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On Γ-ideals, Γ-submonoids and Isomorphism Theorems of Γ-monoids via Γ-submonoids

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Abstract. This study introduces the concept of Γ-ideals and Γ-submonoids of Γ-monoids and investigates their relationships with the existing Γ-order-ideals. Moreover, quotient of Γ-monoids and isomorphism theorems via Γ-submonoids are proved.

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1. Introduction

The talented monoid of a row-finite directed graph $E = (E^0, E^1, r, s)$, denoted by T_E, is the commutative monoid generated by $\{v(i) : v \in E^0, i \in \mathbb{Z}\}$ such that $v(i) = \sum_{e \in s^{-1}(v)} r(e)(i+1)$ for every $i \in \mathbb{Z}$ and every $v \in E^0$ that is not a sink. The additive group $\mathbb{Z}$ of integers acts on T_E by monoid automorphisms by shifting indices: for each $n, i \in \mathbb{Z}$ and $v \in E^0$, define ${}^n v(i) = v(i+n)$, which extends to an action of $\mathbb{Z}$ on T_E [3]. Monoids with a group Γ acting (by monoid automorphisms) on it, called Γ-monoids, was first introduced in the paper of Hazrat and Li [1] as a tool in the study of talented monoids. In the same paper, Γ-order-ideals of Γ-monoids are also introduced. Sebandal and Vilela [5] prove some properties, including the isomorphism theorems for Γ-monoids and Γ-order-ideals are established.

This paper extends the study of Γ-monoids by defining the concept of Γ-ideals and Γ-submonoids and establishing some of their properties. Moreover, this paper studies quotient of Γ-monoids via equivalence classes of Γ-submonoids and proves isomorphism theorems.

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2. Preliminaries

In this section, we present some basic concepts and known results that are useful in this study.

Definition 1. [2] A *semigroup* is a nonempty set M together with a binary operation $*$ on M which is associative, that is, for all $a, b, c \in M$, $a * (b * c) = (a * b) * c$.

Definition 2. [2] A *monoid* is a semigroup M which contains an identity element $1_M \in M$ such that $1_M * m = m * 1_M = m$ for all $m \in M$.

For a monoid M with the binary operation $*$, we may also say that M *is a monoid under* $*$. A monoid M is said to be *commutative* if for all $x, y \in M$, $x * y = y * x$.

If no confusion arises, by a monoid M, we shall mean a triple $(M, 1_M, *)$ unless otherwise specified.

Definition 3. [6] Let $(M, *)$ be a monoid. A *submonoid* is a subset S of M which is closed under the binary operation on M and contains the identity 1_M of M.

Definition 4. [6] Let $(M, *)$ and $(N, \cdot)$ be monoids. A *monoid homomorphism* is a mapping $\varphi : M \to N$ such that $\varphi(a * b) = \varphi(a) \cdot \varphi(b)$ and $\varphi(1_M) = 1_N$ for all $a, b \in M$ where 1_M and 1_N are the identities in M and N, respectively.

Example 1. Consider the monoids $M = (\mathbb{N}, +)$ and $N = (\mathbb{N}, \cdot)$ and the mapping $\varphi : M \to N$ defined by $\varphi(x) = b^x$, where $b \in \mathbb{N} \setminus \{0\}$. For any $x, y \in M$, we have $\varphi(x + y) = b^{x+y} = b^x \cdot b^y = \varphi(x) \cdot \varphi(y)$ and $\varphi(0) = b^0 = 1$. Therefore, φ is a monoid homomorphism.

Definition 5. [6] A *congruence* on a monoid M is an equivalence relation ρ on M which satisfies the condition: For all $u, v, x, y \in M$, if $x\rho y$, then $(u * x * v)\rho(u * y * v)$.

Proposition 1. [6] *Let ρ be a congruence on a monoid M. Then M/ρ is a monoid with binary operation $\circ$ given by $\rho(x) \circ \rho(y) = \rho(x * y)$ for all $x, y \in M$.*

Definition 6. [4] Let M be a commutative monoid. For any submonoid H of M, we define a binary relation ρ_H in M by $x\rho_H y$ if and only if $(x * H) \cap (y * H) \neq \varnothing$.

Remark 1. [4] For any submonoid H of a commutative monoid M, ρ_H is an equivalence relation on M.

Definition 7. [2] An *action* of a group $(G, \circ)$ in a set S is a function $\phi : G \times S \longrightarrow S$ such that for all $x \in S$, and $g_1, g_2 \in G$: $\phi((1_G, x)) = x$ and $\phi((g_1 \circ g_2, x)) = \phi((g_1, \phi((g_2, x))))$. When such an action is given, G is said to *act on the set S.*

Example 2. Consider the group $G = \mathbb{Z}$ under the usual addition and the set $S = \mathbb{R}$ of real numbers and the mapping $\phi : G \times S \to S$ given by $\phi((g, x)) = 2^g x$. Let $(g, x), (h, y) \in G \times S$ such that $(g, x) = (h, y)$. Then $g = h$ and $x = y$. Thus, we have $\phi((g, x)) = 2^g x = 2^h y = \phi((h, y))$ and ϕ is well-defined. Now, for any $g_1, g_2 \in G$ and $x \in S$, we have $\phi((0, x)) = 2^0 x = x$ and $\phi((g_1 + g_2, x)) = 2^{g_1+g_2} x = 2^{g_1} 2^{g_2} x = \phi((g_1, \phi((g_2, x))))$. Therefore, ϕ is an action.

Definition 8. [3] Let M be a monoid and Γ a group. M is said to be a Γ-*monoid* if there is an action $\phi : \Gamma \times M \to M$ of Γ on M via monoid automorphism, that is, ϕ is an action which satisfies: for all $\alpha \in \Gamma$ and $x, y \in M$, $\phi((\alpha, x * y)) = \phi((\alpha, x)) * \phi((\alpha, y))$. For $\alpha \in \Gamma$ and $a \in M$, the action of α on a shall be denoted by ${}^{\alpha}a$.

Example 3. Consider $\Gamma = \mathbb{Z}$ a group of integers under the usual addition and the set $M = \mathbb{R}$ with the usual addition as its binary operation. Then, $(M, +)$ is a monoid with identity 0. Consider the action $\phi : \Gamma \times M \to M$ given by $\phi((\alpha, x)) = 2^{\alpha}x$ in Example 2. Now, let $\alpha \in \Gamma$ and $x, y \in M$. Then we have $\phi((\alpha, x + y)) = 2^{\alpha}(x + y) = 2^{\alpha}x + 2^{\alpha}y = \phi((\alpha, x)) + \phi((\alpha, y))$. Therefore, M is a Γ-monoid.

Example 4. Let Γ be a group of integers under addition and let $T = M_2(\mathbb{R})$ under matrix addition. Consider the mapping $\phi : \Gamma \times T \to T$ given by $\left(\alpha, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \mapsto {}^{\alpha}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 2^{\alpha}a & 2^{\alpha}b \\ 2^{\alpha}c & 2^{\alpha}d \end{pmatrix}$. Let $\left(\alpha, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right), \left(\beta, \begin{pmatrix} e & f \\ g & h \end{pmatrix}\right) \in \Gamma \times T$ such that $\left(\alpha, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \left(\beta, \begin{pmatrix} e & f \\ g & h \end{pmatrix}\right)$. Then $\alpha = \beta$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$. Thus, $\begin{pmatrix} 2^{\alpha}a & 2^{\alpha}b \\ 2^{\alpha}c & 2^{\alpha}d \end{pmatrix} = \begin{pmatrix} 2^{\beta}e & 2^{\beta}f \\ 2^{\beta}g & 2^{\beta}h \end{pmatrix}$ and ϕ is well-defined. Now, for any $\alpha, \beta \in \Gamma$ and $a, b, c, d \in \mathbb{R}$, we have $\phi\left(\left(0, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)\right) = {}^{0}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 2^0a & 2^0b \\ 2^0c & 2^0d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and

$$\begin{aligned}
\phi\left(\left(\alpha + \beta, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)\right) &= {}^{\alpha+\beta}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \\
&= \begin{pmatrix} 2^{\alpha+\beta}a & 2^{\alpha+\beta}b \\ 2^{\alpha+\beta}c & 2^{\alpha+\beta}d \end{pmatrix} \\
&= \begin{pmatrix} 2^{\alpha}2^{\beta}a & 2^{\alpha}3^{\beta}b \\ 2^{\alpha}4^{\beta}c & 2^{\alpha}5^{\beta}d \end{pmatrix} \\
&= \phi\left(\left(\alpha, \begin{pmatrix} 2^{\beta}a & 2^{\beta}b \\ 2^{\beta}c & 2^{\beta}d \end{pmatrix}\right)\right) \\
&= \phi\left(\left(\alpha, \varphi\left(\left(\beta, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)\right)\right)\right).
\end{aligned}$$

Thus, ϕ is an action.

Now, let $\alpha \in \Gamma$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} e & f \\ g & h \end{pmatrix} \in T$. Then we have

$$\begin{aligned}
\phi\left(\left(\alpha, \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix}\right)\right) &= \phi\left(\left(\alpha, \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}\right)\right) \\
&= \begin{pmatrix} 2^{\alpha}(a+e) & 2^{\alpha}(b+f) \\ 2^{\alpha}(c+g) & 2^{\alpha}(d+h) \end{pmatrix}
\end{aligned}$$

$$
\begin{aligned}
&= \begin{pmatrix} 2^\alpha a + 2^\alpha e & 2^\alpha b + 2^\alpha f \\ 2^\alpha c + 2^\alpha g & 2^\alpha d + 2^\alpha h \end{pmatrix} \\
&= \begin{pmatrix} 2^\alpha a & 2^\alpha b \\ 2^\alpha c & 2^\alpha d \end{pmatrix} + \begin{pmatrix} 2^\alpha e & 2^\alpha f \\ 2^\alpha g & 2^\alpha h \end{pmatrix} \\
&= \phi\left(\left(\alpha, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)\right) + \phi\left(\left(\alpha, \begin{pmatrix} e & f \\ g & h \end{pmatrix}\right)\right).
\end{aligned}
$$

Therefore, T is a Γ-monoid.

Example 5. Consider the set $M = \{1, a, b, c, d, e\}$ and an operation $*$ given by

$*$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	a	a	a	a	a	a
b	b	b	b	b	b	b
c	c	c	c	c	c	c
d	d	d	d	d	d	d
e	e	e	e	e	e	e

The operation $*$ is closed and associative since for all $x, y \in M$, $x * y = x$ holds for all $x \neq 1$. Clearly, 1 is an identity in M. Thus, M is a monoid. With a group Γ acting trivially on M, we obtain that M is a Γ-monoid.

Definition 9. [1] Let M, M_1 and M_2 be monoids and let Γ be a group acting on M, M_1 and M_2.

(i) A Γ-*monoid homomorphism* is a monoid homomorphism $\phi : M_1 \longrightarrow M_2$ that respects the action of Γ, this means $\phi({}^\alpha a) = {}^\alpha\phi(a)$.

(ii) A Γ-*order-ideal* of a monoid M is a subset I of M such that for any $\alpha, \beta \in \Gamma$, ${}^\alpha a * {}^\beta b \in I$ if and only if $a, b \in I$.

Remark 2. [1] A Γ-order-ideal is a submonoid I of M which is closed under the action of Γ.

