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The Homotopy Perturbation Method for Solving Nonlocal Initial-Boundary Value Problems for Parabolic and Hyperbolic Partial Differential Equations

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Abstract. To obtain approximate-exact solutions to nonlocal initial-boundary value problems (IBVPs) of linear and nonlinear parabolic and hyperbolic partial differential equations (PDEs) subject to initial and nonlocal boundary conditions of integral type, the homotopy perturbation method (HPM) is utilized in this study. The HPM is used to solve the specified nonlocal IBVPs, which are then transformed into local Dirichlet IBVPs. Some examples demonstrate how accurate and efficient the HPM.

2020 Mathematics Subject Classifications: 35-XX, 35K20, 35L04, 35L20

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1. Introduction

The transport equation, often known as the one-way wave equation, is an illustration of a first-order linear partial differential equation with constant coefficient:

$$v_\tau - k v_\xi = 0, \quad c \le \xi \le d, \quad \tau \ge 0,$$

in which k is a fixed number that specifies constant-speed motion. We establish $v(\tau,\xi)$ at time τ, which we set to 0, i.e. $v(0,\xi)$ is equal to a specific function $v_0(\xi)$ on $c \le \xi \le d$, and the boundary conditions (BCs) known as the nonlocal BCs of integral type which connect the solution of the differential equation to data of the integral type $\int_c^d v(\tau,\xi)\, d\xi = \gamma(\tau)$, in which $v(\tau,\xi)$ indicates the pollutants concentration in gr/cm (ratio of mass to length) at time τ_0 and $\int_c^d v(\tau,\xi)\, d\xi$ indicates the pollutants amount in the interval $[c,d]$ at time τ.

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In the case of beginning data and nonlocal BCs, a nonlocal IBVP is the problem of finding a solution to a PDE. The nonlocal IBVPs with integral BCs can be used to describe a wide range of problems in conduction of heat [14], engineering with chemicals [20], thermo-elasticity [17], and physics of plasma [3]. The parabolic PDE with nonlocal BCs has been studied in [4, 16, 21, 22, 26] and for hyperbolic PDEs [4, 23]. These topics were looked into, and proper existence and uniqueness theorems were established.

In the last three decades, semi-analytical approximation methods have emerged, such as HPM, homotopy analysis method (HAM), Adomian decomposition method (ADM), and variational iteration method (VIM), etc. to solve linear and nonlinear (algebraic, differential, partial differential, integral, etc.) equations. It has been shown that these methods yield a rapid convergence of the solutions series.

Ji-Huan He proposed the HPM in 1998. Many authors have relied on him to solve linear/non-linear ordinary differential equations (ODEs) and PDEs of integer and fractional order [7, 8, 11–13, 18, 19]. If the exact answer exists, the approach converges to it through repeated approximations.

Recently, Al-Hayani and Younis [2] have applied the HPM with green's function to solve the fuzzy system of boundary value problems. Ahmed et al. [1] have solved the nonlinear system of Volterra integral equations and applied the genetic algorithm to enhance the solutions by the HAM. Hamoud and Ghadle have used HAM for solving the first order fuzzy Volterra-Fredholm integro-differential equations [9] and fractional Volterra-Fredholm integro-differential equation of the second kind [10]. Fiza et al. [6] have applied the multistep optimal homotopy asymptotic method to some nonlinear KdV-equations. Younis and Al-Hayani [25] utilized the ADM to solve a fuzzy system of volterra integro-differential equations. Turkyilmazoglu [24] has proven the accelerating the convergence of ADM. Finally, Dawood et al. [5] have exercised VIM and MHPM to solve higher-order integro differential equations.

The principal goal of this study is to use the HPM to find approximate-exact solutions to solve nonlocal IBVPs for linear/non-linear parabolic and hyperbolic PDEs with initial and nonlocal BCs of integral type.

2. Applications of the HPM

The HPM will be used to solve nonlocal IBVPs for linear/non-linear variable-coefficient parabolic and hyperbolic PDEs in this section. There will be five instances shown.

2.1. Nonlocal IBVP for the linear/non-linear parabolic PDE

Let us consider the inhomogeneous linear/non-linear parabolic PDE

$$v_{\tau} - m(\tau,\xi)\, v_{\xi\xi} + n(\tau,\xi)\, v = h(\tau,\xi) + F(v), \quad c \le \xi \le d, \quad \tau \ge 0, \tag{1}$$

subjecting to the IC

$$v(0,\xi) = \alpha(\xi), \tag{2}$$

and the inhomogeneous nonlocal integral type BCs

$$\int_c^d \psi_1(\xi)\, v(\tau,\xi)\, d\xi = \gamma_1(\tau) \quad \text{and} \quad \int_c^d \psi_2(\xi)\, v(\tau,\xi)\, d\xi = \gamma_2(\tau), \tag{3}$$

in which $\psi_i(\xi)$, $\gamma_i(\tau)$, $i = 1, 2$ and $\alpha(\xi)$ are specified as continuous functions.

Converting Equations (1)-(3) into local IBVP by using the method of introducing a function $w(\tau,\xi)$ so that

$$w(\tau,\xi) = \int_c^\xi \psi(\xi)\, v(\tau,\xi)\, d\xi, \tag{4}$$

in which $\psi(\xi) = \psi_1(\xi) + \psi_2(\xi)$. Thus, we have

$$v(\tau,\xi) = \frac{1}{\psi(\xi)} w_\xi(\tau,\xi), \quad \psi(\xi) \neq 0, \tag{5}$$

$$v_\tau(\tau,\xi) = \frac{1}{\psi(\xi)} w_{\tau\xi}(\tau,\xi), \qquad v_{\tau\tau}(\tau,\xi) = \frac{1}{\psi(\xi)} w_{\tau\tau\xi}(\tau,\xi), \tag{6}$$

$$v_\xi(\tau,\xi) = \frac{1}{\psi(\xi)} w_{\xi\xi}(\tau,\xi) + \left(\frac{1}{\psi(\xi)}\right)' w_\xi(\tau,\xi), \tag{7}$$

$$\begin{aligned} v_{\xi\xi}(\tau,\xi) &= \frac{1}{\psi(\xi)} w_{\xi\xi\xi}(\tau,\xi) + 2\left(\frac{1}{\psi(\xi)}\right)' w_{\xi\xi}(\tau,\xi) \\ &\quad + \left(\frac{1}{\psi(\xi)}\right)'' w_\xi(\tau,\xi). \end{aligned} \tag{8}$$

