1,2,\\
a_{m-(n-2),\varrho} &= 0; \quad \varrho=0,1\\
a_{m-(n-1),0} &= 0,
\end{aligned}$$

then, condition (4) becomes true. From Eqs. (12), (16), (20), (24), (28) and (31) can be rewritten as

$$\sum_{j=\gamma_3-m+\varrho_1}^{n}(-1)^{j-\varrho_1}\binom{j}{\varrho_1}\prod_{s=\varrho_1}^{j-1}(m-\varrho_1-\gamma_3+j+\alpha-s)a_{m-\varrho_1-\gamma_3+j,j}=0,$$

for $\varrho_1=0,1,2,\ldots,n-1$ and $\gamma_3=0,1,2,\ldots,m-(n-\varrho_1+2),m-(n-\varrho_1+1),m-(n-\varrho_1)$. Thus, condition (5) holds by replacing $\gamma_3-m+\varrho_1=\rho_2$. Finally, the conditions as stated in Eqs. (11), (15), (19), (23), (27), and (30) are properly equated to condition (6). The proof is completed.

Note that (i) no. COEs is the number of polynomial coefficients in Eq. (3) and (ii) no. CONs is the number of conditions according to conditions (4)-(6) for solving Eq. (3) using the G_α-transform.

Corollary 1. *Given $i=0,1,2,\ldots,m$, $j=0,1,2,\ldots,n$, where $m\geq n$, $a_{i,j}$ are the polynomial coefficients of $\sum_{j=0}^{n}a_{i,j}\tau^j$ with $a_{m,n}\neq 0$ in Eq. (3), the following statements hold:*

(I) no. COEs is $(m+1)(n+1)$,

(II) no. CONs is $mn + \frac{n(n+3)}{2}$,
(III) no. COEs = no. CONs iff $m = \frac{(n-1)(n+2)}{2}$.

Proof. Assume that $i = 0, 1, 2, \ldots, m, \quad j = 0, 1, 2, \ldots, n$, where $m \geq n$, $a_{i,j}$ are the polynomial coefficients of $\sum_{j=0}^{n} a_{i,j}\tau^j$ with $a_{m,n} \neq 0$ in Eq. (3) and by letting $\varrho_1 = 0, 1, 2, \ldots, n-1$, and $\rho_1 = m-(n-\varrho_1-1), m-(n-\varrho_1-2), m-(n-\varrho_1-3), \ldots, m-2, m-1, m, \rho_2 = \varrho_1+1, \varrho_1+2, \varrho_1+3, \ldots, n-2, n-1, n, \rho_3 = 0, 1, 2, \ldots, m-(n-\varrho_1+2), m-(n-\varrho_1+1), m-(n-\varrho_1)$.
(I) For each $i = 0, 1, 2, \ldots, m$, the numbers of polynomial coefficients of all orders of differential equations are equal to $n+1$, then no. COEs is $(m+1)(n+1)$.
(II) From condition (4), we obtain
if $\varrho_1 = 0$, then $\rho_1 = m-(n-1), m-(n-2), m-(n-3), \ldots, m-2, m-1, m$, no. CONs is n,
if $\varrho_1 = 1$, then $\rho_1 = m-(n-2), m-(n-3), m-(n-4), \ldots, m-2, m-1, m$, no. CONs is $n-1$,
if $\varrho_1 = 2$, then $\rho_1 = m-(n-3), m-(n-4), m-(n-5), \ldots, m-2, m-1, m$, no. CONs is $n-2$,

$$\vdots$$

if $\varrho_1 = n-3$, then $\rho_1 = m-2, m-1, m$, no. CONs is 3,
if $\varrho_1 = n-2$, then $\rho_1 = m-1, m$, no. CONs is 2,
if $\varrho_1 = n-1$, then $\rho_1 = m$, no. CONs is 1.
Consequently, the number of conditions in conditon (4) is $\frac{n(n+1)}{2}$.
From condition (5), we have
if $\varrho_1 = 0$, then $\rho_2 = 1, 2, 3, \ldots, n-2, n-1, n$, no. CONs is n,
if $\varrho_1 = 1$, then $\rho_2 = 2, 3, \ldots, n-2, n-1, n$, no. CONs is $n-1$,
if $\varrho_1 = 2$, then $\rho_2 = 3, \ldots, n-2, n-1, n$, no. CONs is $n-2$,