Example 6. Let a group Γ acts trivially on both monoids $M = (\mathbb{N}, +)$ and $N = (\mathbb{N}, \cdot)$, that is, for all $\alpha \in \Gamma$, we have $\phi((\alpha, m)) = {}^\alpha m = m$ and $\phi((\alpha, n)) = {}^\alpha n = n$ for all $m \in M$ and $n \in N$. Now, let $\alpha \in \Gamma$ and $x, y \in M$. Then, $\phi((\alpha, x + y)) = {}^\alpha(x + y) = x + y = {}^\alpha x + {}^\alpha y = \phi((\alpha, x)) + \phi((\alpha, y))$. Thus, M and N are Γ-monoids. Consider the monoid homomorphism $\varphi : M \to N$ defined by $\varphi(x) = b^x$, where $b \in \mathbb{N} \setminus \{0\}$ in Example 1. For all $\alpha \in \Gamma$ and $a \in M$, we have $\varphi({}^\alpha a) = \varphi(a) = {}^\alpha\varphi(a)$. Thus, by Definition 9(ii), φ is a Γ-monoid homomorphism.

Example 7. Consider the Γ-monoid $M = \mathbb{R}$ under the usual addition in Example 3 and the Γ-monoid $T = M_2(\mathbb{R})$ under matrix addition in Example 4. Define a mapping

$\phi : T \to M$ by $\phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = 2(a+b+c+d)$. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} e & f \\ g & h \end{pmatrix} \in T$ such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$. Then $a = e$, $b = f$, $c = g$ and $d = h$. Thus, $2(a+b+c+d) = 2(e+f+g+h)$ and ϕ is well-defined. Now, for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} e & f \\ g & h \end{pmatrix} \in T$, we have $\phi\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right) = 2(0+0+0+0) = 2(0) = 0$ and

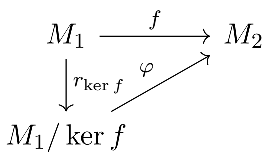
$$\begin{aligned} \phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix}\right) &= \phi\left(\begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}\right) \\ &= 2((a+e)+(b+f)+(c+g)+(d+h)) \\ &= 2((a+b+c+d)+(e+f+g+h)) \\ &= 2(a+b+c+d)+2(e+f+g+h) \\ &= \phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) + \phi\left(\begin{pmatrix} e & f \\ g & h \end{pmatrix}\right). \end{aligned}$$

Thus, ϕ is a monoid homomorphism. Also, for all $\alpha \in \Gamma$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in T$, we have

$$\begin{aligned} \phi\left(\alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) &= \phi\left(\begin{pmatrix} 2^\alpha a & 2^\alpha b \\ 2^\alpha c & 2^\alpha d \end{pmatrix}\right) \\ &= 2(2^\alpha a + 2^\alpha b + 2^\alpha c + 2^\alpha d) \\ &= 2^\alpha 2(a+b+c+d) \\ &= {}^\alpha\phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right). \end{aligned}$$

Hence, ϕ is a Γ-monoid homomorphism.

Theorem 1. [4] *Let M_1 and M_2 be commutative monoids and let $f : M_1 \longrightarrow M_2$ be a homomorphism. There exists a unique homomorphism $\varphi : M_1/\ker f \longrightarrow M_2$ such that the following diagram is commutative*



that is, $\varphi \circ r_{\ker f} = f$, where $r_{\ker f}(x) := \rho_{\ker f}(x)$. Moreover, φ is onto and it has a trivial kernel, namely, $\ker\varphi = \{\ker f\}$. However, φ is an isomorphism if and only if $\rho_f = \rho_{\ker f}$.

3. Γ-ideals

In this section, we discuss the properties of Γ-ideals of Γ-monoids.

Let M be a Γ-monoid and $x \in M$. By Definition 8, for all $\alpha \in \Gamma$, ${}^{\alpha}x * {}^{\alpha}1_M = {}^{\alpha}(x * 1_M) = {}^{\alpha}x$ and ${}^{\alpha}1_M * {}^{\alpha}x = {}^{\alpha}(1_M * x) = {}^{\alpha}x$. By uniqueness of the identity element in M, ${}^{\alpha}1_M = 1_M$.

Remark 3. For a Γ-monoid M and $\alpha \in \Gamma$, ${}^{\alpha}1_M = 1_M$.

Definition 10. Let M be a Γ-monoid. A *left* Γ*-ideal* (respectively, *right* Γ*-ideal*) of M is a subset I of M such that for any $\alpha, \beta \in \Gamma$, for all $a \in I$ and $m \in M$, ${}^{\alpha}m * {}^{\beta}a \in I$ (respectively, ${}^{\alpha}a * {}^{\beta}m \in I$). A Γ*-ideal* of M is a subset I of M such that I is both a left and right Γ-ideal of M.

Let $(M, *)$ be a Γ-monoid and A a Γ-ideal of M with $a \in A$. Then for all $\alpha, \beta \in \Gamma$, we have ${}^{\alpha}a = {}^{\alpha}a * {}^{\alpha}1_M \in A$. Thus, we have the following remark.

Remark 4. Let $(M, *)$ be a Γ-monoid and A be a Γ-ideal of M.

(i) M is a Γ-ideal.

(ii) For all $\alpha \in \Gamma$ and for all $a \in A$, ${}^{\alpha}a \in A$.

Lemma 1. *Let A and B be Γ-ideals of a Γ-monoid M. Then $A * B$ is a Γ-ideal of M.*

Proof. Let A and B be Γ-ideals of a Γ-monoid M. Clearly, $A * B \subseteq M$. Let $x \in A * B$ and $m \in M$. Then $x = a * b$ for some $a \in A$ and $b \in B$. Now, for all $\alpha, \beta \in \Gamma$, ${}^{\alpha}x * {}^{\beta}m = {}^{\alpha}(a * b) * {}^{\beta}m = {}^{\alpha}a * {}^{\alpha}b * {}^{\beta}m = {}^{\alpha}a * ({}^{\alpha}b * {}^{\beta}m) \in A * B$ by Remark 4(ii) and Definition 10. Similarly, for all $\alpha, \beta \in \Gamma$, ${}^{\alpha}m * {}^{\beta}x \in A * B$. Therefore, $A * B$ is a Γ-ideal of M. □

The following example shows that a Γ-ideal is not necessarily a Γ-order-ideal.

Example 8. Consider the set $M = \{1, n, h, s\}$ and operation $*$ given by

$*$	1	n	h	s
1	1	n	h	s
n	n	n	h	s
h	h	h	h	s
s	s	s	s	s

Clearly, the operation is commutative. It can be verified that $*$ is associative. Since $1 * 1 = 1$, $1 * n = n$, $1 * h = h$ and $1 * s = s$, it follows that 1 is the identity in M. Thus, M is a commutative monoid. Let Γ be a group and the mapping $\phi : \Gamma \times M \longrightarrow M$ given by $(\alpha, a) \mapsto {}^{\alpha}a = a$. For any $\alpha, \beta \in \Gamma$ and $a \in M$, we have $\phi((0, a)) = {}^{0}a = a$ and

$$\phi((\alpha + \beta, a)) = {}^{\alpha+\beta}a = a = \phi(\beta, a) = {}^{\beta}a = \phi((\alpha, {}^{\beta}a)) = \phi((\alpha, \phi((\beta, a)))).$$

Thus, ϕ is an action. Now, let $\alpha \in \Gamma$ and $a, b \in M$. Then
$\phi((\alpha, a * b)) = {}^{\alpha}(a * b) = a * b = {}^{\alpha}a * {}^{\alpha}b = \phi((\alpha, a)) * \phi((\alpha, b))$. Hence, M is a Γ-monoid.

Let $C = \{n, h, s\}$. Then for any $\alpha, \beta \in \Gamma$, we have for all $a \in C$ and $m \in M$,

$$
\begin{array}{ll}
{}^{\alpha}a * {}^{\beta}m = {}^{\alpha}n * {}^{\beta}1 = n * 1 = n \in C, & {}^{\alpha}a * {}^{\beta}m = {}^{\alpha}n * {}^{\beta}n = n * n = n \in C; \\
{}^{\alpha}a * {}^{\beta}m = {}^{\alpha}n * {}^{\beta}h = n * h = h \in C, & {}^{\alpha}a * {}^{\beta}m = {}^{\alpha}n * {}^{\beta}s = n * s = s \in C; \\
{}^{\alpha}a * {}^{\beta}m = {}^{\alpha}h * {}^{\beta}1 = h * 1 = h \in C, & {}^{\alpha}a * {}^{\beta}m = {}^{\alpha}h * {}^{\beta}n = h * n = h \in C; \\
{}^{\alpha}a * {}^{\beta}m = {}^{\alpha}h * {}^{\beta}h = h * h = h \in C, & {}^{\alpha}a * {}^{\beta}m = {}^{\alpha}h * {}^{\beta}s = h * s = s \in C; \\
{}^{\alpha}a * {}^{\beta}m = {}^{\alpha}s * {}^{\beta}1 = s * 1 = s \in C, & {}^{\alpha}a * {}^{\beta}m = {}^{\alpha}s * {}^{\beta}n = s * n = s \in C; \\
{}^{\alpha}a * {}^{\beta}m = {}^{\alpha}s * {}^{\beta}h = s * h = s \in C, & {}^{\alpha}a * {}^{\beta}m = {}^{\alpha}s * {}^{\beta}s = s * s = s \in C.
\end{array}
$$

Since M is commutative, ${}^{\beta}m * {}^{\alpha}a = {}^{\alpha}a * {}^{\beta}m \in C$. Thus, by Definition 10, C is a Γ-ideal. However, the identity $1 \notin C$. Thus, C is not a Γ-order-ideal of M.

The following example shows that Γ-order-ideal is not necessarily a Γ-ideal.

Example 9. Consider the Γ-monoid $M = \{1, n, h, s\}$ in Example 8. Let $A = \{1, n, h\}$. Now, suppose that for all $a, b \in M$ and for all $\alpha, \beta \in \Gamma$, ${}^{\alpha}a * {}^{\beta}b \in A$. Then $a * b \in A$. We consider the following three cases.
Case 1. $a * b = 1$. Then $a = 1$ and $b = 1$. Thus $a, b \in A$.
Case 2. $a * b = n$. Then $a * b = 1 * n = n * 1 = n * n$. Clearly, $a, b \in A$.
Case 3. $a * b = h$. Then $a * b = 1 * h = n * h = h * 1 = h * n$. Clearly, $a, b \in A$.
Thus, $a, b \in A$.

Now, suppose that $a, b \in A$. Then, we have

$$
\begin{array}{ll}
{}^{\alpha}a * {}^{\beta}b = {}^{\alpha}1 * {}^{\beta}1 = 1 * 1 = 1 \in A; & {}^{\alpha}a * {}^{\beta}b = {}^{\alpha}n * {}^{\beta}n = n * n = n \in A; \\
{}^{\alpha}a * {}^{\beta}b = {}^{\alpha}1 * {}^{\beta}n = 1 * n = n \in A; & {}^{\alpha}a * {}^{\beta}b = {}^{\alpha}n * {}^{\beta}h = n * h = h \in A; \\
{}^{\alpha}a * {}^{\beta}b = {}^{\alpha}1 * {}^{\beta}h = 1 * h = h \in A; & {}^{\alpha}a * {}^{\beta}b = {}^{\alpha}h * {}^{\beta}h = h * h = h \in A.
\end{array}
$$

Thus, ${}^{\alpha}a * {}^{\beta}b \in A$. Hence, A is a Γ-order-ideal of M.