Replacing Equations (5)-(8) into Equation (1) we conclude

Lemma 1. The nonlocal IBVP (1)-(3) may be reduced to a local IBVP of the form

$$\begin{cases} w_{\tau\xi} + r(\tau,\xi)\, w_\xi + s(\tau,\xi)\, w_{\xi\xi} - m(\tau,\xi)\, w_{\xi\xi\xi} = g(\tau,\xi) + N(w), \\ w_\xi(0,\xi) = h_1(\xi), \quad w(\tau,c) = 0, \quad w(\tau,d) = \gamma(\tau), \end{cases} \tag{9}$$

in which

$$r(\tau,\xi) = -m(\tau,\xi)\,\psi(\xi)\left(\frac{1}{\psi(\xi)}\right)'' + n(\tau,\xi), \tag{10}$$

$$s(\tau,\xi) = -2m(\tau,\xi)\,\psi(\xi)\left(\frac{1}{\psi(\xi)}\right)', \tag{11}$$

$$g(\tau,\xi) = \psi(\xi)\, h(\tau,\xi), \tag{12}$$

$$h_1(\xi) = \psi(\xi)\, \alpha(\xi), \tag{13}$$

$$\gamma(\tau) = \gamma_1(\tau) + \gamma_2(\tau). \tag{14}$$

and the non-linear term $N(w) = \psi(\xi) F\left(\dfrac{w_\xi}{\psi(\xi)}\right)$ is assumed to be an analytic function. This problem's solution will lead to the original problem's solution, in which $v(\tau, \xi)$ is provided by Eq (5). By the HPM [7, 8, 11–13, 18, 19], we write

$$\begin{aligned} w_{\tau\xi} - (v_0)_{\tau\xi} + p \Big[(v_0)_{\tau\xi} + r(\tau,\xi) w_\xi + s(\tau,\xi) w_{\xi\xi} \\ -m(\tau,\xi) w_{\xi\xi\xi} - g(\tau,\xi) - N(w) \Big] = 0, \end{aligned} \tag{15}$$

Define the solution $w(\tau, \xi)$ by an infinite series in the form

$$w(\tau,\xi) = \sum_{j=0}^{\infty} p^j w_j, \tag{16}$$

and the non-linear term $N(w)$ can be decomposed as

$$N(w(\tau,\xi)) = \sum_{j=0}^{\infty} p^j H_j(w), \tag{17}$$

in which the H_j are He's polynomials of $w_0, w_1, \ldots, w_j$ and are calculated by the definitional formula [8, 15]

$$H_j(w_0, w_1, \ldots, w_j) = \frac{1}{j!} \frac{\partial^j}{\partial p^j} \left[N\left(\sum_{i=0}^{\infty} p^i w_i \right) \right]_{p=0}, \quad j = 0, 1, \ldots . \tag{18}$$

in which $p \in [0,1]$ is an embedding parameter. Replacing (16) and (17) into Equation (15), we get

$$\begin{aligned} \sum_{j=0}^{\infty} p^j (w_j)_{\tau\xi} - (v_0)_{\tau\xi} + p \Bigg[(v_0)_{\tau\xi} + r(\tau,\xi) \sum_{j=0}^{\infty} p^j (w_j)_\xi + s(\tau,\xi) \sum_{j=0}^{\infty} p^j (w_j)_{\xi\xi} \\ -m(\tau,\xi) \sum_{j=0}^{\infty} p^j (w_j)_{\xi\xi\xi} - g(\tau,\xi) - \sum_{j=0}^{\infty} p^j H_j(w) \Bigg] = 0, \end{aligned} \tag{19}$$

and when we combine the terms in the same power of p, we get

$$\begin{aligned} p^0 &: \begin{cases} (w_0)_{\tau\xi} - (v_0)_{\tau\xi} = 0, \\ (w_0)_\xi (0,\xi) = h_1(\xi), \quad w_0(\tau, c) = 0, \quad w_0(\tau, d) = \gamma(\tau), \end{cases} \\ p^1 &: \begin{cases} (w_1)_{\tau\xi} + (v_0)_{\tau\xi} + r(\tau,\xi)(w_0)_\xi + s(\tau,\xi)(w_0)_{\xi\xi} - m(\tau,\xi)(w_0)_{\xi\xi\xi} \\ \qquad -g(\tau,\xi) - H_0(w) = 0, \\ (w_1)_\xi (0,\xi) = 0, \quad w_1(\tau,c) = 0, \quad w_1(\tau,d) = 0, \end{cases} \\ p^j &: \begin{cases} (w_j)_{\tau\xi} + r(\tau,\xi)(w_{j-1})_\xi + s(\tau,\xi)(w_{j-1})_{\xi\xi} - m(\tau,\xi)(w_{j-1})_{\xi\xi\xi} - H_{j-1}(w) = 0, \\ (w_j)_\xi (0,\xi) = 0, \quad w_j(\tau,c) = 0, \quad w_j(\tau,d) = 0, \quad j \geq 2 \end{cases} \end{aligned} \tag{20}$$

