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Spectral Analysis of Splitting Signed Graph

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Abstract. An ordered pair $\Sigma = (\Sigma^u,\sigma)$ is called the *signed graph*, where $\Sigma^u = (V, E)$ is a *underlying graph* and σ is a signed mapping, called *signature*, from E to the sign set $\{+, -\}$. The *splitting signed graph* $\Gamma(\Sigma)$ of a signed graph Σ is defined as, for every vertex $u \in V(\Sigma)$, take a new vertex u'. Join u' to all the vertices of Σ adjacent to u such that $\sigma_\Gamma(u'v) = \sigma(u'v),\ u \in N(v)$. The objective of this paper is to propose an algorithm for the generation of a splitting signed graph, a splitting root signed graph from a given signed graph using Matlab. Additionally, we conduct a spectral analysis of the resulting graph. Spectral analysis is performed on the adjacency and laplacian matrices of the splitting signed graph to study its eigenvalues and eigenvectors. A relationship between the energy of the original signed graph Σ and the energy of the splitting signed graph $\Gamma(\Sigma)$ is established.

2020 Mathematics Subject Classifications: 05C22, 05C50, 05C90

Key Words and Phrases: Signed graph, splitting signed graph, spectrum, energy

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1. Introduction

The initial notation and terminology used in this paper have been sourced from Harary [10], Zaslavsky [20] and West [19]. The graphs examined in this paper are finite and simple. A *signed graph*, $\Sigma = (\Sigma^u, \sigma)$, is composed of an *underlying graph*, $\Sigma^u = (V, E)$, where $|V| = n$ & $|E| = m$, and a *signature*, $\sigma : E \to \{+, -\}$, which labels each edge of Σ^u as either '+' or '−'. In this paper, edges labeled with '+' are considered positive and are depicted using solid lines, while edges labeled with '−' are considered negative and are depicted using dashed lines. If all edges in Σ are signed '+' or '−', the signed graph is referred to as *homogeneous*, otherwise, it is *heterogeneous*. Graphs can be thought of as homogeneous signed graphs with each edge being labeled as '+'. A cycle in a signed graph Σ is considered *positive* if it includes an even number of negative edges. If every cycle in Σ is positive, then Σ is defined as *balanced signed graph*.
An ordered pair (Σ, μ) is known as a *marked signed graph* where $\Sigma = (\Sigma^u, \sigma)$ is a signed graph, and $\mu : V(\Sigma^u) \to \{+, -\}$ is a function defined on the vertex set $V(\Sigma^u)$ of Σ^u. The function μ assigns each vertex of Σ^u to either the positive or negative sign from the set

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$\{+,-\}$, and is called the *marking of* Σ.
The adjacency matrix of Σ, whose vertices are v_1, v_2, ..., v_n is the $n \times n$ matrix $A(\Sigma) = [a_{i,j}]$ where

$$a_{i,j} = \begin{cases} 0 & \text{if } v_i \text{ and } v_j \text{ are not adjacent} \\ 1 & \text{if } \sigma(v_i, v_j) \text{ is positive} \\ -1 & \text{if } \sigma(v_i, v_j) \text{ is negative} \end{cases} \tag{1}$$

The *spectrum* of a matrix is a list of its eigenvalues along with their multiplicities. Since, $A(\Sigma)$ is a symmetric matrix with real entries so all its eigenvalues are real. Let $\lambda_1(\Sigma) > \lambda_2(\Sigma) > ... > \lambda_k(\Sigma)$ are distinct eigenvalues of $A(\Sigma)$ along with their multiplicities $m_1, m_2, ..., m_k$, $1 \leq k \leq n$, then the list of eigenvalues of adjacency matrix is called *adjacency spectrum* of the signed graph Σ and usually denoted as:

$$Sp(\Sigma) = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ m_1 & m_2 & \cdots & m_k \end{pmatrix}$$