Observe that there exist $n \in A$ and $s \in M$ such that for any $\alpha, \beta \in \Gamma$, ${}^{\alpha}n * {}^{\beta}s = n * s = s \notin A$. Thus, by Definition 10, A is not a Γ-ideal.

Remark 5. If I is a Γ-ideal, in general I is not necessarily a Γ-order-ideal. Similarly, if I is a Γ-order-ideal, in general I is not necessarily a Γ-ideal.

Lemma 2. *Let I be a Γ-ideal of a Γ-monoid M. Then the identity $1_M \in I$ if and only if $I = M$.*

Proof. Let I is a Γ-ideal of M. Suppose that the identity $1_M \in I$ and $m \in M$. Then for any $\alpha, \beta \in \Gamma$, we have ${}^{\alpha}1_M * {}^{\beta}m \in I$. For $\alpha = \beta = 0$, we have ${}^{0}1_M * {}^{0}m = 1_M * m = m \in I$. Thus, $M \subseteq I$. Consequently, $I = M$. Conversely, suppose that $I = M$. Thus, the identity $1_M \in I$. □

Theorems 2 and 3 imply that there exists no proper Γ-order-ideal which is also a Γ-ideal and vice versa.

Theorem 2. *Let I be a Γ-ideal of a Γ-monoid M. Then I is a Γ-order-ideal of M if and only if $I = M$.*

Proof. Let I be a Γ-ideal of M. Suppose that I is a Γ-order-ideal of M. Then the identity $1_M \in I$. By Lemma 2, $I = M$. Conversely, suppose that $I = M$. Thus, I is a Γ-order-ideal. □

Theorem 3. *Let I be a Γ-order-ideal of a Γ-monoid M. Then I is a Γ-ideal of M if and only if $I = M$.*

Proof. Let I be a Γ-order-ideal of a Γ-monoid M. Then $1_M \in I$ since I is also a submonoid. Suppose that I is a Γ-ideal of M. By Lemma 2, $I = M$. Conversely, suppose that $I = M$. Thus, by Remark 4(i), I is a Γ-ideal. □

Lemma 3. *Let A and B be Γ-ideals of a Γ-monoid M. Then $A \cap B$ and $A \cup B$ are Γ-ideals of M.*

Proof. Let A and B be Γ-ideals of M. Let $x \in A \cap B$ and $m \in M$. Then $x \in A$ and $x \in B$. Since A and B are Γ-ideals of M, for all $\alpha, \beta \in \Gamma$, we have ${}^{\alpha}x * {}^{\beta}m, {}^{\alpha}m * {}^{\beta}x \in A$ and ${}^{\alpha}x * {}^{\beta}m, {}^{\alpha}m * {}^{\beta}x \in B$. Hence, for all $\alpha, \beta \in \Gamma$, ${}^{\alpha}x * {}^{\beta}m, {}^{\alpha}m * {}^{\beta}x \in A \cap B$. Therefore, $A \cap B$ is a Γ-ideal of M. Now, let $x \in A \cup B$ and $m \in M$. Then $x \in A$ or $x \in B$. Since A and B are Γ-ideals of M, for all $\alpha, \beta \in \Gamma$, we have ${}^{\alpha}x * {}^{\beta}m, {}^{\alpha}m * {}^{\beta}x \in A$ or ${}^{\alpha}x * {}^{\beta}m, {}^{\alpha}m * {}^{\beta}x \in B$. Hence, for all $\alpha, \beta \in \Gamma$ ${}^{\alpha}x * {}^{\beta}m, {}^{\alpha}m * {}^{\beta}x \in A \cup B$. Therefore, $A \cup B$ is a Γ-ideal of M. □

Theorem 4. *Let I be a Γ-order-ideal of a Γ-monoid M and J a Γ-ideal of M.*

(i) *If $J \cap I \neq \varnothing$, then $J \cap I$ is a Γ-ideal of I.*

(ii) *If M is commutative, then $J \cup I$ is a Γ-order-ideal of M.*

Proof. Let I be a Γ-order-ideal of M and J a Γ-ideal of M.

(i) Let $x \in J \cap I$ and $a \in I$. Then $x \in J$ and $x \in I$. Since J is a Γ-ideal of M, for all $\alpha, \beta \in \Gamma$, ${}^{\alpha}x * {}^{\beta}a, {}^{\alpha}a * {}^{\beta}x \in J$. Also, since I is a Γ-order-ideal of M, for all $\alpha, \beta \in \Gamma$, ${}^{\alpha}x * {}^{\beta}a, {}^{\alpha}a * {}^{\beta}x \in I$. Thus, for all $\alpha, \beta \in \Gamma$, ${}^{\alpha}x * {}^{\beta}a, {}^{\alpha}a * {}^{\beta}x \in J \cap I$. Therefore, $J \cap I$ is a Γ-ideal of I.

(ii) Suppose that ${}^{\alpha}x * {}^{\beta}a \in J \cup I$ for all $\alpha, \beta \in \Gamma$. Then, ${}^{\alpha}x * {}^{\beta}a \in J$ or ${}^{\alpha}x * {}^{\beta}a \in I$. Since I is a Γ-order-ideal of M, it follows that $x, a \in I \subseteq J \cup I$. Now, suppose that $x, a \in J \cup I$. Consider the following cases.

Case 1. $x, a \in I$. Then, since I is a Γ-order-ideal of M, for all $\alpha, \beta \in \Gamma$, ${}^{\alpha}x * {}^{\beta}a \in I \subseteq J \cup I$.

Case 2. $x \in I$, $a \in J$. Then, since J is a Γ-ideal of M and M is commutative, for all $\alpha, \beta \in \Gamma$, we have ${}^{\alpha}x * {}^{\beta}a = {}^{\beta}a * {}^{\alpha}x \in J \subseteq J \cup I$.

Case 3. $x \in J, a \in I$. Then, since J is a Γ-ideal of M, for all $\alpha, \beta \in \Gamma$, we have ${}^{\alpha}x * {}^{\beta}a \in J \subseteq J \cup I$.

Case 4. $x, a \in J$. Then, since J is a Γ-ideal of M, for all $\alpha, \beta \in \Gamma$, we have ${}^{\alpha}x * {}^{\beta}a \in J \subseteq J \cup I$.

Thus, $J \cup I$ is a Γ-order-ideal of M. □

Definition 11. Let $(M, *)$ and $(N, \cdot)$ be Γ-monoids and $\varphi : M \to N$ a Γ-monoid homomorphism. The *kernel of* φ is denoted and defined by $\ker\varphi = \{m \in M : \varphi(m) = 1_N\}$.

Proposition 2. *Let* $(M, *)$ *and* $(N, \cdot)$ *be* Γ*-monoids and* $\varphi : M \to N$ *a* Γ*-monoid homomorphism.*

(i) *If* φ *is surjective and* I *is a* Γ*-ideal of* M*, then* $\varphi(I)$ *is a* Γ*-ideal of* N*.*

(ii) *If* J *is a* Γ*-ideal of* N*, then* $\varphi^{-1}(J)$ *is a* Γ*-ideal of* M*.*

Proof. Let $\varphi : M \to N$ be a Γ-monoid homomorphism.

(i) Let $x \in \varphi(I)$ and $z \in N$. Since φ is surjective, $z = \varphi(n)$ for some $n \in M$ and $x = \varphi(y)$ for some $y \in I$. Then for all $\alpha, \beta \in \Gamma$,

$$ {}^{\alpha}x * {}^{\beta}z = {}^{\alpha}\varphi(y) \cdot {}^{\beta}\varphi(n) = \varphi({}^{\alpha}y) \cdot \varphi({}^{\beta}n) = \varphi({}^{\alpha}y * {}^{\beta}n). $$

Since I is a Γ-ideal of M, ${}^{\alpha}y * {}^{\beta}n \in I$, so, ${}^{\alpha}x * {}^{\beta}z \in \varphi(I)$. Similarly, for all $\alpha, \beta \in \Gamma$, ${}^{\alpha}z * {}^{\beta}x \in \varphi(I)$. Therefore, $\varphi(I)$ is a Γ-ideal of N.

(ii) Let $y \in \varphi^{-1}(J)$ and $m \in M$. Then $\varphi(y) \in J$ and $\varphi(m) \in N$. Thus, for all $\alpha, \beta \in \Gamma$, $\varphi({}^{\alpha}y * {}^{\beta}m) = \varphi({}^{\alpha}y) \cdot \varphi({}^{\beta}m) = {}^{\alpha}\varphi(y) \cdot {}^{\beta}\varphi(m) \in J$, since J is a Γ-ideal of N. Hence, ${}^{\alpha}y * {}^{\beta}m \in \varphi^{-1}(J)$ for all $\alpha, \beta \in \Gamma$. Similarly, for all $\alpha, \beta \in \Gamma$, ${}^{\alpha}m * {}^{\beta}y \in \varphi^{-1}(J)$. Therefore, $\varphi^{-1}(J)$ is a Γ-ideal of M. □

Example 10. Consider the Γ-monoid homomorphism $\varphi : M \to N$ defined by $\varphi(x) = b^x$, where $b \neq 0$ in Example 6. Note that

$$ \ker\varphi = \{x \in M : \varphi(x) = 1\} = \{x \in M : b^x = 1\} = \{x \in M : b = 1 \text{ or } x = 0\}. $$

Take $x = 0 \in \ker\varphi$, $m = 2 \in M$, and $b = 2$. Then for all $\alpha, \beta \in \Gamma$, $\varphi({}^{\alpha}x + {}^{\beta}m) = \varphi({}^{\alpha}0 + {}^{\beta}2) = \varphi(0 + 2) = \varphi(2) = 2^2 \neq 1$. This implies that ${}^{\alpha}x + {}^{\beta}m \notin \ker\varphi$. By Definition 10, $\ker\varphi$ is not a Γ-ideal of M.

Remark 6. For any Γ-monoids M and N, the kernel of a Γ-monoid homomorphism $\varphi : M \to N$ is not necessarily a Γ-ideal of M.

Proposition 3. *Let* $(M, *)$ *and* $(N, \cdot)$ *be* Γ*-monoids and* $\varphi : M \to N$ *a* Γ*-monoid homomorphism. Then* $\ker\varphi$ *is a* Γ*-ideal of* M *if and only if* $\ker\varphi = M$*.*

Proof. Let $\varphi : M \to N$ be a Γ-monoid homomorphism. Then $1_M \in \ker\varphi$. Suppose that $\ker\varphi$ is a Γ-ideal of M. Then by Lemma 2, $\ker\varphi = M$. Now, suppose that $\ker\varphi = M$. Then by Remark 4(i), $\ker\varphi$ is a Γ-ideal of M. □

By Proposition 3, $\ker\varphi$ is a Γ-ideal if and only if φ is a zero map. Thus, isomorphism theorems via Γ-ideals are irrelevant.

4. Γ-submonoids

This section presents the discussions on Γ-submonoids of Γ-monoids.

Definition 12. Let $(M, *)$ be a Γ-monoid. A Γ-*submonoid* is a subset S of M such that the identity $1_M \in S$ and, for all $\alpha, \beta \in \Gamma$ and for all $s, t \in S$, ${}^{\alpha}s * {}^{\beta}t \in S$.