Solving the Equations (20) with choosing the initial approximation $v_0 = \alpha(\xi)$. Applying the inverse linear operators $L^{-1}_{c,\tau\xi}(\cdot) = \int_c^\xi \int_0^\tau (\cdot)\, d\tau d\xi$ to both sides of Equations (20), we obtain

$$\begin{aligned}
w_0(\tau,\xi) &= \int_c^\xi h_1(\xi)\, d\xi + L^{-1}_{c,\tau\xi}(v_0)_{\tau\xi}, \\
w_1(\tau,\xi) &= L^{-1}_{c,\tau\xi}\Big[m(\tau,\xi)(w_0)_{\xi\xi\xi} - s(\tau,\xi)(w_0)_{\xi\xi} - r(\tau,\xi)(w_0)_\xi \\
&\quad + g(\tau,\xi) + H_0(w)\Big], \\
w_j(\tau,\xi) &= L^{-1}_{c,\tau\xi}\Big[m(\tau,\xi)(w_{j-1})_{\xi\xi\xi} - s(\tau,\xi)(w_{j-1})_{\xi\xi} \\
&\quad - r(\tau,\xi)(w_{j-1})_\xi + H_{j-1}(w)\Big], \\
j &\geq 2
\end{aligned} \tag{21}$$

Applying the inverse linear operator $L^{-1}_{d,\tau\xi}(\cdot) = \int_\xi^d \int_0^\tau (\cdot)\, d\tau d\xi$ to both sides of Equations (20), as previously, we get

$$\begin{aligned}
w_0(\tau,\xi) &= \gamma(\tau) - \int_\xi^d h_1(\xi)\, d\xi + L^{-1}_{d,\tau\xi}(v_0)_{\tau\xi}, \\
w_1(\tau,\xi) &= -L^{-1}_{d,\tau\xi}\Big[m(\tau,\xi)(w_0)_{\xi\xi\xi} - s(\tau,\xi)(w_0)_{\xi\xi} - r(\tau,\xi)(w_0)_\xi \\
&\quad + g(\tau,\xi) + H_0(w)\Big], \\
w_j(\tau,\xi) &= -L^{-1}_{d,\tau\xi}\Big[m(\tau,\xi)(w_{j-1})_{\xi\xi\xi} - s(\tau,\xi)(w_{j-1})_{\xi\xi} \\
&\quad - r(\tau,\xi)(w_{j-1})_\xi + H_{j-1}(w)\Big], \\
j &\geq 2
\end{aligned} \tag{22}$$

By combining the relationships in (21) and (22) and dividing by 2, we arrive at the equal-weight average as the solution

$$\begin{aligned}
w_0(\tau,\xi) &= \frac{1}{2}\left[\int_c^\xi h_1(\xi)\, d\xi + \gamma(\tau) - \int_\xi^d h_1(\xi)\, d\xi\right] + \frac{1}{2}\left[L^{-1}_{c,\tau\xi}(v_0)_{\tau\xi} + L^{-1}_{d,\tau\xi}(v_0)_{\tau\xi}\right], \\
w_1(\tau,\xi) &= \frac{1}{2}L^{-1}_{c,\tau\xi}\left[m(\tau,\xi)(w_0)_{\xi\xi\xi} - s(\tau,\xi)(w_0)_{\xi\xi} - r(\tau,\xi)(w_0)_\xi + g(\tau,\xi) + H_0(w)\right] \\
&\quad - \frac{1}{2}L^{-1}_{d,\tau\xi}\left[m(\tau,\xi)(w_0)_{\xi\xi\xi} - s(\tau,\xi)(w_0)_{\xi\xi} - r(\tau,\xi)(w_0)_\xi + g(\tau,\xi) + H_0(w)\right], \\
w_j(\tau,\xi) &= \frac{1}{2}L^{-1}_{c,\tau\xi}\left[m(\tau,\xi)(w_{j-1})_{\xi\xi\xi} - s(\tau,\xi)(w_{j-1})_{\xi\xi} - r(\tau,\xi)(w_{j-1})_\xi + H_{j-1}(w)\right]
\end{aligned}$$

$$-\frac{1}{2}L_{d,\tau\xi}^{-1}\left[m(\tau,\xi)(w_{j-1})_{\xi\xi\xi}-s(\tau,\xi)(w_{j-1})_{\xi\xi}-r(\tau,\xi)(w_{j-1})_{\xi}+H_{j-1}(w)\right],$$
$$j \geq 2 \tag{23}$$

The best approximation for the solution is

$$w(\tau,\xi)=\lim_{p\to 1}\sum_{j=0}^{\infty}p^j w_j = w_0+w_1+w_2+w_3+\cdots .$$

we may use Equation (5) to return to the original dependent variable $v(\tau,\xi)$ once the function $w(\tau,\xi)$ has been determined.

2.2. Nonlocal IBVP for the linear/non-linear hyperbolic PDE

We consider the inhomogeneous linear/non-linear hyperbolic PDE

$$v_{\tau\tau}-m(\tau,\xi)v_{\xi\xi}+n(\tau,\xi)v=h(\tau,\xi)+F(v), \quad c\leq\xi\leq d, \quad \tau\geq 0, \tag{24}$$

subjecting to the ICs

$$v(0,\xi)=\alpha_1(\xi), \qquad v_\tau(0,\xi)=\alpha_2(\xi) \tag{25}$$

with the BCs (3). Replacing Equations (5)-(8) into Equation (24) we conclude

Lemma 2. The nonlocal IBVP (24) subjecting to (25) and (3) may be reduced to a local IBVP of the form

$$\begin{cases} w_{\tau\tau\xi}+r(\tau,\xi)w_\xi+s(\tau,\xi)w_{\xi\xi}-m(\tau,\xi)w_{\xi\xi\xi}=g(\tau,\xi)+N(w), \\ w_\xi(0,\xi)=h_2(\xi), \ w_{\tau\xi}(0,\xi)=h_3(\xi), \ w(\tau,c)=0, \ w(\tau,d)=\gamma(\tau), \end{cases} \tag{26}$$

in which $h_i(\xi)=\psi(\xi)\alpha_i(\xi), \ i=2,3.$

This problem's solution will lead to the original problem's solution, in which $v(\tau,\xi)$ is provided by Eq (5). By the HPM, we write

$$\begin{aligned} w_{\tau\tau\xi}-(v_0)_{\tau\tau\xi}+p\Big[&(v_0)_{\tau\tau\xi}+r(\tau,\xi)w_\xi+s(\tau,\xi)w_{\xi\xi} \\ &-m(\tau,\xi)w_{\xi\xi\xi}-g(\tau,\xi)-N(w)\Big]=0, \end{aligned} \tag{27}$$