Let $D(\Sigma) = [d_{i,j}]$ be a diagonal matrix of order n such that the entry (i, j) is $deg(u_i)$ if $i = j$ and 0 otherwise, where $deg(u_i)$ denotes the degree of the vertex u_i. $D(\Sigma)$ is called *degree matrix* of the signed graph Σ. The *Laplacian matrix* $L(\Sigma) = [l_{i,j}]$ of a signed graph Σ is a square matrix of order n such that $l_{i,j} = d_{i,j} - a_{i,j}$ for $0 \leq i, j \leq n$. The eigenvalues of the laplacian matrix is called *laplacian spectrum* and is denoted by $SLp(\Sigma)$. Two signed graphs are said to be *co-spectral* if they have same spectrum. The largest eigenvalue $\lambda_1(\Sigma)$ is called the *index* of Σ, whereas the largest absolute eigenvalue is called *spectral radius* $\rho(\Sigma)$, i.e.

$$\rho = max\{\lambda_1(\Sigma), -\lambda_k(\Sigma)\}. \tag{2}$$

The study of graph spectrum is of significant importance in the field of graph theory, and spectral graph-theoretic techniques have been applied in a range of fields including quantum physics, chemistry, computer science, and more. In recent years, researchers have explored the spectral properties of graphs constructed through graph operations such as disjoint union, Cartesian product, Kronecker product, strong product, lexicographic product, corona, edge corona, and neighbourhood corona. A comprehensive overview of results on the spectra of these graphs can be found in the literature [4–7, 9, 14–16]. In [**?**] authors presented the idea of the splitting graph $\Gamma(\Sigma^u)$ for a given graph Σ^u. The process of creating the splitting graph $\Gamma(\Sigma^u)$ involves taking a new vertex v' for each vertex v in graph Σ^u. The new vertex v' is then connected to all vertices in Σ^u that are adjacent to v. The resulting graph is referred to as the *splitting graph* $\Gamma(\Sigma^u)$ of graph Σ^u. Recently, a variation of this concept has been applied in the analysis of online social networks (OSNs), where v' is also connected to v. This variation of the concept is referred to as the "*clone*" of v. For the purpose of convenience, the term "*clone*" is adopted for v' in the splitting graph $\Gamma(\Sigma^u)$ as well.

Gutman [11] introduced the concept of *energy* of a graph Σ^u in 1978 as the sum of the absolute values of its eigenvalues, denoted by $E(\Sigma^u)$, *i.e.*,

$$E(\Sigma^u) = \sum_{i=1}^{n} |\lambda_i|$$

Later, in 2004, Bapat et.al [3] proved that the energy of a graph can only be an even integer if it is a rational number. Pirzada et.al [12], on the other hand, demonstrated that the energy of a given graph can never be the square root of an odd integer. Graph energy is briefly discussed in [2], while in [18] authors established a relationship between the energy of a graph and its splitting graph. The concept of graph energy has been widely studied in graph theory and has significant applications in various fields. The study of graph energy can provide insights into the structural properties of the graph and is often used in the design and analysis of communication networks, molecular chemistry, and social networks. Additionally, the energy of a graph is closely related to its spectrum and can be used to investigate various graph invariants, such as chromatic number, clique number, and independence number [5, 7, 8, 12, 16].

Sinha et.al [13] introduced the splitting signed graphs as an extension of the splitting graph concept. The splitting signed graph of a signed graph $\Sigma = (V, E, \sigma)$, denoted as $\Gamma(\Sigma) = (V_\Gamma, E_\Gamma, \sigma_\Gamma)$, is obtained by creating a new vertex v' for each vertex $v \in V(\Sigma)$, and connecting v' to all vertices in Σ adjacent to v such that the sign of the corresponding edges is preserved, *i.e.*, $\sigma_\Gamma(v'u) = \sigma(vu)$ for all $u \in N(v)$. This construction is depicted in Figure 1. A signed graph Σ is called a splitting signed graph if it is isomorphic to the splitting signed graph $\Gamma(U)$ of some signed graph U, where U is referred to as the splitting root signed graph of Σ.