Let S be a Γ-submonoid of M. Then $1_M \in S$ and for all $\alpha, \beta \in \Gamma$ and for all $s, t \in S$, we have ${}^{\alpha}s * {}^{\beta}t \in S$. Take $\alpha = \beta = 0$. Thus, we have $s * t = {}^{0}s * {}^{0}t \in S$. Hence, S is a submonoid of M.

Remark 7. Let S be a Γ-submonoid of a Γ-monoid M.

(i) S is a submonoid of M, hence a monoid itself.

(ii) For all $s \in S$ and for all $\alpha \in \Gamma$, ${}^{\alpha}s \in S$.

(iii) M is a Γ-submonoid of M.

Let S be a Γ-submonoid of a Γ-monoid M and let $\phi : \Gamma \times M \to M$ be the action (by monoid automorphism) of a group Γ on M. By Remark 7, S is a monoid. Moreover, by restricting the action ϕ to S, ϕ acts on S by monoid automorphism and hence, S is a Γ-monoid.

Remark 8. A Γ-submonoid of a Γ-monoid is itself a Γ-monoid.

Example 11. Consider the set $M = \{0, 1, x, y, z, s, b\}$ and an operation $+$ given by

$+$	0	1	x	y	z	s	b
0	0	1	x	y	z	s	b
1	1	1	1	s	s	s	b
x	x	1	1	s	s	s	b
y	y	s	s	y	y	s	b
z	z	s	s	y	y	s	b
s	s	s	s	s	s	s	b
b	b	b	b	b	b	b	s

It was shown in [5] that M is a commutative Γ-monoid with identity 0, where the trivial group $\Gamma = \{0\}$ acts trivially on M. Let $S = \{0, y, s, b\}$, $U = \{0, 1, y, s, b\}$, $V = \{0, 1, x\}$ and $W = \{0, y\}$. Note that the identity 0 is in S, U, V and W. Now, we have

$$\begin{aligned} {}^0 0 + {}^0 0 &= 0 + 0 = 0 \in S, & {}^0 0 + {}^0 y &= 0 + y = y \in S; \\ {}^0 0 + {}^0 s &= 0 + s = s \in S, & {}^0 0 + {}^0 b &= 0 + b = b \in S; \\ {}^0 y + {}^0 y &= y + y = y \in S, & {}^0 y + {}^0 s &= y + s = s \in S; \\ {}^0 y + {}^0 b &= y + b = b \in S, & {}^0 s + {}^0 s &= s + s = s \in S; \\ {}^0 s + {}^0 b &= s + b = b \in S, & {}^0 b + {}^0 b &= b + b = b \in S. \end{aligned}$$

Thus, by Definition 12, S is Γ-submonoid of M. Similarly, U, V and W are Γ-submonoids of M. Consider the Γ-submonoid $S = \{0, y, s, b\}$. Now, take $0 \in S$ and $z \in M$. Then $0 * z = z \notin S$. Thus, S is not a Γ-ideal of M.

Remark 9. Let M be a Γ-monoid. A Γ-submonoid of M is not necessarily a Γ-ideal of M.

Theorems 5 and 6 imply that there is no proper Γ-submonoid which is also a Γ-ideal and vice versa.

Theorem 5. *Let S be a Γ-submonoid of a Γ-monoid M. Then S is a Γ-ideal of M if and only if $S = M$.*

Proof. Let S be a Γ-submonoid of M. Suppose that S is a Γ-ideal of M. Since S is a Γ-submonoid, $1_M \in S$ and thus, by Lemma 2, $S = M$. Conversely, suppose that $S = M$. Then, by Remark 4(i), S is a Γ-ideal of M. □

Theorem 6. *Let I be a Γ-ideal of a Γ-monoid M. Then I is a Γ-submonoid of M if and only if $I = M$.*

Proof. Let I be a Γ-ideal of a Γ-monoid M. Suppose that I is a Γ-submonoid of M. Then $1_M \in I$ and $I = M$. Conversely, suppose that $I = M$. By Remark 7(iii), I is a Γ-submonoid of M. □

Example 12. Consider the Γ-submonoid $S = \{0, y, s, b\}$ in Example 11. Note that $x * z = s \in S$. However, $x, z \notin S$. Thus, S is not a Γ-order-ideal of M.

Note that if S is a Γ-order-ideal of a Γ-monoid M, then by Remark 2, S is a submonoid and $1_M \in S$. Also, since S is a Γ-order-ideal, for all $\alpha, \beta \in \Gamma$ and for all $s, t \in S$, we have ${}^\alpha s * {}^\beta t \in S$. Thus, S is a Γ-submonoid of M and the following remark holds.

Remark 10. Every Γ-order-ideal of a Γ-monoid M is a Γ-submonoid of M. However, a Γ-submonoid of M is not necessarily a Γ-order-ideal of M.

The following example shows that a Γ-submonoid is not necessarily a normal submonoid.

Example 13. Consider the Γ-submonoid $U = \{0, 1, y, s, b\}$ in Example 11 which is also commutative. Observe that $y, z \in M$ such that $y, y * z = y \in U$. However, $z \notin U$. Thus, U is not a normal submonoid of M.

Remark 11. In general, a Γ-submonoid of a Γ-monoid M is not necessarily a normal submonoid of M.

Theorem 7. *Let S be a subset of a Γ-monoid M. Then S is a Γ-order-ideal if and only if S is a Γ-submonoid such that $x * y \in S$ implies $x, y \in S$.*

Proof. Let S be a subset of a Γ-monoid M. Suppose S is a Γ-order-ideal of M. Then by Remark 10, S is a Γ-submonoid and for $\alpha = \beta = 0$, we have $x * y = {}^0x * {}^0y \in S$ implies $x, y \in S$ since S is a Γ-order-ideal. Now, suppose S is a Γ-submonoid such that $x * y \in S$ implies $x, y \in S$. Then for all $\alpha, \beta \in \Gamma$ and for all $x, y \in S$, ${}^\alpha x * {}^\beta y \in S$. Suppose for all $\alpha, \beta \in \Gamma$, ${}^\alpha x * {}^\beta y \in S$. Take $\alpha = \beta = 0$. Then $x * y = {}^0x * {}^0y \in S$ which implies that $x, y \in S$. Therefore, S is a Γ-order-ideal. □

Lemma 4. *Let A and B be Γ-submonoids of a Γ-monoid M. Then*

(i) *$A \cap B$ is a Γ-submonoid of M.*

(ii) *If M is commutative and A, B are normal, then $A \cap B$ is a normal Γ-submonoid of M.*

Proof. Let A and B be Γ-submonoids of a Γ-monoid M.

(i) Since A and B are Γ-submonoids of M, the identity $1_M \in A$ and $1_M \in B$. Thus, $1_M \in A \cap B$. Now, let $a, b \in A \cap B$. Then, $a, b \in A$ and $a, b \in B$. Since A and B are Γ-submonoids, for all $\alpha, \beta \in \Gamma$, ${}^\alpha a * {}^\beta b \in A$ and ${}^\alpha a * {}^\beta b \in B$. Hence, ${}^\alpha a * {}^\beta b \in A \cap B$. Therefore, $A \cap B$ is a Γ-submonoid of M.

(ii) By (i), $A \cap B$ is a Γ-submonoid of M. It remains to show that $A \cap B$ is normal. Let $x, x * y \in A \cap B$. Then $x, x * y \in A$ and $x, x * y \in B$. Since A and B are normal, $y \in A$ and $y \in B$. Therefore, $y \in A \cap B$ and $A \cap B$ is a normal Γ-submonoid of M. □

Example 14. Consider the Γ-submonoids $V = \{0, 1, x\}$ and $W = \{0, y\}$ in Example 11. Then, $V \cup W = \{0, 1, x, y\}$. Now, for $x, y \in V \cup W$, we have $x * y = s \notin V \cup W$. Thus, $V \cup W$ is not a Γ-submonoid of M.

Remark 12. The union of two Γ-submonoids of a Γ-monoid M is not necessarily a Γ-submonoid of M.

Theorem 8. *Let $(M, *)$ and $(N, \cdot)$ be Γ-monoids and $\varphi : M \to N$ a Γ-monoid homomorphism.*

(i) *If S is a Γ-submonoid of M, then $\varphi(S)$ is a Γ-submonoid of N. In particular, $\varphi(M)$ is a Γ-submonoid of N.*

(ii) *If T is a Γ-submonoid of N, then $\varphi^{-1}(T)$ is a Γ-submonoid of M.*

(iii) $\ker\varphi$ *is a Γ-submonoid of M.*

(iv) *If M is commutative, then* $\ker\varphi$ *is normal.*

Proof. Let $\varphi : M \to N$ be a Γ-monoid homomorphism.

(i) Let S be a Γ-submonoid of M. Then $1_M \in S$ and $1_N = \varphi(1_M) \in \varphi(S)$. Let $x, y \in \varphi(S)$. Then $x = \varphi(a)$ and $y = \varphi(b)$ for some $a, b \in S$. Since S is a Γ-submonoid, for all $\alpha, \beta \in \Gamma$, ${}^{\alpha}a * {}^{\beta}b \in S$. Now, for all $\alpha, \beta \in \Gamma$, we have ${}^{\alpha}x \cdot {}^{\beta}y = {}^{\alpha}\varphi(a) \cdot {}^{\beta}\varphi(b) = \varphi({}^{\alpha}a) \cdot \varphi({}^{\beta}b) = \varphi({}^{\alpha}a * {}^{\beta}b)$. Since ${}^{\alpha}a * {}^{\beta}b \in S$, it follows that ${}^{\alpha}x \cdot {}^{\beta}y = \varphi({}^{\alpha}a * {}^{\beta}b) \in \varphi(S)$. Thus, $\varphi(S)$ is a Γ-submonoid of N.

(ii) Let T be a Γ-submonoid of N. Then, $\varphi(1_M) = 1_N \in T$ and $1_M \in \varphi^{-1}(T)$. Let $x, y \in \varphi^{-1}(T)$. Then $\varphi(x), \varphi(y) \in T$. Now, for all $\alpha, \beta \in \Gamma$, we have $\varphi({}^{\alpha}x * {}^{\beta}y) = \varphi({}^{\alpha}x) \cdot \varphi({}^{\beta}y) = {}^{\alpha}\varphi(x) \cdot {}^{\beta}\varphi(y) \in T$ since T is a Γ-submonoid of N. This implies that for all $\alpha, \beta \in \Gamma$, we have ${}^{\alpha}x * {}^{\beta}y \in \varphi^{-1}(T)$. Therefore, $\varphi^{-1}(T)$ is a Γ-submonoid of M.

(iii) Since φ is a Γ-monoid homomorphism, $\varphi(1_M) = 1_N$. Thus, $1_M \in \ker\varphi$. Now, let $x, y \in \ker\varphi$. Then $\varphi(x) = 1_N$ and $\varphi(y) = 1_N$. Thus, by Remark 3, for all $\alpha, \beta \in \Gamma$,

$$\varphi({}^{\alpha}x * {}^{\beta}y) = \varphi({}^{\alpha}x) \cdot \varphi({}^{\beta}y) = {}^{\alpha}\varphi(x) \cdot {}^{\beta}\varphi(y) = {}^{\alpha}1_N \cdot {}^{\beta}1_N = 1_N \cdot 1_N = 1_N.$$

Hence, for all $\alpha, \beta \in \Gamma$, ${}^{\alpha}x * {}^{\beta}y \in \ker\varphi$. Therefore, $\ker\varphi$ is a Γ-submonoid of M.