Replacing (16) and (17) into Equation (27), we get

$$\begin{aligned} \sum_{j=0}^{\infty}p^j(w_j)_{\tau\tau\xi}-(v_0)_{\tau\tau\xi}+p\Bigg[&(v_0)_{\tau\tau\xi}+r(\tau,\xi)\sum_{j=0}^{\infty}p^j(w_j)_\xi+s(\tau,\xi)\sum_{j=0}^{\infty}p^j(w_j)_{\xi\xi} \\ &-m(\tau,\xi)\sum_{j=0}^{\infty}p^j(w_j)_{\xi\xi\xi}-g(\tau,\xi)-\sum_{j=0}^{\infty}p^jH_j(w)\Bigg]=0, \end{aligned} \tag{28}$$

and when we combine the terms in the same power of p, we get

$$p^0 \quad : \quad \begin{cases} (w_0)_{\tau\tau\xi} - (v_0)_{\tau\tau\xi} = 0, \\ (w_0)_\xi (0,\xi) = h_2(\xi), \ \ (w_0)_{\tau\xi}(0,\xi) = h_3(\xi), \ w_0(\tau,c) = 0, \ w_0(\tau,d) = \gamma(\tau), \end{cases}$$

$$p^1 \quad : \quad \begin{cases} (w_1)_{\tau\tau\xi} + (v_0)_{\xi\tau} + r(\tau,\xi)(w_0)_\xi + s(\tau,\xi)(w_0)_{\xi\xi} - m(\tau,\xi)(w_0)_{\xi\xi\xi} \\ \qquad -g(\tau,\xi) - H_0(w) = 0, \\ (w_1)_\xi(0,\xi) = 0, \ \ (w_1)_{\tau\xi}(0,\xi) = 0, \ w_1(\tau,c) = 0, \ w_1(\tau,d) = 0, \end{cases} \tag{29}$$

$$p^j \quad : \quad \begin{cases} (w_j)_{\tau\tau\xi} + r(\tau,\xi)(w_{j-1})_\xi + s(\tau,\xi)(w_{j-1})_{\xi\xi} - m(\tau,\xi)(w_{j-1})_{\xi\xi\xi} - H_{j-1}(w) = 0, \\ (w_j)_\xi(0,\xi) = 0, \ \ (w_j)_{\tau\xi}(0,\xi) = 0, \ w_j(\tau,c) = 0, \ w_j(\tau,d) = 0, \ \ j \geq 2 \end{cases}$$

Solving the Equations (29) with choosing the initial approximation $v_0 = \alpha_1(\xi) + \tau\alpha_2(\xi)$. Applying the inverse linear operators $L^{-1}_{c,\tau\tau\xi}(\cdot) = \int_c^\xi \int_0^\tau \int_0^\tau (\cdot)\, d\tau d\xi$ to both sides of Equations (29), we obtain

$$\begin{aligned} w_0(\tau,\xi) &= \int_c^\xi h_2(\xi)\, d\xi + \tau \int_c^\xi h_3(\xi)\, d\xi + L^{-1}_{c,\tau\tau\xi}(v_0)_{\tau\tau\xi}, \\ w_1(\tau,\xi) &= L^{-1}_{c,\tau\tau\xi}\Big[m(\tau,\xi)(w_0)_{\xi\xi\xi} - s(\tau,\xi)(w_0)_{\xi\xi} - r(\tau,\xi)(w_0)_\xi\Big] \\ &\quad + g(\tau,\xi) + H_0(w)], \\ w_j(\tau,\xi) &= L^{-1}_{c,\tau\tau\xi}\Big[m(\tau,\xi)(w_{j-1})_{\xi\xi\xi} - s(\tau,\xi)(w_{j-1})_{\xi\xi}\Big] \\ &\quad - r(\tau,\xi)(w_{j-1})_\xi + H_{j-1}(w)\Big], \\ j &\geq 2 \end{aligned} \tag{30}$$

Applying the inverse linear operator $L^{-1}_{d,\tau\tau\xi}(\cdot) = \int_\xi^d \int_0^\tau \int_0^\tau (\cdot)\, d\tau d\xi$ to both sides of Equations (29), as previously, we get

$$\begin{aligned} w_0(\tau,\xi) &= \gamma(\tau) - \int_\xi^d h_2(\xi)\, d\xi - \tau \int_\xi^d h_3(\xi)\, d\xi + L^{-1}_{d,\tau\tau\xi}(v_0)_{\tau\tau\xi}, \\ w_1(\tau,\xi) &= -L^{-1}_{d,\tau\tau\xi}\Big[m(\tau,\xi)(w_0)_{\xi\xi\xi} - s(\tau,\xi)(w_0)_{\xi\xi} - r(\tau,\xi)(w_0)_\xi \\ &\quad + g(\tau,\xi) + H_0(w)], \\ w_j(\tau,\xi) &= -L^{-1}_{d,\tau\tau\xi}\Big[m(\tau,\xi)(w_{j-1})_{\xi\xi\xi} - s(\tau,\xi)(w_{j-1})_{\xi\xi} \\ &\quad - r(\tau,\xi)(w_{j-1})_\xi + H_{j-1}(w)\Big], \end{aligned}$$

$$j \geq 2 \tag{31}$$

By combining the relationships in (30) and (31) and dividing by 2, we arrive at the equal-weight average as the solution