$$\vdots$$

if $\varrho_1 = n-3$, then $\rho_2 = n-2, n-1, n$, no. CONs is 3,
if $\varrho_1 = n-2$, then $\rho_2 = n-1, n$, no. CONs is 2,
if $\varrho_1 = n-1$, then $\rho_2 = n$, no. CONs is 1,
Consequently, no. CONs in condition (5) is $\frac{n(n+1)}{2}$,
From Eq. (6), we get
if $\varrho_1 = 0$, then $\rho_3 = 0, 1, 2, \ldots, m-n-2, m-n-1, m-n$, no. CONs is $m-n+1$,
if $\varrho_1 = 1$, then $\rho_3 = 0, 1, 2, \ldots, m-n-3, m-n-2, m-n-1$, no. CONs is $m-n+2$,
if $\varrho_1 = 2$, then $\rho_3 = 0, 1, 2, \ldots, m-n-4, m-n-3, m-n-2$, no. CONs is $m-n+3$,

$$\vdots$$

if $\varrho_1 = n-3$, then $\rho_3 = 0, 1, 2, \ldots, m-5, m-4, m-3$, no. CONs is $m-2$,
if $\varrho_1 = n-2$, then $\rho_3 = 0, 1, 2, \ldots, m-4, m-3, m-2$, no. CONs is $m-1$,
if $\varrho_1 = n-1$, then $\rho_3 = 0, 1, 2, \ldots, m-3, m-2, m-1$, no. CONs is m.
Consequently, the number of conditions in Eq. (6) is $mn - \frac{(n-1)n}{2}$, it follows that no. CONs is $mn + \frac{n(n+3)}{2}$.
(III)($\Rightarrow$) If no. COEs = no. CONs, then form (I) and (II), we have $(m+1)(n+1) =$

$mn + \frac{n(n+3)}{2}$, it can be rewritten as $m = \frac{(n-1)(n+2)}{2}$.
($\Leftarrow$) Let $m = \frac{(n-1)(n+2)}{2}$, suppose that no. COEs is not equal to no. CONs, that is $(m+1)(n+1) \neq mn + \frac{n(n+3)}{2}$, implying that $m \neq \frac{(n-1)(n+2)}{2}$, which is a contradiction. Therefore, no. COEs = no. CONs.

4. Applications

According to Theorem 1, the solutions of the higher-order differential equations with polynomial coefficients through the G_α-transform can be solved, as follows:

4.1. The application of fifth-order differential equation with polynomial coefficients where $\alpha = 3$

4.1.1. Process of general solution

Example 1. *Let us consider the fifth-order differential equation with polynomial coefficients in the form of*

$$t^5y^{(5)}(t) + 20t^4y^{(4)}(t) + 120t^3y'''(t) + 240t^2y''(t) + 120ty'(t) = t, \quad t \geq 0. \tag{35}$$

From Eqs. (3) and Eq. (35), we have

$$a_{5,5} = 1, a_{4,4} = 20, a_{3,3} = 120, a_{2,2} = 240, a_{1,1} = 120,$$

and we determine $\alpha = 3$ according to the conditions of Theorem 1, then applying the G_3-transform leads to finding the solution of (35)*. By using the G_3-transform to* (35)*, we have*

$$G_3\{t^5y^{(5)}(t)\} + G_3\{20t^4y^{(4)}(t)\} + G_3\{120t^3y'''(t)\} + G_3\{240t^2y''(t)\} + G_3\{120ty'(t)\}$$
$$= G_3\{t\}.$$

Using Lemma 1 and a little rewriting yields

$$u^5F^{(5)}(u) = u^5,$$
$$F^{(5)}(u) = 1.$$

It follows that

$$F(u) = \frac{u^5}{120} + c_1\frac{u^4}{24} + c_2\frac{u^3}{6} + c_3\frac{u^2}{2} + c_4u + c_5, \tag{36}$$

where $c_1, c_2, c_3, c_4,$ and c_5 are constants.

4.1.2. Graphical analysis

From Matlab, Figure 1 shows the curves of general solutions $y(t)$ *by applying Lemma 2 and the inverse* G_3*-transform. For the classical solutions, we have to set* $c_2 = c_3 = c_4 = c_5 = 0$ *and various cases of* c_1 *in Eq.* (36)*,*
(i) when $c_1 = 1$*, we get* $y(t) = \frac{t}{120} + \frac{1}{24}$ *as a solution to Eq.* (35)*,*
(ii) when $c_1 = 3$*, we get* $y(t) = \frac{t}{120} + \frac{1}{8}$ *as a solution to Eq.* (35)*,*
(iii) when $c_1 = 24$*, we get* $y(t) = \frac{t}{120} + 1$ *as a solution to Eq.* (35)*.*
We can summarize that the general solutions are line graphs with intercept y*-axis at several points depending on* c_1*.*

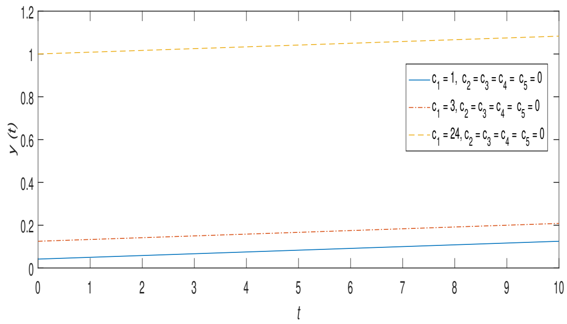

Figure 1: General solutions of Example 1.