(iv) Let $x, x * y \in \ker\varphi$. Then $\varphi(x) = 1_N$ and $\varphi(x * y) = 1_N$. Thus, $\varphi(y) = 1_N \cdot \varphi(y) = \varphi(x) \cdot \varphi(y) = \varphi(x * y) = 1_N$. This implies that $y \in \ker\varphi$ and thus, $\ker\varphi$ is normal. □

Theorem 9. *Let J be a Γ-ideal and S a Γ-submonoid of a Γ-monoid M such that $J \cap S \neq \varnothing$. Then* (i) *$J \cap S$ is a Γ-ideal of S;* (ii) *$J \cup S$ is a Γ-submonoid of M.*

Proof. Let J be a Γ-ideal and S a Γ-submonoid of M such that $J \cap S \neq \varnothing$.

(i) Let $x \in J \cap S$ and $s \in S$. Then $x \in J$ and $x, s \in S$. Since J is a Γ-ideal of M, for all $\alpha, \beta \in \Gamma$, ${}^{\alpha}x * {}^{\beta}s, {}^{\alpha}s * {}^{\beta}x \in J$. Also, since S is a Γ-submonoid of M, for all $\alpha, \beta \in \Gamma$, ${}^{\alpha}x * {}^{\beta}s, {}^{\alpha}s * {}^{\beta}x \in S$. Thus, for all $\alpha, \beta \in \Gamma$, ${}^{\alpha}x * {}^{\beta}s, {}^{\alpha}s * {}^{\beta}x \in J \cap S$ and so, $J \cap S$ is a Γ-ideal of S.

(ii) Let $x, y \in J \cup S$. We consider the following cases.

Case 1. $x, y \in J$. Since J is a Γ-ideal of M, for all $\alpha, \beta \in \Gamma$, ${}^{\alpha}x * {}^{\beta}y \in J \subseteq J \cup S$.

Case 2. $x \in J, y \in S$. Since J is a Γ-ideal of M, for all $\alpha, \beta \in \Gamma$, ${}^{\alpha}x * {}^{\beta}y \in J \subseteq J \cup S$.

Case 3. $x, y \in S$. Since S is a Γ-submonoid of M, for all $\alpha, \beta \in \Gamma$, ${}^{\alpha}x * {}^{\beta}y \in S \subseteq J \cup S$.

Case 4. $y \in J, x \in S$. Since J is a Γ-ideal of M, for all $\alpha, \beta \in \Gamma$, ${}^{\alpha}x * {}^{\beta}y \in J \subseteq J \cup S$.

Also, since S is a Γ-submonoid of M, $1_M \in S \subseteq J \cup S$. Therefore, $J \cup S$ is a Γ-submonoid of M. □

Remark 13. Theorem 4(i) is also a consequence of Theorem9(i).

Lemma 5. *Let A and B be Γ-submonoids of a commutative Γ-monoid M. Then $A * B$ is a Γ-submonoid of M.*

Proof. Let $x, y \in A*B$ and $\alpha, \beta \in \Gamma$. Then $x = a_1*b_1$ and $y = a_2*b_2$ for some $a_1, a_2 \in A$ and $b_1, b_2 \in B$. Since A and B are Γ-submonoids, ${}^{\alpha}a_1 * {}^{\beta}a_2 \in A$ and ${}^{\alpha}b_1 * {}^{\beta}b_2 \in B$. Note that $1_M = 1_M * 1_M \in A * B$. Since M is commutative,

$${}^{\alpha}x * {}^{\beta}y = {}^{\alpha}(a_1 * b_1) * {}^{\beta}(a_2 * b_2) = ({}^{\alpha}a_1 * {}^{\alpha}b_1) * ({}^{\beta}a_2 * {}^{\beta}b_2) = ({}^{\alpha}a_1 * {}^{\beta}a_2) * ({}^{\alpha}b_1 * {}^{\beta}b_2).$$

This implies that ${}^{\alpha}x * {}^{\beta}y \in A * B$. Therefore, $A * B$ is a Γ-submonoid of M. □

Lemma 6. *Let A and B be Γ-submonoids of a commutative Γ-monoid M. Then the map $f : A \to A * B$ defined by $f(a) = a * 1_M$ is a Γ-monoid homomorphism.*

Proof. Let $x, y \in A$ such that $x = y$. Then $f(x) = x * 1_M = x = y = y * 1_M = f(y)$ and f is well-defined. Let $x, y \in A$. Then

(i) $f(x * y) = x * y * 1_M = x * y = (x * 1_M) * (y * 1_M) = f(x) * f(y)$,

(ii) $f(1_M) = 1_M * 1_M$, the identity in $A * B$.

Thus, f is a monoid homomorphism. Now, for all $\alpha \in \Gamma$ and $x \in A$,

$$f({}^{\alpha}x) = {}^{\alpha}x * 1_M = {}^{\alpha}x * {}^{\alpha}1_M = {}^{\alpha}(x * 1_M) = {}^{\alpha}f(x).$$

Thus, f is a Γ-monoid homomorphism. □

5. Quotient Γ-monoids

In [5], the quotient Γ-monoid M/S was established using the equivalence relation in Definition 6 such that the commutative Γ-monoid M and Γ-order-ideal S of M were treated as commutative monoid and submonoid, respectively. Further, the third isomorphism theorem for Γ-monoids via Γ-order-ideals was proved.

Here, we define an equivalence relation and construct quotient Γ-monoids via Γ-submonoids. Moreover, we prove the isomorphism theorems.

Definition 13. Let M be a Γ-monoid. For any Γ-submonoid S of M and for all $x, y \in M$, we define a binary relation ρ_S in M by $x\rho_S y$ if and only if for all $\alpha \in \Gamma$, $({}^{\alpha}x*S) \cap ({}^{\alpha}y*S) \neq \varnothing$.

The next example shows that if a Γ-submonoid S of a Γ-monoid M is not commutative, then ρ_S is not an equivalence relation.

Example 15. Consider the Γ-monoid $M = \{1, a, b, c, d, e\}$ in Example 5 with operation $*$ given by

$*$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	a	a	a	a	a	a
b	b	b	b	b	b	b
c	c	c	c	c	c	c
d	d	d	d	d	d	d
e	e	e	e	e	e	e

Let $S = \{1, a, b\}$. Then, by routine calculation, S is a Γ-submonoid of M. Also, S is not commutative since $a*b = a \neq b = b*a$. Now, for all $\alpha \in \Gamma$, we have ${}^{\alpha}1*S = 1*S = \{1, a, b\}$, ${}^{\alpha}a * S = a * S = \{a\}$ and ${}^{\alpha}b * S = b * S = \{b\}$. Thus, $({}^{\alpha}a * S) \cap ({}^{\alpha}1 * S) = \{a\} \neq \varnothing$ which implies that $a\rho_S 1$. Also, $({}^{\alpha}1 * S) \cap ({}^{\alpha}b * S) = \{b\} \neq \varnothing$ which implies that $1\rho_S b$. However, $({}^{\alpha}a * S) \cap ({}^{\beta}b * S) = \varnothing$ which implies that a is not related to b under ρ_S, that is, ρ_S is not transitive, hence not an equivalence relation.

The following result tells us that ρ_S is an equivalence relation for any commutative Γ-submonoid S of a Γ-monoid M. Further, if M is commutative, then ρ_S is a congruence relation on M.

Theorem 10. *Let S be a commutative Γ-submonoid of a Γ-monoid M. Then*

(i) *ρ_S is an equivalence relation on M.*

(ii) *If M is commutative, then ρ_S is a congruence relation on M.*

Proof. Let S be a commutative Γ-submonoid of a Γ-monoid M.

(i) Let $x \in M$ and S a Γ-submonoid of M. Then, for $\alpha \in \Gamma$, we have $({}^{\alpha}x*S) \cap ({}^{\alpha}x*S) = {}^{\alpha}x * S \neq \varnothing$ since ${}^{\alpha}x = {}^{\alpha}x * 1_M \in {}^{\alpha}x * S$. Thus, $x\rho_S x$ and ρ_S is reflexive.

Let $x\rho_S y$. Then, for all $\alpha \in \Gamma$, $({}^{\alpha}x * S) \cap ({}^{\alpha}y * S) \neq \varnothing$. Thus, $({}^{\alpha}y * S) \cap ({}^{\alpha}x * S) = ({}^{\alpha}x * S) \cap ({}^{\alpha}y * S) \neq \varnothing$. Hence, $y\rho_S x$ and ρ_S is symmetric.

Now, let $x\rho_S y$ and $y\rho_S z$. Then, for all $\alpha, \beta \in \Gamma$, $({}^{\alpha}x * S) \cap ({}^{\alpha}y * S) \neq \varnothing$ and $({}^{\beta}y * S) \cap ({}^{\beta}z * S) \neq \varnothing$. Thus, we have ${}^{\alpha}x * s_1 = {}^{\alpha}y * s_2$ and ${}^{\beta}y * s_3 = {}^{\beta}z * s_4$ for some $s_1, s_2, s_3, s_4 \in S$. Hence, for all $\alpha \in \Gamma$, ${}^{\alpha}x * s_1 * s_3 = {}^{\alpha}y * s_2 * s_3 = {}^{\alpha}z * s_2 * s_4$ and $s_1 * s_3, s_2 * s_4 \in S$ since S is a Γ-submonoid. Hence, $({}^{\alpha}x * S) \cap ({}^{\alpha}z * S) \neq \varnothing$ and $x\rho_S z$. Therefore, ρ_S is transitive. Consequently, ρ_S is an equivalence relation on M.

(ii) Let M be a commutative Γ-monoid. Suppose that $x\rho_S y$ and $u, v \in M$. Then, we have for all $\alpha, \beta \in \Gamma$, $({}^{\alpha}x * S) \cap ({}^{\alpha}y * S) \neq \varnothing$ and thus, ${}^{\alpha}x * s_1 = {}^{\alpha}y * s_2$ for some $s_1, s_2 \in S$. Hence, $({}^{\alpha}x * s_1) * {}^{\alpha}(u * v) = ({}^{\alpha}y * s_2) * {}^{\alpha}(u * v)$. Since M is commutative, for all $\alpha \in \Gamma$, ${}^{\alpha}(u * x * v) * s_1 = {}^{\alpha}(u * y * v) * s_2$ and $(u * x * v)\rho_S(u * y * v)$. Thus, ρ_S is a congruence relation on M. □

Definition 14. Let S be a commutative Γ-submonoid of a Γ-monoid M. Then for all $x \in M$, the *equivalence class* of x is denoted and defined by $\rho_S(x) = \{y \in M : x\rho_S y\}$.

Let S be a commutative Γ-submonoid of a Γ-monoid M and let $m \in M$. Then for all $\alpha \in \Gamma$, $({}^{\alpha}m * S) \cap ({}^{\alpha}m * S) = {}^{\alpha}m * S \neq \varnothing$ since for $\alpha = 0$, $m = m * 1_M \in m * S$. Thus, $m \in \rho_S(m)$. Hence, the following remark holds.