$$\begin{aligned}
w_0(\tau,\xi) &= \frac{1}{2}\left[\int_c^{\xi} h_2(\xi)\,d\xi + \tau\int_c^{\xi} h_3(\xi)\,d\xi + \gamma(\tau) - \int_{\xi}^{d} h_2(\xi)\,d\xi - \tau\int_{\xi}^{d} h_3(\xi)\,d\xi\right] \\
&\quad + \frac{1}{2}\left[L_{c,\tau\tau\xi}^{-1}(v_0)_{\tau\tau\xi} + L_{d,\tau\tau\xi}^{-1}(v_0)_{\tau\tau\xi}\right], \\
w_1(\tau,\xi) &= \frac{1}{2}L_{c,\tau\tau\xi}^{-1}\left[m(\tau,\xi)(w_0)_{\xi\xi\xi} - s(\tau,\xi)(w_0)_{\xi\xi} - r(\tau,\xi)(w_0)_{\xi} + g(\tau,\xi) + H_0(w)\right] \\
&\quad - \frac{1}{2}L_{d,\tau\tau\xi}^{-1}\left[m(\tau,\xi)(w_0)_{\xi\xi\xi} - s(\tau,\xi)(w_0)_{\xi\xi} - r(\tau,\xi)(w_0)_{\xi} + g(\tau,\xi) + H_0(w)\right], \\
w_j(\tau,\xi) &= \frac{1}{2}L_{c,\tau\tau\xi}^{-1}\left[m(\tau,\xi)(w_{j-1})_{\xi\xi\xi} - s(\tau,\xi)(w_{j-1})_{\xi\xi} - r(\tau,\xi)(w_{j-1})_{\xi} + H_{j-1}(w)\right] \\
&\quad - \frac{1}{2}L_{d,\tau\tau\xi}^{-1}\left[m(\tau,\xi)(w_{j-1})_{\xi\xi\xi} - s(\tau,\xi)(w_{j-1})_{\xi\xi} - r(\tau,\xi)(w_{j-1})_{\xi} + H_{j-1}(w)\right], \\
j &\geq 2
\end{aligned} \tag{32}$$

Similarly, once the function $w(\tau,\xi)$ has been established, we may utilize Equation (5) to go back to the initial dependant variable $v(\tau,\xi)$.

3. Problems

Problem 1. We first consider the linear nonlocal inhomogeneous IBVP [4]

$$\begin{cases}
v_\tau - v_{\xi\xi} + v = 0, \quad 0 \leq \xi \leq \pi, \quad \tau \geq 0, \\
v(0,\xi) = \sin(\xi), \\
\int_0^{\pi} \xi v(\tau,\xi)\,d\xi = \pi e^{-2\tau}, \\
\int_0^{\pi} (1-\xi)\,v(\tau,\xi)\,d\xi = (2-\pi)\,e^{-2\tau},
\end{cases} \tag{33}$$

in which $c = 0$, $d = \pi$, $m(\tau,\xi) = 1$, $n(\tau,\xi) = 1$, $h(\tau,\xi) = 0$, $\alpha(\xi) = \sin(\xi)$, $\gamma(\tau) = 2e^{-2\tau}$ and $\psi(\xi) = 1$. Replacing Equations (5)-(8) into Equation (33), we get a local inhomogeneous IBVP of the form

$$w_{\tau\xi} + w_\xi - w_{\xi\xi\xi} = 0, \quad w_\xi(0,\xi) = \sin(\xi), \quad w(\tau,0) = 0, \quad w(\tau,\pi) = 2e^{-2\tau}$$

in which $r(\tau,\xi) = 1$, $s(\tau,\xi) = 0$, $m(\tau,\xi) = 1$, $g(\tau,\xi) = 0$ and $h_1(\xi) = \sin(\xi)$. Following the algorithm (23), the iterations are

$$w_0(\tau,\xi) = \frac{1}{2}\left[\int_0^{\xi} h_1(\xi)\,d\xi + \gamma(\tau) - \int_{\xi}^{\pi} h_1(\xi)\,d\xi\right] = -\cos(\xi) + e^{-2\tau},$$

$$\begin{aligned} w_1(\tau,\xi) &= \frac{1}{2}L_{0,\tau\xi}^{-1}\left[(w_0)_{\xi\xi\xi}-(w_0)_{\xi}+g(\tau,\xi)\right]-\frac{1}{2}L_{\pi,\tau\xi}^{-1}\left[(w_0)_{\xi\xi\xi}-(w_0)_{\xi}+g(\tau,\xi)\right]=2\tau\cos(\xi), \\ w_j(\tau,\xi) &= \frac{1}{2}L_{0,\tau\xi}^{-1}\left[(w_{j-1})_{\xi\xi\xi}-(w_{j-1})_{\xi}\right]-\frac{1}{2}L_{\pi,\tau\xi}^{-1}\left[(w_{j-1})_{\xi\xi\xi}-(w_{j-1})_{\xi}\right] \\ &= \frac{(-1)^{j-1}}{j!}(2\tau)^j\cos(\xi), \quad j\geq 2. \end{aligned}$$

Thus, the series form's approximate solution is

$$w(\tau,\xi)=-\left(1-2\tau+2\tau^2-\frac{4}{3}\tau^3+\frac{2}{3}\tau^4-\frac{4}{15}\tau^5+\cdots\right)\cos(\xi)+e^{-2\tau}.$$

This series has been written in closed-form.

$$w(\tau,\xi)=-e^{-2\tau}\cos(\xi)+e^{-2\tau}.$$

Using Equation (5) to return to the original dependent variable, we get

$$v(\tau,\xi)=\frac{w_{\xi}(\tau,\xi)}{\psi(\xi)}=e^{-2\tau}\sin(\xi),$$

is the exact solution of the nonlocal IBVP (33) compatible with ADM.