4.2. The application of fifth-order differential equation with polynomial coefficients where $\alpha = -1$

4.2.1. Process of general solution

Example 2. *Let us consider the fifth-order differential equation with polynomial coefficients in the form of*

$$t^3 y^{(5)}(t) + (t^3 + 6t^2) y^{(4)}(t) + (3t^2 + 6t) y'''(t) = \frac{t^2}{2} + t, \quad t \geq 0. \tag{37}$$

From Eqs.(3) and (37), we have

$$a_{5,3} = 1, a_{4,3} = 1, a_{4,2} = 6, a_{3,2} = 3, a_{3,1} = 6,$$

and we determine $\alpha = -1$ *according to the conditions of Theorem 1, so applying the* G_{-1}*-transform leads to finding the solution of Eq.* (37)*. By using the* G_{-1}*-transform to* (37)*, we have*

$$G_{-1}\{t^3 y^{(5)}(t)\} + G_{-1}\{t^3 y^{(4)}(t)\} + G_{-1}\{6t^2 y^{(4)}(t)\} + G_{-1}\{3t^2 y'''(t)\} + G_{-1}\{6ty'''(t)\}$$
$$= G_{-1}\{\frac{t^2}{2}\} + G_{-1}\{t\}.$$

Using Lemma 1 and simplifying the above equation, we have the following

$$(u^2+u)F'''(u) = u^2+u,$$
$$F'''(u) = 1.$$

Then, we have

$$F(u) = \frac{u^3}{6} + c_1\frac{u^2}{2} + c_2 u + c_3, \tag{38}$$

where $c_1, c_2,$ and c_3 are constants.

4.2.2. Graphical analysis

From Matlab, Figure 2 draws the curve of general solution $y(t)$ by applying Lemma 2 and the inverse G_{-1}-transform. For the classical solutions, we have to set $c_1 = c_2 = c_3 = 0$ in Eq. (38). It is straightforward to illustrate that $y(t) = \frac{t^3}{36}$ satisfies Eq. (37).

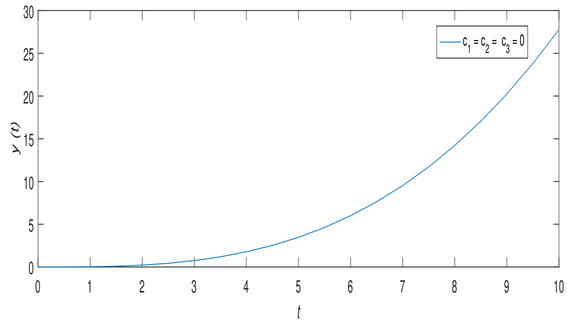

Figure 2: General solutions of Example 2.

4.3. The application of seventh-order differential equation with polynomial coefficients where $\alpha = -5$

4.3.1. Process of general solution

Example 3. *Let us consider the seventh-order differential equation with polynomial coefficients in the form of*

$$t^4y^{(7)}(t) - 4t^3y^{(6)}(t) + 12t^2y^{(5)}(t) + (t^4-24t)y^{(4)}(t) + (-16t^3+24)y'''(t) + 120t^2y''(t)$$
$$- 480ty'(t) + 840y(t) = t^8 + 336t^5, \quad t \geq 0. \tag{39}$$

From Eqs. (3) and (39), we have

$$a_{7,4} = 1, a_{6,3} = -4, a_{5,2} = 12, a_{4,4} = 1, a_{4,1} = -24, a_{3,3} = -16, a_{3,0} = 24, a_{2,2} = 120,$$
$$a_{1,1} = -480, a_{0,0} = 840,$$

and we determine $\alpha = -5$ according to the conditions of Theorem 1, so applying the G_{-5}-transform leads to finding the solution of Eq. (39). *By using the G_{-5}-transform to* (39), *we have*

$$G_{-5}\{t^4y^{(7)}(t)\} - G_{-5}\{4t^3y^{(6)}(t)\} + G_{-5}\{12t^2y^{(5)}(t)\} + G_{-5}\{t^4y^{(4)}(t)\} - G_{-5}\{24ty^{(4)}(t)\}$$
$$- G_{-5}\{16t^3y'''(t)\} + G_{-5}\{24y'''(t)\} + G_{-5}\{120t^2y''(t)\} - G_{-5}\{480ty'(t)\} + G_{-5}\{840y(t)\}$$
$$= G_{-5}\{t^8\} + G_{-5}\{336t^5\}.$$

Using Lemma 1 and the above equation, this can be rewritten as

$$(u^4 + u)F^{(4)}(u) = 8!(u^4 + u),$$
$$F^{(4)}(u) = 8!.$$

Then, we have

$$F(u) = 8!\frac{u^4}{24} + c_1\frac{u^3}{6} + c_2\frac{u^2}{2} + c_3u + c_4, \tag{40}$$

where c_1, c_2, c_3, and c_4 are constants.