Remark 14. Let S be a commutative Γ-submonoid of a Γ-monoid M and let $m_1, m_2 \in M$.

(i) For all $m \in M$, $m \in \rho_S(m)$.

(ii) $\rho_S(m_1) = \rho_S(m_2)$ if and only if $({}^{\alpha}m_1 * S) \cap ({}^{\alpha}m_2 * S) \neq \varnothing$ for all $\alpha \in \Gamma$.

The quotient M/S using equivalence relation in Definition 6, where M is a monoid and S is a submonoid of M is different from M/S using the equivalence relation in Definition 13, where M is a Γ-monoid and S is a Γ-submonoid as shown in the following example.

Example 16. Let $\Gamma = \mathbb{Z}$ the additive group of integers and $M = \mathbb{Z}_8 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}, \overline{6}, \overline{7}\}$ under addition modulo 8. Then M is a monoid with identity $\overline{0}$. Consider a mapping $\phi : \Gamma \times M \to M$ given by $\phi((\alpha, \overline{m})) = \overline{7^{\alpha}m}$. Let $(\alpha, \overline{x}), (\beta, \overline{y}) \in \Gamma \times M$ such that $(\alpha, \overline{x}) = (\beta, \overline{y})$. Then $\alpha = \beta$ and $\overline{x} = \overline{y}$. Thus, $\overline{7^{\alpha}x} = \overline{7^{\beta}y}$ and ϕ is well-defined. Now, let $\alpha, \beta \in \Gamma$ and $m \in M$. Observe that

(i) $\phi((0, \overline{m})) = \overline{7^0 m} = \overline{m}$;

(ii) $\phi((\alpha + \beta, \overline{m})) = \overline{7^{\alpha+\beta}m} = \overline{7^{\alpha}7^{\beta}m} = \phi((\alpha, \phi((\beta, \overline{m}))))$.

This implies that ϕ is an action. Now, let $\alpha \in \Gamma$ and $\overline{x}, \overline{y} \in M$. Then

$$\phi((\alpha, \overline{x} +_8 \overline{y})) = \phi((\alpha, \overline{x +_8 y})) = \overline{7^{\alpha}(x +_8 y)} = \overline{7^{\alpha}x} +_8 \overline{7^{\alpha}y} = \phi((\alpha, \overline{x})) +_8 \phi((\alpha, \overline{y})).$$

Therefore, M is a Γ-monoid.

Let $S = \{\overline{0}, \overline{4}\}$. Observe that the identity $\overline{0} \in S$ and $\overline{0} +_8 \overline{0} = \overline{0}$, $\overline{0} +_8 \overline{4} = \overline{4} +_8 \overline{0} = \overline{4}$, $\overline{4} +_8 \overline{4} = \overline{0} \in S$. This implies that S is a submonoid of M. Now, note that

$$\begin{aligned}
\overline{0} +_8 S &= \overline{0} +_8 \{\overline{0}, \overline{4}\} = \{\overline{0}, \overline{4}\}, & \overline{4} +_8 S &= \overline{4} +_8 \{\overline{0}, \overline{4}\} = \{\overline{0}, \overline{4}\};\\
\overline{1} +_8 S &= \overline{1} +_8 \{\overline{0}, \overline{4}\} = \{\overline{1}, \overline{5}\}, & \overline{5} +_8 S &= \overline{5} +_8 \{\overline{0}, \overline{4}\} = \{\overline{1}, \overline{5}\};\\
\overline{2} +_8 S &= \overline{2} +_8 \{\overline{0}, \overline{4}\} = \{\overline{2}, \overline{6}\}, & \overline{6} +_8 S &= \overline{6} +_8 \{\overline{0}, \overline{4}\} = \{\overline{2}, \overline{6}\};\\
\overline{4} +_8 S &= \overline{4} +_8 \{\overline{0}, \overline{4}\} = \{\overline{0}, \overline{4}\}, & \overline{7} +_8 S &= \overline{7} +_8 \{\overline{0}, \overline{4}\} = \{\overline{3}, \overline{7}\}.
\end{aligned}$$

Moreover, $\rho_S(\overline{0}) = \{\overline{0}, \overline{4}\}$, $\rho_S(\overline{1}) = \{\overline{1}, \overline{5}\}$, $\rho_S(\overline{2}) = \{\overline{2}, \overline{6}\}$, $\rho_S(\overline{3}) = \{\overline{3}, \overline{7}\}$, $\rho_S(\overline{4}) = \{\overline{0}, \overline{4}\}$, $\rho_S(\overline{5}) = \{\overline{1}, \overline{5}\}$, $\rho_S(\overline{6}) = \{\overline{2}, \overline{6}\}$, and $\rho_S(\overline{7}) = \{\overline{3}, \overline{7}\}$. Thus, the quotient $M/S = \{\rho_S(\overline{0}), \rho_S(\overline{1}), \rho_S(\overline{2}), \rho_S(\overline{3})\}$ using the equivalence relation in Definition 6.

Now, observe that for all $\alpha \in \Gamma$, $\overline{7^{\alpha}} = \overline{1}$ or $\overline{7^{\alpha}} = \overline{7}$. Note that the identity $\overline{0} \in S$ and for all $\alpha, \beta \in \Gamma$,

$${}^{\alpha}\overline{0} +_8 {}^{\beta}\overline{0} = \overline{7^{\alpha}0} +_8 \overline{7^{\beta}0} = \overline{0} +_8 \overline{0} \in S;$$

$$^{\alpha}\overline{0} +_8 {}^{\beta}\overline{4} = \overline{7^{\alpha}0} +_8 \overline{7^{\beta}4} = \overline{7^{\beta}4} = \overline{4} \in S;$$
$$^{\alpha}\overline{4} +_8 {}^{\beta}\overline{4} = \overline{7^{\alpha}4} +_8 \overline{7^{\beta}4} = \overline{0} \text{ or } \overline{4} \in S.$$

This implies that S is a Γ-submonoid of M. Now, note that for all $\alpha \in \Gamma$,

$$^{\alpha}\overline{0} +_8 S = {}^{\alpha}\overline{0} +_8 \{\overline{0}, \overline{4}\} = \overline{7^{\alpha}0} +_8 \{\overline{0}, \overline{4}\} = \{\overline{0}, \overline{4}\};$$
$$^{\alpha}\overline{1} +_8 S = {}^{\alpha}\overline{1} +_8 \{\overline{0}, \overline{4}\} = \overline{7^{\alpha}1} +_8 \{\overline{0}, \overline{4}\} = \{\overline{1}, \overline{5}\} \text{ or } \{\overline{3}, \overline{7}\};$$
$$^{\alpha}\overline{2} +_8 S = {}^{\alpha}\overline{2} +_8 \{\overline{0}, \overline{4}\} = \overline{7^{\alpha}2} +_8 \{\overline{0}, \overline{4}\} = \{\overline{2}, \overline{6}\};$$
$$^{\alpha}\overline{3} +_8 S = {}^{\alpha}\overline{3} +_8 \{\overline{0}, \overline{4}\} = \overline{7^{\alpha}3} +_8 \{\overline{0}, \overline{4}\} = \{\overline{1}, \overline{5}\} \text{ or } \{\overline{3}, \overline{7}\};$$
$$^{\alpha}\overline{4} +_8 S = {}^{\alpha}\overline{4} +_8 \{\overline{0}, \overline{4}\} = \overline{7^{\alpha}4} +_8 \{\overline{0}, \overline{4}\} = \{\overline{0}, \overline{4}\};$$
$$^{\alpha}\overline{5} +_8 S = {}^{\alpha}\overline{5} +_8 \{\overline{0}, \overline{4}\} = \overline{7^{\alpha}5} +_8 \{\overline{0}, \overline{4}\} = \{\overline{1}, \overline{5}\} \text{ or } \{\overline{3}, \overline{7}\};$$
$$^{\alpha}\overline{6} +_8 S = {}^{\alpha}\overline{6} +_8 \{\overline{0}, \overline{4}\} = \overline{7^{\alpha}6} +_8 \{\overline{0}, \overline{4}\} = \{\overline{2}, \overline{6}\};$$
$$^{\alpha}\overline{7} +_8 S = {}^{\alpha}\overline{7} +_8 \{\overline{0}, \overline{4}\} = \overline{7^{\alpha}7} +_8 \{\overline{0}, \overline{4}\} = \{\overline{1}, \overline{5}\} \text{ or } \{\overline{3}, \overline{7}\}.$$

Moreover, we have $\rho_S(\overline{0}) = \rho_S(\overline{4}) = \{\overline{0}, \overline{4}\}$, $\rho_S(\overline{2}) = \rho_S(\overline{6}) = \{\overline{2}, \overline{6}\}$, and $\rho_S(\overline{1}) = \rho_S(\overline{3}) = \rho_S(\overline{5}) = \rho_S(\overline{7}) = \{\overline{1}, \overline{3}, \overline{5}, \overline{7}\}$. Thus, the quotient $M/S = \{\rho_S(\overline{0}), \rho_S(\overline{1}), \rho_S(\overline{2})\}$ using the Definition 13. Observe that M/S yield is not equal to M/S above. Moreover, $\rho_S(\overline{0})$ is the same with $\rho_S(\overline{0})$ above, however, $\rho_S(\overline{1})$s are different. This implies that their equivalence classes are not equal. Hence, M/S via Γ-submonoid is different from M/S via submonoid, where M is a monoid.

Theorem 11. *If M is a commutative Γ-monoid and S a Γ-submonoid of M, then M/S is a Γ-monoid.*

Proof. Let M be a commutative Γ-monoid and S a Γ-submonoid of M. By Proposition 1, since ρ_S is a congruence on M, we have $M/\rho_S = M/S$ is a monoid with binary operation $\circ$ given by $\rho_S(x) \circ \rho_S(y) = \rho_S(x * y)$ with identity $\rho_S(1_M)$. Consider a mapping $\phi : \Gamma \times M/S \longrightarrow M/S$ given by $(\alpha, \rho_S(x)) \mapsto {}^{\alpha}\rho_S(x) = \rho_S({}^{\alpha}x)$ for all $\alpha \in \Gamma$ and $x \in M$. Let $(\alpha, \rho_S(x))$, $(\beta, \rho_S(y)) \in \Gamma \times M/S$ such that $(\alpha, \rho_S(x)) = (\beta, \rho_S(y))$. Then $\alpha = \beta$ and $\rho_S(x) = \rho_S(y)$. Thus, by Remark 14(ii), $({}^{\alpha'}x * S) \cap ({}^{\alpha'}y * S) \neq \varnothing$ for all $\alpha' \in \Gamma$, which implies that ${}^{\alpha'}x * s_1 = {}^{\alpha'}y * s_2$ for some $s_1, s_2 \in S$. Accordingly, ${}^{\alpha}({}^{\alpha'}x * s_1) = {}^{\alpha}({}^{\alpha'}x) * {}^{\alpha}s_1 = {}^{\alpha}({}^{\alpha'}y * s_2) = {}^{\alpha}({}^{\alpha'}y) * {}^{\alpha}s_2$. Since S is a Γ-submonoid, ${}^{\alpha}s_1, {}^{\alpha}s_2 \in S$ and $({}^{\alpha+\alpha'}x * S) \cap ({}^{\alpha+\alpha'}y * S) \neq \varnothing$. This means that

$$\phi(\alpha, \rho_S(x)) = {}^{\alpha}\rho_S(x) = \rho_S({}^{\alpha}x) = \rho_S({}^{\beta}y) = {}^{\beta}\rho_S(y) = \phi(\beta, \rho_S(y)).$$

Hence, ϕ is well-defined.