Problem 2. Let us consider the linear nonlocal inhomogeneous IBVP [4]

$$\begin{cases} v_{\tau}-v_{\xi\xi}=\sin(\xi), \quad 0\leq\xi\leq\pi, \quad \tau\geq 0, \\ v(0,\xi)=\cos(\xi), \\ \int_0^{\pi}\xi v(\tau,\xi)\,d\xi=-(2+\pi)e^{-\tau}+\pi, \\ \int_0^{\pi}(k-\xi)v(\tau,\xi)\,d\xi=(2+\pi-2k)e^{-\tau}+2k-\pi, \end{cases} \tag{34}$$

in which $c=0$, $d=\pi$, $m(\tau,\xi)=1$, $n(\tau,\xi)=0$, $h(\tau,\xi)=\sin(\xi)$, $\alpha(\xi)=\cos(\xi)$, $\gamma(\tau)=2k(1-e^{-\tau})$ and $\psi(\xi)=k$, k constant. Replacing Equations (5)-(8) into Equation (34), we get a local inhomogeneous IBVP of the form

$$w_{\tau\xi}-w_{\xi\xi\xi}=k\sin(\xi), \quad w_{\xi}(0,\xi)=k\cos(\xi), \quad w(\tau,0)=0, \quad w(\tau,\pi)=2k(1-e^{-\tau})$$

in which $r(\tau,\xi)=0$, $s(\tau,\xi)=0$, $m(\tau,\xi)=1$, $g(\tau,\xi)=k\sin(\xi)$ and $h_1(\xi)=k\cos(\xi)$. Utilizing the algorithm (23), the iterations are

$$\begin{aligned} w_0(\tau,\xi) &= \frac{1}{2}\left[\int_0^{\xi}h_1(\xi)\,d\xi+\gamma(\tau)-\int_{\xi}^{\pi}h_1(\xi)\,d\xi\right]=k\sin(\xi)+k\left(1-e^{-\tau}\right), \\ w_1(\tau,\xi) &= \frac{1}{2}L_{0,\tau\xi}^{-1}\left[(w_0)_{\xi\xi\xi}+g(\tau,\xi)\right]-\frac{1}{2}L_{\pi,\tau\xi}^{-1}\left[(w_0)_{\xi\xi\xi}+g(\tau,\xi)\right]=-k\tau(\sin(\xi)+\cos(\xi)), \end{aligned}$$

$$w_j(\tau,\xi) = \frac{1}{2}L^{-1}_{0,\tau\xi}\left[(w_{j-1})_{\xi\xi\xi}\right] - \frac{1}{2}L^{-1}_{\pi,\tau\xi}\left[(w_{j-1})_{\xi\xi\xi}\right] = \frac{(-1)^j}{j!}k\tau^j(\sin(\xi)+\cos(\xi)), \quad j\geq 2.$$

Thus, the series form's approximate solution is

$$w(\tau,\xi) = k\left(1-\tau+\frac{\tau^2}{2!}-\frac{\tau^3}{3!}+\cdots\right)\sin(\xi) - k\left(\tau-\frac{\tau^2}{2!}+\frac{\tau^3}{3!}+\cdots\right)\cos(\xi) + k\left(1-e^{-\tau}\right).$$

This series has been written in closed-form

$$w(\tau,\xi) = ke^{-\tau}\sin(\xi) - k\left(1-e^{-\tau}\right)\cos(\xi) + k\left(1-e^{-\tau}\right).$$

Using Equation (5) to return to the original dependent variable, we get

$$v(\tau,\xi) = \frac{w_\xi(\tau,\xi)}{\psi(\xi)} = e^{-\tau}\cos(\xi) + \left(1-e^{-\tau}\right)\sin(\xi),$$

is the exact solution of the nonlocal IBVP (34) compatible with ADM.

Problem 3. We consider the linear nonlocal inhomogeneous IBVP [4]

$$\begin{cases} v_{\tau\tau} - v_{\xi\xi} = 0, \quad 0\leq\xi\leq 1, \quad \tau\geq 0, \\ v(0,\xi) = \xi^2, \quad v_\tau(0,\xi) = 0, \\ \int_0^1 v(\tau,\xi)\,d\xi = \frac{1}{3}+\tau^2, \\ \int_0^1 \xi v(\tau,\xi)\,d\xi = \frac{1}{4}+\frac{1}{2}\tau^2, \end{cases} \tag{35}$$

in which $c=0$, $d=1$, $m(\tau,\xi)=1$, $n(\tau,\xi)=0$, $h(\tau,\xi)=0$, $\alpha_1(\xi)=\xi^2$, $\alpha_2(\xi)=0$, $\gamma(\tau)=\frac{7}{12}+\frac{3}{2}\tau^2$ and $\psi(\xi)=\xi+1$. Replacing Equations (5)-(8) into Equation (35), we get a local inhomogeneous IBVP of the form

$$\begin{cases} w_{\tau\tau\xi} - \frac{2}{(\xi+1)^2}w_\xi + \frac{2}{\xi+1}w_{\xi\xi} - w_{\xi\xi\xi} = 0, \\ w_\xi(0,\xi) = \xi^3+\xi^2, \quad w_{\tau\xi}(0,\xi) = 0, \\ w(\tau,0) = 0, \quad w(\tau,1) = \frac{7}{12}+\frac{3}{2}\tau^2, \end{cases}$$

in which $r(\tau,\xi) = \frac{-2}{(\xi+1)^2}$, $s(\tau,\xi) = \frac{2}{\xi+1}$, $m(\tau,\xi)=1$, $g(\tau,\xi)=0$, $h_2(\xi)=\xi^3+\xi^2$ and $h_3(\xi)=0$. Using the algorithm (32), the iterations are

$$w_0(\tau,\xi) = \frac{1}{2}\left[\int_0^\xi h_2(\xi)\,d\xi + \tau\int_0^\xi h_3(\xi)\,d\xi + \gamma(\tau) - \int_\xi^1 h_2(\xi)\,d\xi - \tau\int_\xi^1 h_3(\xi)\,d\xi\right]$$