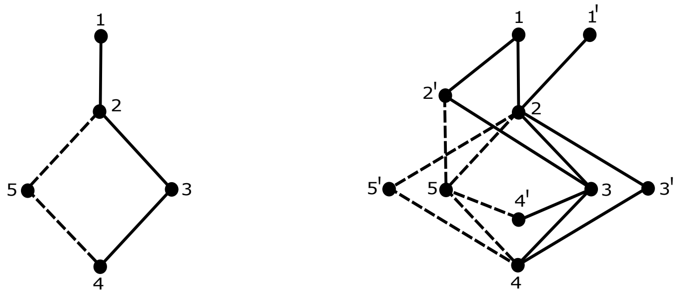

Figure 1: Signed graph Σ and its splitting signed graph $\Gamma(\Sigma)$

Algorithmic characterization of splitting signed graph by Sinha et.al appears in the proceedings of the International Conference on Current Trends in Graph Theory and Computation in [17]. Here, in this research paper, we give an algorithm to generate the splitting signed graph and its variant, the splitting root signed graph, from a given signed graph by using Matlab. We also perform spectral analysis on the adjacency and Laplacian matrices of the splitting signed graph to investigate its eigenvalues and eigenvectors. Furthermore, we establish a relationship between the energy of the original signed graph and the energy of the splitting signed graph. Our study sheds light on the properties of the splitting signed graph, and its potential applications in various domains.

The *Kronecker product (or tensor product)* of matrices U and W is a matrix defined as follows:

$$U \otimes W = \begin{bmatrix} a_{11}W & a_{12}W & \cdots & a_{1,n}W \\ a_{21}W & a_{22}W & \cdots & a_{2,n}W \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}W & a_{m2}W & \cdots & a_{m,n}W \end{bmatrix}$$

where, $U \in \mathbb{R}^{m\times n}$ and $W \in \mathbb{R}^{p\times q}$.

Theorem 1. *[8] Let U and W are two square matrices such that $U \in \mathbb{M}^m$ and $W \in \mathbb{M}^n$. If μ_i is an eigenvalue of U with its corresponding eigenvector y_i, and λ_j is an eigenvalue of W with its corresponding eigenvector x_i, then $\mu_i\lambda_j$ is an eigenvalue of the Kronecker product $U \otimes W$, with the corresponding eigenvector $y_i \otimes x_j$.*

2. Generating splitting signed graph

The procedure for generating a splitting signed graph can be described as follows: Given a graph with n vertices, the first step is to encode an $n \times n$ symmetric adjacency matrix for the graph. Since a new vertex is created for each vertex in the original graph, the splitting graph will have a total of $2n$ vertices. The non-zero entries in the first row of the adjacency matrix indicate the vertices which are adjacent to the first vertex *i.e.* v_1. These entries in first row are also considered adjacent to vertex v_{n+1} and are updated in the output matrix. This process is repeated for each row until all rows have been processed. As a result, a $2n \times 2n$ output matrix is generated. The adjacency matrices of the original signed graph Σ and its splitting signed graph $\Gamma(\Sigma)$ can be represented as follows:

$$A(\Sigma) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & -1 & 0 \end{bmatrix}$$

and

$$A(\Gamma(\Sigma)) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & -1 & 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

It has been observed that the adjacency matrix, of order $2n$, of the splitting signed graph $\Gamma(\Sigma)$ can be partitioned into four equal matrices each of order n, where the initial three matrices are identical and the fourth is always zero. Additionally, it has been discovered that the original signed graph Σ is an induced subgraph of its splitting signed graph $\Gamma(\Sigma)$.

Algorithm 1 Algorithm to derive the splitting signed graph $\Gamma(\Sigma)$ of a signed graph Σ

```
1:  Input: number of vertices (N)
2:  Input: adjacency matrix of the signed graph A(Σ) = a(i, j)
3:  for i = 1 : 2N do
4:     for j = 1 : 2N do
5:        check
6:        if i > N&&j > N then
7:           Assign a(i,j)=0;
8:        else
9:           check
10:          if i > N&&j ≤ N then
11:             Assign a(i,j) = a(i-N,j);
12:          else
13:             check
14:             if i ≤ N&&j > N then
15:                Assign a(i,j) = a(i,j-N);
16:             else
17:                Assign a(i,j) = a(i,j);
18:             end if
19:          end if
20:       end if
21:    end for
22: end for
23: Output Generate 2N × 2N matrix i.e. adjacency matrix of signed split graph
```

Computational complexity:

Computational complexity analysis is an essential aspect of evaluating the performance of an algorithm. In this regard, we analyze the complexity involved in Steps 3 and 4 of our algorithm. In these steps, we traverse each vertex of the signed graph and examine its adjacency with all other vertices. As a result, the complexity involved in these steps is $O(n^2)$.