4.3.2. Graphical analysis

From Matlab, Figure 3 shows the curves of general solutions $y(t)$ by applying Lemma 2 and the inverse G_{-5}-transform. We let c_1, c_2, c_3, and c_4 in Eq. (40) *in various ways as follows:*

(i) when $c_1 = c_2 = c_3 = c_4 = 0$, we get $y(t) = \dfrac{t^8}{24}$ as a solution of Eq. (39),

(ii) when $c_1 = 7!, c_2 = 6!, c_3 = 5!, c_4 = 4!$, we get $y(t) = \dfrac{t^8}{24} + \dfrac{t^7}{6} + \dfrac{t^6}{2} + t^5 + t^4$ as a solution of Eq. (39),

(iii) when $c_1 = 6 \times 7!, c_2 = 2 \times 6!, c_3 = 5!, c_4 = 4!$, we get $y(t) = \dfrac{t^8}{24} + t^7 + t^6 + t^5 + t^4$ as a solution of Eq. (39).

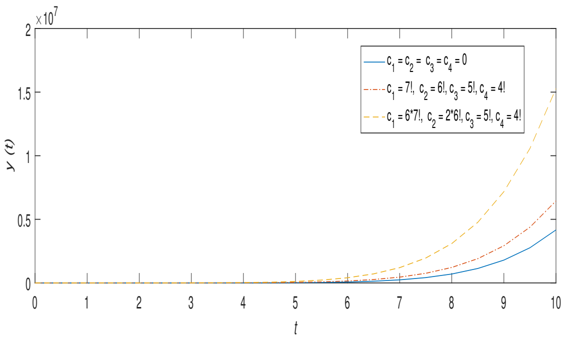

Figure 3: General solutions of Example 3.

Remark 1. *(I) For $m = 5$ and $n = 5$, by Corollary 1, no. COEs = 36 and no. CONs = 45 that is no. COEs ≠ no. CONs. The solutions of HODEPCs under conditions (4)-(5) by the G_3-transform are infinite solutions. In particular, in Example 1., the polynomial coefficients $a_{5,5} = 1, a_{4,4} = 20, a_{3,3} = 120, a_{2,2} = 240, a_{1,1} = 120$, otherwise, 0 can be solved.*
(II) For $m = 5$ and $n = 3$, by Corollary 1, no. COEs = no. CONs = 24. Example 2. is one of the solutions under conditions (4)-(5) which can be solved by the G_{-1}-transform.
(III) For $m = 7$ and $n = 4$, by Corollary 1, no. COEs = 40 and no. CONs = 42 that is no. COEs ≠ no. CONs. In Example 3., the polynomial coefficients $a_{7,4} = 1, a_{6,3} = -4, a_{5,2} = 12, a_{4,4} = 1, a_{4,1} = -24, a_{3,3} = -16, a_{3,0} = 24, a_{2,2} = 120, a_{1,1} = -480, a_{0,0} = 840$, otherwise, 0 which can be solved using the solutions of HODEPCs under conditions (4)-(5) by the G_{-5}-transform.

In fact, the solutions of HODEPCs under conditions (4)-(5) by the G_α-transform can be solved using in various examples. If not, the HODEPCs can not find the solutions.

Remark 2. *From Example 2. if $a_{3,1}$ is equal to 1, then we have*

$$t^3y^{(5)}(t) + (t^3 + 6t^2)y^{(4)}(t) + (3t^2 + t)y'''(t) = \frac{t^2}{2} + t, \quad t \geq 0. \tag{41}$$

The conditions do not satisfy Theorem 1. If we take G_{-1}-transform both sides of Eq. (41), we obtain

$$(u^2 + u)Y'''(u) - 5\frac{Y'(u)}{u} + 10\frac{Y(u)}{u^2} = u^2 + u.$$

Observe that Eq. (41) transformed into a third-order differential equation with variable coefficients. As a result, setting $\alpha = -1$ did not lead to the solution of (41), including for α equaling all other values.

5. Conclusions

Solutions to higher-order differential equations with polynomial coefficients (HODEPCs) were introduced using the G_α-transform. The theorem and corollary of the HODEPCs were obtained to guarantee that they can be corrected by the G_α-transform. Next, we showed some examples according to the theorem which is a strength of the G_α-transform in solving the HODEPCs by selecting an appropriate value for α and proper coefficients of the polynomial.

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