$$\begin{aligned}
&= \frac{1}{4}\xi^4 + \frac{1}{3}\xi^3 + \frac{3}{4}\tau^2, \\
w_1(\tau,\xi) &= \frac{1}{2}L^{-1}_{0,\tau\tau\xi}\left[(w_0)_{\xi\xi\xi} - \frac{2}{\xi+1}(w_0)_{\xi\xi} + \frac{2}{(\xi+1)^2}(w_0)_\xi + g(\tau,\xi)\right] \\
&\quad -\frac{1}{2}L^{-1}_{1,\tau\tau\xi}\left[(w_0)_{\xi\xi\xi} - \frac{2}{\xi+1}(w_0)_{\xi\xi} + \frac{2}{(\xi+1)^2}(w_0)_\xi + g(\tau,\xi)\right] \\
&= \frac{1}{2}\tau^2\xi^2 + \tau^2\xi - \frac{3}{4}\tau^2, \\
w_j(\tau,\xi) &= \frac{1}{2}L^{-1}_{0,\tau\tau\xi}\left[(w_{j-1})_{\xi\xi\xi} - \frac{2}{\xi+1}(w_{j-1})_{\xi\xi} + \frac{2}{(\xi+1)^2}(w_{j-1})_\xi\right] \\
&\quad -\frac{1}{2}L^{-1}_{1,\tau\tau\xi}\left[(w_{j-1})_{\xi\xi\xi} - \frac{2}{\xi+1}(w_{j-1})_{\xi\xi} + \frac{2}{(\xi+1)^2}(w_{j-1})_\xi\right] = 0, \quad j \geq 2.
\end{aligned}$$

Thus, the series form's approximate solution is

$$w(\tau,\xi) = \frac{1}{4}\xi^4 + \frac{1}{3}\xi^3 + \frac{1}{2}\tau^2\xi^2 + \tau^2\xi,$$

Using Equation (5) to return to the original dependent variable, we get

$$v(\tau,\xi) = \frac{w_\xi(\tau,\xi)}{\psi(\xi)} = \xi^2 + \tau^2,$$

is the exact solution of the nonlocal IBVP (35) compatible with ADM.

Problem 4. Consider the non-linear nonlocal inhomogeneous IBVP [4]

$$\begin{cases} v_{\tau\tau} - \xi v_{\xi\xi} = 1 - v^2, \quad 0 \leq \xi \leq 1, \quad \tau \geq 0, \\ v(0,\xi) = 1, \quad v_\tau(0,\xi) = 0, \\ \int_0^1 v(\tau,\xi)\,d\xi = 1, \quad \int_0^1 (\xi - 1)\,v(\tau,\xi)\,d\xi = -\frac{1}{2}, \end{cases} \tag{36}$$

in which $c = 0$, $d = 1$, $m(\tau,\xi) = \xi$, $n(\tau,\xi) = 0$, $h(\tau,\xi) = 1$, $F(v) = -v^2$, $\alpha_1(\xi) = 1$, $\alpha_2(\xi) = 0$, $\gamma(\tau) = \frac{1}{2}$ and $\psi(\xi) = \xi$. Replacing Equations (5)-(8) into Equation (36), we get a local inhomogeneous IBVP of the form

$$\begin{cases} w_{\tau\tau\xi} - \frac{2}{\xi}w_\xi + 2w_{\xi\xi} - \xi w_{\xi\xi\xi} = \xi - \frac{1}{\xi}(w_\xi)^2, \\ w_\xi(0,\xi) = \xi, \quad w_{\tau\xi}(0,\xi) = 0, \\ w(\tau,0) = 0, \quad w(\tau,1) = \frac{1}{2}, \end{cases}$$

in which $r(\tau,\xi) = \dfrac{-2}{\xi}$, $s(\tau,\xi) = 2$, $m(\tau,\xi) = \xi$, $g(\tau,\xi) = \xi$, $h_2(\xi) = \xi$, $h_3(\xi) = 0$ and the non-linear term $N(w) = -\dfrac{1}{\xi}(w_\xi)^2$ is given by Equation (17). The corresponding He's polynomials by the formula Equation (18) are given by

$$H_j(w) = -\frac{1}{\xi}\sum_{i=0}^{j}(w_\xi)_{j-i}(w_\xi)_i, \quad j \geq i, \quad j = 0, 1, \ldots \ .$$

due to the fact that the nonlinear component $N(w)$ exhibits quadratic nonlinearity in w_ξ. It should be noted that only the dependent variable w and its derivatives are parametrized in p, whereas τ and ξ are not. Following the algorithm (32), the iterations are

$$\begin{aligned}
w_0(\tau,\xi) &= \frac{1}{2}\left[\int_0^\xi h_2(\xi)\,d\xi + \tau\int_0^\xi h_3(\xi)\,d\xi + \gamma(\tau) - \int_\xi^1 h_2(\xi)\,d\xi - \tau\int_\xi^1 h_3(\xi)\,d\xi\right] = \frac{1}{2}\xi^2, \\
w_1(\tau,\xi) &= \frac{1}{2}L^{-1}_{0,\tau\tau\xi}\left[\xi(w_0)_{\xi\xi\xi} - 2(w_0)_{\xi\xi} + \frac{2}{\xi}(w_0)_\xi + g(\tau,\xi) - H_0(w)\right] \\
&\quad -\frac{1}{2}L^{-1}_{1,\tau\tau\xi}\left[\xi(w_0)_{\xi\xi\xi} - 2(w_0)_{\xi\xi} + \frac{2}{\xi}(w_0)_\xi + g(\tau,\xi) - H_0(w)\right] = 0, \\
w_j(\tau,\xi) &= \frac{1}{2}L^{-1}_{0,\tau\tau\xi}\left[\xi(w_{j-1})_{\xi\xi\xi} - 2(w_{j-1})_{\xi\xi} + \frac{2}{\xi}(w_{j-1})_\xi - H_{j-1}(w)\right] \\
&\quad -\frac{1}{2}L^{-1}_{1,\tau\tau\xi}\left[\xi(w_{j-1})_{\xi\xi\xi} - 2(w_{j-1})_{\xi\xi} + \frac{2}{\xi}(w_{j-1})_\xi - H_{j-1}(w)\right] = 0, \quad j \geq 2.
\end{aligned}$$

Thus, the series form's approximate solution is

$$w(\tau,\xi) = \frac{1}{2}\xi^2,$$

Using Equation (5) to return to the original dependent variable, we get

$$v(\tau,\xi) = \frac{w_\xi(\tau,\xi)}{\psi(\xi)} = 1,$$

is the exact solution of the nonlocal IBVP (36) compatible with ADM.