Considering the overall algorithm, the total complexity involved is the sum of the complexities of all the steps. As the complexity in Steps 3 and 4 is the highest, the total complexity is also $O(n^2)$.

Therefore, the complexity of the proposed algorithm for finding a signed split graph with a given adjacency matrix is $O(n^2)$, where n represents the number of vertices in the signed graph.

3. Structural characterization to derive splitting root signed graph

Sampathkumar and Walikar [**?**] presented the following characterization of splitting graphs, which is utilized to derive the splitting root graph.

Theorem 2. *[**?**] A graph Σ^u can be characterized as a splitting graph if and only if its vertex set, $V(\Sigma^u)$, can be divided into two sets, V_1 and V_2, such that:*

(a) there exists a bijective mapping between V_1 and V_2, with v mapping to v', and

(b) the neighbours of v', $N(v')$, are equal to the intersection of the neighbours of v, $N(v)$, and the set V_1.

In [17] authors presented a structural characterization of a splitting signed graph that can be used to derive the splitting root signed graph.

Theorem 3. *[17] Let Σ be a connected signed graph, then Σ is splitting signed graph if and only if the following two conditions hold:*

(a) The underlying graph Σ^u is splitting graph

(b) the vertex set of Σ can be divided into two sets, V_1 and V_2, such that for each $u' \in V_2$ there exist a vertex $u \in V_1$ that holds the condition $\sigma(u'v) = \sigma(uv)$ for every $v \in V_1 \cap N(v)$.

Generating splitting root signed graph using Theorem 3

The process for deriving the splitting root signed graph involves several steps. Firstly, it is necessary for the number of vertices, denoted as "n", to be even in order for a splitting root signed graph to exist. If "n" is odd, then it is impossible to construct a splitting root signed graph. Once it has been established that "n" is even, an n dimensional matrix is generated to represent the given signed graph. The matrix is examined to count the number of positive and negative edges, and it is determined whether both counts are divisible by 3. If this condition is met, a splitting root signed graph can be created; otherwise, it cannot.

The next step involves calculating the number of negative and positive edges for each vertex by counting the number of $-1s$ and $1s$ in each row. If it is possible to partition the vertex set $V(\Sigma)$ into two sets, such that the number of negative and positive edges in one set is exactly double of the other set, then a splitting root signed graph can be constructed.

If a splitting root signed graph exists, it's adjacency matrix of order n can be partitioned into four matrices of equal order $\frac{n}{2}$. Let $[a_{i,j}]$, $[b_{i,j}]$, $[c_{i,j}]$ and $[d_{i,j}]$ are the four matrices of order $\frac{n}{2}$, then $a_{i,j} = b_{i,j} = c_{i,j}$ for $0 \leq i, j \leq \frac{n}{2}$ and $d_{i,j} = 0$ for $0 \leq i, j \leq \frac{n}{2}$ i.e. $[d_{i,j}]$ is zero matrix. To make the matrix identical and zero, it is necessary to interchange rows and columns. This will result in a matrix of size $\frac{n}{2} \times \frac{n}{2}$.

An example of the computation of a splitting root signed graph from a given signed graph will be provided in the following section.

$$A(\Sigma_1) = \begin{bmatrix} 0 & -1 & 0 & -1 & -1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \end{bmatrix}$$

The adjacency matrix of this signed graph, $A(\Sigma_1)$ is a 8×8 matrix. Here the number of vertices in a signed graph is even, so we can find its splitting root signed graph. However, if the vertices are odd in numbers, then splitting root signed graph of the given signed graph will not exist. In this case, there are 12 occurrences of the value -1 and 12 occurrences of the value 1 in the adjacency matrix. Clearly, here both the values are divisible by 3, so the splitting root signed graph can be computed.