Now, for any $\alpha, \beta \in \Gamma$ and $x \in M$, $\phi((0, \rho_S(x))) = {}^{0}\rho_S(x) = \rho_S({}^{0}x) = \rho_S(x)$ and $\phi((\alpha + \beta, \rho_S(x))) = {}^{\alpha+\beta}\rho_S(x) = {}^{\alpha}({}^{\beta}\rho_S(x)) = \phi((\alpha, \phi((\beta, \rho_S(x)))))$. Thus, ϕ is an action.

Now, let $\alpha \in \Gamma$ and $x, y \in M$. Then

$$\begin{aligned} \phi((\alpha, \rho_S(x) \circ \rho_S(y))) &= {}^{\alpha}(\rho_S(x) \circ \rho_S(y)) \\ &= {}^{\alpha}(\rho_S(x * y)) \end{aligned}$$

$$\begin{aligned} &= \rho_S({}^{\alpha}(x * y)) \\ &= \rho_S({}^{\alpha}x * {}^{\alpha}y) \\ &= \rho_S({}^{\alpha}x) \circ \rho_S({}^{\alpha}y) \\ &= {}^{\alpha}\rho_S(x) \circ {}^{\alpha}\rho_S(y) \\ &= \phi((\alpha, \rho_S(x))) \circ \phi((\alpha, \rho_S(y))). \end{aligned}$$

Therefore, M/S is a Γ-monoid. □

Proposition 4. *Let S be a normal Γ-submonoid of a commutative Γ-monoid M. Then $\rho_S(h) = \rho_S(1_M)$ if and only if $h \in S$.*

Proof. Suppose $h \in S$. Let $x \in \rho_S(h)$. Then, for all $\alpha \in \Gamma$, $({}^{\alpha}x*S) \cap ({}^{\alpha}h*S) \neq \varnothing$. This implies that there exist $h_1, h_2 \in S$ such that ${}^{\alpha}x * h_1 = {}^{\alpha}h * h_2 \in S$. Since S is a normal Γ-submonoid and $h_1, {}^{\alpha}x * h_1 \in S$, it follows that ${}^{\alpha}x \in S$ for all $\alpha \in \Gamma$. Accordingly, for all $\alpha \in \Gamma$, ${}^{\alpha}x * {}^{\alpha}1_M = {}^{\alpha}1_M * {}^{\alpha}x$ implies $({}^{\alpha}x*S) \cap ({}^{\alpha}1_M*S) \neq \varnothing$. Hence, $x \rho_S 1_M$ and $x \in \rho_S(1_M)$. It follows that $\rho_S(h) \subseteq \rho_S(1_M)$. Let $x \in \rho_S(1_M)$. Then, $({}^{\alpha}x * S) \cap ({}^{\alpha}1_M * S) \neq \varnothing$ for all $\alpha \in \Gamma$. Thus, there exist $h_1, h_2 \in S$ such that for all $\alpha \in \Gamma$, ${}^{\alpha}x * h_1 = {}^{\alpha}1_M * h_2 \in S$. Since S is a normal Γ-submonoid and $h_1, {}^{\alpha}x * h_1 \in S$, it follows that ${}^{\alpha}x \in S$. Observe that for all $\alpha \in \Gamma$, ${}^{\alpha}h = {}^{\alpha}h * 1_M \in S$ since S is a Γ-submonoid. Accordingly, ${}^{\alpha}x * {}^{\alpha}h = {}^{\alpha}h * {}^{\alpha}x$ implies $({}^{\alpha}x * S) \cap ({}^{\alpha}h * S) \neq \varnothing$. Hence, $x \rho_S h$ and $x \in \rho_S(h)$. Consequently, $\rho_S(1_M) \subseteq \rho_S(h)$. Therefore, $\rho_S(1_M) = \rho_S(h)$.

Now, suppose $\rho_S(1_M) = \rho_S(h)$. Then, by Remark 14(ii), $({}^{\alpha}1_M * S) \cap ({}^{\alpha}h * S) \neq \varnothing$ for all $\alpha \in \Gamma$. Thus, there exist $h_1, h_2 \in S$ such that ${}^{\alpha}h * h_2 = {}^{\alpha}1_M * h_1 \in S$ for all $\alpha \in \Gamma$. Since S is a normal Γ-submonoid and $h_2, {}^{\alpha}h * h_2 \in S$, it follows that ${}^{\alpha}h \in S$ for all $\alpha \in \Gamma$. Therefore, $h \in S$. □

Proposition 5. *Let S be a normal Γ-submonoid of a commutative Γ-monoid M. Then $M = S$ if and only if $M/S = \{\rho_S(1_M)\}$.*

Proof. Suppose $M = S$. Let $x \in M/S = M/M$. Then $x = \rho_M(y)$ for some $y \in M$. By Proposition 4, we have $\rho_M(1_M) = \rho_M(y) = x$. Hence, $M/M = M/S = \{\rho_M(1_M)\}$. Conversely, suppose $M/S = \{\rho_S(1_M)\}$. Let $x \in M$. Then $\rho_S(x) \in M/S$. Thus, $\rho_S(x) = \rho_S(1_M)$. By Proposition 4, $x \in S$. Hence, $M \subseteq S$. Accordingly, $M = S$. □

Proposition 6. *Let S be a normal Γ-submonoid of a commutative Γ-monoid M. Every Γ-submonoid of M/S is of the form R/S, where R is a Γ-submonoid of M containing S.*

Proof. Let H be a Γ-submonoid of M/S. Then $H \subseteq M/S$. Let $R = \{m \in M : \rho_S(m) \in H\}$. We show that R is a Γ-submonoid of M. Note that the identity in M/S is $\rho_S(1_M) \in H$ and thus, $1_M \in R$. Now, let $x, y \in R$ and $\alpha, \beta \in \Gamma$. Then $\rho_S(x), \rho_S(y) \in H$ and ${}^{\alpha}\rho_S(x) * {}^{\beta}\rho_S(y) \in H$ since H is a Γ-submonoid. Accordingly, we have $\rho_S({}^{\alpha}x * {}^{\beta}y) = \rho_S({}^{\alpha}x) \circ \rho_S({}^{\beta}y) = {}^{\alpha}\rho_S(x) \circ {}^{\beta}\rho_S(y) \in H$. It follows that ${}^{\alpha}x * {}^{\beta}y \in R$. Accordingly, R is a Γ-submonoid of M. Now, we show that $S \subseteq R$. Let $x \in S$. Then by Proposition 4, we have $\rho_S(x) = \rho_S(1_M)$. Since $\rho_S(1_M)$ is the identity in M/S and H is a Γ-submonoid of M/S, we must have $\rho_S(x) = \rho_S(1_M) \in H$. Thus, $x \in R$. Therefore, $S \subseteq R$. □

Theorem 12. *Let M be a commutative Γ-monoid and S a normal Γ-submonoid of M. Then the mapping $\pi_S : M \longrightarrow M/S$ given by $\pi_S(x) = \rho_S(x)$ is a Γ-monoid epimorphism with kernel S.*

Proof. Let $x, y \in M$ such that $x = y$. Then, $\pi_S(x) = \rho_S(x) = \rho_S(y) = \pi(y)$. Thus, π_S is well-defined. Now, let $x, y \in M$. Then, we have
$\pi_S(x * y) = \rho_S(x * y) = \rho_S(x) \circ \rho_S(y) = \pi_S(x) \circ \pi_S(y)$ and $\pi_S(1_M) = \rho_S(1_M)$. Thus, by Definition 4, π_S is a monoid homomorphism. Since ${}^{\alpha}\pi_S(x) = {}^{\alpha}\rho_S(x) = \rho_S({}^{\alpha}x) = \pi_S({}^{\alpha}x)$, by Definition, π_S is a Γ-monoid homomorphism. Now, let $b \in M/S$. Then, $b = \rho_S(a)$ for some $a \in M$. Thus, $b = \rho_S(a) = \pi_S(a)$ and so, π is surjective. Therefore, π_S is an epimorphism. Now, since S is normal, by Proposition 4 we have

$$\ker \pi_S = \{m \in M : \rho_S(m) = \rho_S(1_M)\} = \{m \in M : m \in S\} = S \cap M = S$$

as desired. □

The map π_S in Theorem 12 is called the *canonical epimorphism.*

Proposition 7. *Let M be a Γ-monoid. Then for any $A \subseteq M$ and S a commutative Γ-submonoid of M, $\pi_S^{-1}(\pi_S(A)) = \bigcup_{x \in A} \rho_S(x)$.*

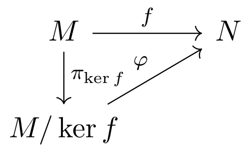
Proof. Suppose $y \in \pi_S^{-1}(\pi_S(A))$. Then $\rho_S(y) = \pi_S(y) \in \pi_S(A)$. Since π_S is an epimorphism, there exists an $x \in A$ such that $\pi_S(x) = \rho_S(y)$. Hence, $\rho_S(x) = \rho_S(y)$. By Remark 14(ii), $({}^{\alpha}x * S) \cap ({}^{\alpha}y * S) \neq \varnothing$ for all $\alpha \in \Gamma$, that is, $x\rho_S y$. This implies that $y \in \rho_S(x)$ for some $x \in A$. It follows that $y \in \bigcup_{x \in A} \rho_S(x)$ so that $\pi_S^{-1}(\pi_S(A)) \subseteq \bigcup_{x \in A} \rho_S(x)$. Conversely, suppose $y \in \bigcup_{x \in A} \rho_S(x)$. Then $y \in \rho_S(x)$ for some $x \in A$. This implies that $y\rho_S x$, that is, $({}^{\alpha}y * S) \cap ({}^{\alpha}x * S) \neq \varnothing$ for all $\alpha \in \Gamma$. By Remark 14(ii), $\rho_S(y) = \rho_S(x)$. Thus, $\pi_S(y) = \pi_S(x)$. Since $\pi_S(x) \in \pi_S(A)$, it follows that $\pi_S(y) \in \pi_S(A)$ implying that $y \in \pi_S^{-1}(\pi_S(A))$. Hence, $\bigcup_{x \in A} \rho_S(x) \subseteq \pi_S^{-1}(\pi_S(A))$. Therefore, $\pi_S^{-1}(\pi_S(A)) = \bigcup_{x \in A} \rho_S(x)$. □

6. Isomorphism Theorems

In [5], the isomorphism theorems for Γ-monoids via Γ-order-ideals are established. Here, we prove isomorphism theorems for Γ-monoids via Γ-submonoids.

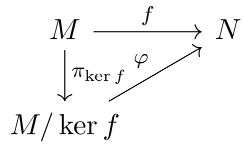
As shown already in Example 16, the quotient M/S in our discussion is not the same with the quotient discussed in [5].