Problem 5. Finally, we consider the non-linear nonlocal inhomogeneous IBVP [4]

$$\begin{cases}
v_\tau - \xi v_{\xi\xi} = -v v_\xi, \quad 0 \leq \xi \leq 1, \quad \tau \geq 0, \\
v(0,\xi) = \xi, \\
\int_0^1 v(\tau,\xi)\,d\xi = \dfrac{1}{2(1+\tau)}, \\
\int_0^1 \left(e^\xi - 1\right) v(\tau,\xi)\,d\xi = \dfrac{1}{2(1+\tau)},
\end{cases} \tag{37}$$

in which $c = 0$, $d = \pi$, $m(\tau,\xi) = \xi$, $n(\tau,\xi) = 0$, $h(\tau,\xi) = 0$, $F(v) = -vv_\xi$, $\alpha(\xi) = \xi$, $\gamma(\tau) = \dfrac{1}{1+\tau}$ and $\psi(\xi) = e^\xi$. Replacing Equations (5)-(8) into Equation (37), we get a local inhomogeneous IBVP of the form

$$\begin{cases} w_{\tau\xi} - \xi w_\xi + 2\xi w_{\xi\xi} - \xi w_{\xi\xi\xi} = e^{-\xi}\left[(w_\xi)^2 - w_\xi w_{\xi\xi}\right], \\ w_\xi(0,\xi) = \xi e^\xi, \quad w(\tau,0) = 0, \quad w(\tau,1) = \dfrac{1}{1+\tau} \end{cases}$$

in which $r(\tau,\xi) = -\xi$, $s(\tau,\xi) = 2\xi$, $m(\tau,\xi) = \xi$, $g(\tau,\xi) = 0$, $h_1(\xi) = \xi e^\xi$ and the non-linear term $N(w) = e^{-\xi}\left[(w_\xi)^2 - w_\xi w_{\xi\xi}\right]$ is given by Equation (17). The corresponding He's polynomials by the formula Equation (18) are given by

$$H_j(w) = e^{-\xi}\left[\sum_{i=0}^{j}(w_\xi)_{j-i}(w_\xi)_i - \sum_{i=0}^{j}(w_\xi)_{j-i}(w_{\xi\xi})_i\right], \quad j \geq i, \quad j = 0,1,\ldots .$$

because the non-linear term $N(w)$ is the difference between a quadratic nonlinearity in w_ξ and a product nonlinearity in w_ξ and $w_{\xi\xi}$. Utilizing the algorithm (23), the iterations are

$$\begin{aligned} w_0(\tau,\xi) &= \frac{1}{2}\left[\int_0^\xi h_1(\xi)\,d\xi + \gamma(\tau) - \int_\xi^1 h_1(\xi)\,d\xi\right] = \frac{1}{2} + e^\xi(\xi-1) + \frac{1}{2(1+\tau)}, \\ w_1(\tau,\xi) &= \frac{1}{2}L_{0,\tau\xi}^{-1}\left[\xi(w_0)_{\xi\xi\xi} - 2\xi(w_0)_{\xi\xi} + \xi(w_0)_\xi + g(\tau,\xi) + H_0(w)\right] \\ &\quad -\frac{1}{2}L_{1,\tau\xi}^{-1}\left[\xi(w_0)_{\xi\xi\xi} - 2\xi(w_0)_{\xi\xi} + \xi(w_0)_\xi + g(\tau,\xi) + H_0(w)\right] \\ &= -\frac{1}{2}\tau - \tau e^\xi(\xi-1), \\ w_j(\tau,\xi) &= \frac{1}{2}L_{0,\tau\xi}^{-1}\left[\xi(w_{j-1})_{\xi\xi\xi} - 2\xi(w_{j-1})_{\xi\xi} + \xi(w_{j-1})_\xi + H_{j-1}(w)\right] \\ &\quad -\frac{1}{2}L_{1,\tau\xi}^{-1}\left[\xi(w_{j-1})_{\xi\xi\xi} - 2\xi(w_{j-1})_{\xi\xi} + \xi(w_{j-1})_\xi + H_{j-1}(w)\right] \\ &= (-1)^j\left[\frac{1}{2}\tau^j + \tau^j e^\xi(\xi-1)\right], \quad j \geq 2. \end{aligned}$$

Thus, the series form's approximate solution is

$$w(\tau,\xi) = \frac{1}{2}\left(1 - \tau + \tau^2 - \tau^3 + \cdots\right) + \left(1 - \tau + \tau^2 - \tau^3 + \cdots\right)e^\xi(\xi-1) + \frac{1}{2(1+\tau)}.$$

This series has been written in closed-form

$$w(\tau,\xi) = \frac{1}{1+\tau}e^\xi(\xi-1) + \frac{1}{1+\tau}, \quad |\tau| < 1.$$

Using Equation (5) to return to the original dependent variable, we get

$$v(\tau,\xi) = \frac{w_\xi(\tau,\xi)}{\psi(\xi)} = \frac{\xi}{1+\tau}, \quad |\tau| < 1$$

is the exact solution of the nonlocal IBVP (37) compatible with ADM.

4. Conclusion

To obtain approximate-exact solutions, the HPM was effectively employed to resolve nonlocal IBVPs for linear/non-linear parabolic and hyperbolic PDEs subjecting to initial and nonlocal BCs of integral type. The presented nonlocal integral IBVPs for linear/non-linear parabolic and hyperbolic PDEs have been turned into local Dirichlet IBVPs. The HPM has proven to be useful in dealing with these models, broadening its applicability. The method was put to the test by using it on five different Problems. The results obtained in each Problem show that this strategy is reliable and efficient for handling this type of nonlocal IBVPs.

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