The next step is to count the number of $1s$ and $-1s$ in each row of the matrix:

Vertex v_i	no. of $1s$	no. of $-1s$
1	0	4
2	1	1
3	4	0
4	1	1
5	2	2
6	2	0
7	2	2
8	0	2

With this knowledge, the vertex set is splitted into two sets, V_1 and V_2, where the number of $1s$ and $-1s$ in V_1 is double that of in V_2. In this example, V_1 is $\{1, 3, 5, 7\}$ and V_2 is $\{2, 4, 6, 8\}$. If the partition is successful, the splitting root signed graph exists, otherwise it does not.

Next, the splitting root graph is computed. The adjacency matrix $A(\Gamma(\Sigma))$, of order $2n$, of the splitting signed graph can be partitioned into four equal matrices of the order n. Let $[a_{i,j}]$, $[b_{i,j}]$, $[c_{i,j}]$ and $[d_{i,j}]$ are the four matrices of order n, then $a_{i,j} = b_{i,j} = c_{i,j}$ for $0 \le i, j \le n$ and $d_{i,j} = 0$ for $0 \le i, j \le n$ i.e. $[d_{i,j}]$ is zero matrix.

In this example, the 8×8 input matrix is divided into four equal matrices of size 4×4. The fourth matrix is made zero and the first, second, and third matrices are made identical through row and column transformations. The output matrix is 4×4.

After applying transformations such that $R2 \Leftrightarrow R7$ and $C2 \Leftrightarrow C7$, we get:

Vertex v_i	1	7	3	4	5	6	2	8
1	0	-1	0	-1	-1	0	-1	0
7	-1	0	1	0	0	1	0	-1
3	0	1	0	1	1	0	1	0
4	-1	0	1	0	0	0	0	0
5	-1	0	1	0	0	1	0	-1
6	0	1	0	0	1	0	0	0
2	-1	0	1	0	0	0	0	0
8	0	-1	0	0	-1	0	0	0

and then $R4 \Leftrightarrow R5$ and $C4 \Leftrightarrow C5$, we get:

Vertex v_i	1	7	3	4	5	6	2	8
1	0	-1	0	-1	-1	0	-1	0
7	-1	0	1	0	0	1	0	-1
3	0	1	0	1	1	0	1	0
4	-1	0	1	0	0	1	0	-1
5	-1	0	1	0	0	0	0	0
6	0	1	0	1	0	0	0	0
2	-1	0	1	0	0	0	0	0
8	0	-1	0	-1	0	0	0	0

The first, second, and third matrices can be seen to be the same, and the fourth matrix is zero, which satisfies the requirements for a splitting root signed graph. The final output matrix is the splitting root signed graph:

Vertex v_i	1	7	3	5
1	0	-1	0	-1
7	-1	0	1	0
3	0	1	0	1
5	-1	0	1	0

The splitting signed graph S_1 and its corresponding splitting root signed graph S are depicted in the Figure 2.