Theorem 13. *Let $(M, *)$ and $(N, \cdot)$ be commutative Γ-monoids and let $f : M \to N$ be a Γ-monoid homomorphism. There exists a unique Γ-monoid homomorphism $\varphi : M/\ker f \to N$ such that the following diagram is commutative*



that is, $\varphi \circ \pi_{\ker f} = f$*, where* $\pi_{\ker f}(x) := \rho_{\ker f}(x)$*. Moreover,* φ *is onto and it has a trivial kernel, namely,* $\ker \varphi = \{\ker f\}$*. However,* φ *is a* Γ*-monoid isomorphism if and only if* $\rho_f = \rho_{\ker f}$*.*

Proof. Let $(M, *)$ and $(N, \cdot)$ be commutative Γ-monoids and let $f : M \to N$ be a Γ-monoid homomorphism. Since Γ-monoids are monoids and Γ-monoid homomorphism is a monoid homomorphism, by Theorem 1, there exists a unique monoid homomorphism $\varphi : M/\ker f \to N$ such that the following diagram is commutative



that is, $\varphi \circ \pi_{\ker f} = f$, where $\pi_{\ker f}(x) := \rho_{\ker f}(x)$. Moreover, φ is onto and it has a trivial kernel, namely, $\ker \varphi = \{\ker f\}$. However, φ is an isomorphism if and only if $\rho_f = \rho_{\ker f}$. Thus, it remains to show that φ is a Γ-monoid homomorphism. Now, let $\rho_{\ker f}(x) \in M/\ker f$ and $\alpha \in \Gamma$. Since f is a Γ-monoid homomorphism, we have

$$\varphi({}^{\alpha}\rho_{\ker f}(x)) = \varphi(\rho_{\ker f}({}^{\alpha}x)) = f({}^{\alpha}x) = {}^{\alpha}f(x) = {}^{\alpha}\varphi(\rho_{\ker f}(x)).$$

Hence, φ is a Γ-monoid homomorphism. □

Corollary 1. *Let* M *and* N *be commutative* Γ*-monoids and* $f : M \to N$ *be a* Γ*-monoid homomorphism. Then* f *induces a* Γ*-monoid isomorphism* $M/\ker f \cong Imf$*.*

Proof. Suppose $f : M \to N$ is a Γ-monoid homomorphism. Then, by Theorem 13, there exists a Γ-monoid homomorphism $\varphi : M/\ker f \to N$. If we set $N = \mathrm{Im} f$, then $\varphi : M/\ker f \to \mathrm{Im} f$ is a Γ-monoid epimorphism. Thus, $\ker \varphi = \{\rho_{\ker f}(x) : f(x) = 1_N\} = \{\ker f\}$ implies that $\rho_{\ker f}(x) = \ker f$ and $x \in \ker f$. Hence, by Proposition 4, $\rho_{\ker f}(x) = \rho_{\ker f}(1_M)$ which implies that $\ker \varphi = \{\rho_{\ker f}(1_M)\}$ and φ is injective. Accordingly, $M/\ker f \cong \mathrm{Im} f$. □

Corollary 2. *Let* K *and* L *be normal* Γ*-submonoids of a commutative* Γ*-monoid* M*. Then* $K/(K \cap L) \cong (K * L)/L$*.*

Proof. Consider the map $f : K \to K * L$ defined by $f(k) = k * 1_M$ and $\pi_L : K * L \to (K*L)/L$ defined by $\pi_L(k*l) = \rho_L(k*l)$. Then $\varphi : K \to (K*L)/L$ defined by $\varphi(k) = \rho_L(k)$ is a Γ-monoid homomorphism. Let $x \in (K * L)/L$. Then $x = \rho_L(k * l)$ for some $k \in K$ and $l \in L$. Observe that $x = \rho_L(k*l) = \rho_L(k) \circ \rho_L(l) = \rho_L(k) \circ \rho_L(1_M) = \rho_L(k)$. So, there is a $k \in K$ such that $\varphi(k) = \rho_L(k) = x$ and φ is onto. Moreover,

$$\ker \varphi = \{k \in K : \rho_L(k) = \rho_L(1_M)\} = \{k \in K : k \in L\} = K \cap L.$$

By Corollary 1, $K/\ker \varphi \cong \mathrm{Im}\varphi = (K * L)/L$. □

The following theorem is the counterpart to the third isomorphism theorem of groups for Γ-monoids via Γ-submonoids.

Theorem 14. *Let S and T be normal Γ-submonoids of a commutative Γ-monoid M with $S \subseteq T$. Then $(M/S)/(T/S) \cong M/T$.*

Proof. Define $f : M/S \to M/T$ by $f(\rho_S(h)) = \rho_T(h)$ for all $\rho_S(h) \in M/S$. Let $\rho_S(h_1), \rho_S(h_2) \in M/S$ and suppose that $\rho_S(h_1) = \rho_S(h_2)$. Then, $({}^{\alpha}h_1 * S) \cap ({}^{\alpha}h_2 * S) \neq \varnothing$ for all $\alpha \in \Gamma$. Thus, ${}^{\alpha}h_1 * w_1 = {}^{\alpha}h_2 * w_2$ for some $w_1, w_2 \in S \subseteq T$. Thus, $({}^{\alpha}h_1 * T) \cap ({}^{\alpha}h_2 * T) \neq \varnothing$ for all $\alpha \in \Gamma$. By Remark 14(ii), $\rho_T(h_1) = \rho_T(h_2)$. Thus, $f(\rho_S(h_1)) = f(\rho_S(h_2))$. Hence, f is well-defined.

Let $\rho_S(h_1), \rho_S(h_2) \in M/S$. Then

$$f(\rho_S(h_1) \circ \rho_S(h_2)) = f(\rho_S(h_1 * h_2)) = \rho_T(h_1) \circ \rho_T(h_2) = f(\rho_S(h_1)) \circ f(\rho_S(h_2)).$$

Hence, f is a homomorphism.

Let $\rho_S(h) \in \ker f$. Then $f(\rho_S(h)) = \rho_T(1_M)$, the identity in M/T. Thus, $\rho_T(h) = \rho_T(1_M)$. By Proposition 4, $h \in T$. Hence, $\rho_S(h) \in T/S$. Thus, $\ker f \subseteq T/S$. Let $\rho_S(h) \in T/S$. Then $h \in T$. By Proposition 4, $\rho_T(h) = \rho_T(1_M)$. Thus, $f(\rho_S(h)) = \rho_T(h) = \rho_T(1_M)$. Accordingly, $\rho_S(h) \in \ker f$. Hence, $T/S \subseteq \ker f$. So, $T/S = \ker f$.

For $\rho_S(x), \rho_S(y) \in M/S$ and $\alpha \in \Gamma$, recall that $\rho_S(x)\rho_f\rho_S(y)$ if and only if $f({}^{\alpha}\rho_S(x)) = f({}^{\alpha}\rho_S(y))$. We claim that $\rho_f = \rho_{\ker f}$.

Let $\rho_S(z) \in M/S$. We show that $\rho_f(\rho_S(z)) = \rho_{\ker f}(\rho_S(z))$.

Let $\rho_S(w) \in \rho_{\ker f}(\rho_S(z))$. Then $({}^{\alpha}\rho_S(z) \circ \ker f) \cap ({}^{\alpha}\rho_S(w) \circ \ker f) \neq \varnothing$. Thus, there exist $y_1, y_2 \in \ker f$ such that ${}^{\alpha}\rho_S(z) \circ y_1 = {}^{\alpha}\rho_S(w) \circ y_2$. Hence, $f({}^{\alpha}\rho_S(z)) = f({}^{\alpha}\rho_S(z)) \circ \rho_T(1_M) = f({}^{\alpha}\rho_S(z)) \circ f(y_1) = f({}^{\alpha}\rho_S(z) \circ y_1)$ and
$f({}^{\alpha}\rho_S(w)) = f({}^{\alpha}\rho_S(w)) \circ \rho_T(1_M) = f({}^{\alpha}\rho_S(w)) \circ f(y_2) = f({}^{\alpha}\rho_S(w) \circ y_2)$. So, by well-definedness of f, we have $f({}^{\alpha}\rho_S(z)) = f({}^{\alpha}\rho_S(z) \circ y_1) = f({}^{\alpha}\rho_S(w) \circ y_2) = f({}^{\alpha}\rho_S(w))$. Accordingly, $\rho_S(w) \in \rho_f(\rho_S(z))$. Thus, $\rho_{\ker f}(\rho_S(z)) \subseteq \rho_f(\rho_S(z))$.

Now, let $\rho_S(w) \in \rho_f(\rho_S(z))$ and $\alpha \in \Gamma$. Then $f({}^{\alpha}\rho_S(z)) = f({}^{\alpha}\rho_S(w))$, that is, ${}^{\alpha}\rho_T(z) = {}^{\alpha}\rho_T(w)$. Thus, $\rho_T({}^{\alpha}z) = \rho_T({}^{\alpha}w)$ implies $({}^{\alpha}w * T) \cap ({}^{\alpha}z * T) \neq \varnothing$. Thus, there exist $h_1, h_2 \in T$ such that ${}^{\alpha}w * h_1 = {}^{\alpha}z * h_2$. Hence, $\rho_S(h_1), \rho_S(h_2) \in T/S = \ker f$. Consequently, $\rho_S({}^{\alpha}w) \circ \rho_S(h_1) = \rho_S({}^{\alpha}w * h_1) = \rho_S({}^{\alpha}z * h_2) = \rho_S({}^{\alpha}z) \circ \rho_S(h_2)$ for all $\alpha \in \Gamma$. This implies that $({}^{\alpha}\rho_S(w) \circ \ker f) \cap ({}^{\alpha}\rho_S(z) \circ \ker f) \neq \varnothing$. Hence, $\rho_S(w) \in \rho_{\ker f}(\rho_S(z))$. Accordingly, $\rho_f(\rho_S(z)) \subseteq \rho_{\ker f}(\rho_S(z))$.

Therefore, $\rho_f(\rho_S(z)) = \rho_{\ker f}(\rho_S(z))$ for all $\rho_S(z) \in M/S$, that is, $\rho_f = \rho_{\ker f}$. By Theorem 13, these all imply that $(M/S)/(T/S) = (M/S)/\ker f \cong M/T$. □

7. Conclusion

: In this paper, we have shown that Γ-ideals and Γ-submonoids of a Γ-monoid M are not equivalent to the existing Γ-order-ideals of M. For any Γ-monoids M and N, we proved that the kernel of a Γ-monoid homomorphism $\varphi : M \to N$ is a Γ-submonoid of M. Also, for any Γ-submonoid S of a Γ-monoid M, ρ_S is a congruence relation if M is commutative and thus, $M/S = M/\rho_S$ is defined for commutative Γ-monoid M. Moreover, isomorphism theorems for Γ-monoids via Γ-submonoids were proved.

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References

[1] R. Hazrat and H. Li. The talented monoid of a Leavitt path algebra. *Journal of Algebra* 547, pages 430–455, 2020.

[2] T. W. Hungerford. *Algebra*. 1980.

[3] D. Gonçalvez L. G. Cordeiro and R. Hazrat. The talented monoid of a directed graph with applications to graph algebras. *Rev. Mat. Iberoam*, 38:223–256, 2022.

[4] Y. Give' on. *Normal monoids and factor monoids of commutative monoids*. 1963.

[5] A. Sebandal and J. Vilela. The Jordan-Hölder theorem for monoids with group action. *Journal of Algebra and Its Application*, 22(4):1–18, 2023.

[6] B. Steinberg. *Representation Theory of Finite Monoids*. 2016.