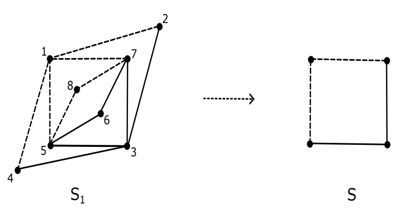

Figure 2: Signed graph S_1 and S is its splitting root signed graph

Algorithm 2 Algorithm to derive the splitting root signed graph of a given signed graph

```
Input: number of vertices (N)
Input: adjacency matrix of the signed graph A(Σ) = a(i, j)
Check
if N mod 2 = 1 then
    Print: matrix is of odd order, splitting root signed graph dose not exist
    Terminate
else
    for i = 1 : N do
        for j = 1 : N do
            check
            if a(i,j)==1 then
                countp = countp + 1
            else
                if a(i,j)== -1 then
                    countq = countq + 1
                end if
            end if
        end for
    end for
end if
check
if countp mod 3 == 0 && countq mod 3 == 0  then
    Print: Splitting root signed graph is possible
else
    Print: Splitting root signed graph is not possible and Terminate
end if
Assign: positive[i] = countp
        negative[i] = countq
        boole[i] = F
for i = 1 : N do
    for j = i + 1 : N do
        Assign: k = i
        if boole[i] == T then
            no-found = 1
            if positive[i] == 2 positive[j] && negative[i] == 2 negative[j] && boole[i] == T then
```

```
            Assign: vertex[i] = 2
                    vertex[j] = 1
                    boole[i] = T
                    boole[i] = T
                    no-found = 0
        end if
        if positive[i] == positive[j]/2 && negative[i] == negative[j]/2 && boole[i] == T then
            Assign: vertex[i] = 1
                    vertex[j] = 2
                    boole[i] = T
                    boole[i] = T
                    no-found = 0
        end if
        if no-found == 0 then
            Assign: j=n
            if no-found == 0 then
                Print: Splitting graph is not possible as vertex division is not proper and Terminate
            end if
        end if
        for i = 1 : N do
            if vertex[i] == 2 then
                Assign: arrayhigh[o1] = i
                count o1 = o1+1
            else
                if vertex[i] == 1 then
                    Assign: arrayhigh[o2] = i
                    count o2 = o2+1
                end if
            end if
        end for
    end if
    Assign: j = 1
    for i = 1 : N/2 do
        Assign: val = arrayhigh[i]
        if val > N/2 then
            Assign: val-low = arraylow[j]
            if val - low <= n/2 then
                Apply row transformation
            end if
        end if
    end for
    Assign: j = j+1
    for i = 1 : 2N do
        for j = 1 : 2N do
```

```
80:         Assign: b(i,j) = a(i,j)
81:         Divide the matrix into four equal blocks
82:         Set: val-ret = Above divided matrix
83:         if val-ret == 0 then
84:            Assign: j=N+1 and Terminate
85:         else
86:            Assign: a(i,j) = b(i,j)
87:         end if
88:      end for
89:   end for
90: end for
91: end for
92: Output: Generate [a_{i,j}] matrix of order N/2
```

Computational complexity:

The algorithm calculates the total number of positive and negative edges and traverses every entry of the matrix. As a result, the complexity to count the edges is $O(n^2)$.

In Steps 30 to 52, the vertex set is partitioned into two sets such that number of positive and negative edges in one set is exactly double of the other. As a result, the complexity is $O(n^2)$.

If the function is denoted by fun then the complexity required for the row and column transformations from Step 67 to Step 77 to make all the three matrices identical is $O(n^3)$. If the fourth submatrix is already zero in the steps 78 to 91 then we apply row operation to make all other sub matrices identical. We traverse each vertex of the signed graph and examine every entry if it is identical. As a result, the complexity involved in these steps is $O(n^2 \times n \times n) = O(n^4)$. Therefore, the complexity of the proposed algorithm for finding a root signed split graph with a given adjacency matrix is $O(n^2)+O(n^3)+O(n^4)+O(n^2) = O(n^4)$., where n represents the number of vertices in the signed graph.

4. Spectrum of splitting signed graph

In this section, we aim to find out the spectrum of the Adjacency matrix and Laplacian matrix of a splitting signed graph $\Gamma(\Sigma)$. Let $A(\Sigma)$ be the adjacency matrix of the signed graph Σ on n vertices, and is given as follow

$$A(\Sigma) = \begin{bmatrix} 0 & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & 0 & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & 0 \end{bmatrix}$$

Let v'_i be the vertex corresponding to v_i, $1 \leq i \leq n$, which is added in Σ to construct $\Gamma(\Sigma)$, such that $N(v'_i) = N(v_i)$, for $1 \leq i \leq n$. Then the adjacency matrix of $\Gamma(\Sigma)$, $A(\Gamma(\Sigma))$, can be expressed as a block matrix with blocks as follows

$$A(\Gamma(\Sigma)) = \begin{bmatrix} 0 & a_{1,2} & \cdots & a_{1,n} & 0 & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & 0 & \cdots & a_{2,n} & a_{2,1} & 0 & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & 0 & a_{n,1} & a_{n,2} & \cdots & 0 \\ 0 & a_{1,2} & \cdots & a_{1,n} & 0 & 0 & \cdots & 0 \\ a_{2,1} & 0 & \cdots & a_{2,n} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Let $\lambda_1(\Sigma), \lambda_2(\Sigma), ..., \lambda_n(\Sigma)$ are the eigenvalues of the signed graph Σ. We can write above adjacency matrix as,

$$A(\Gamma(\Sigma)) = \begin{bmatrix} A(\Sigma) & A(\Sigma) \\ A(\Sigma) & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \otimes A(\Sigma)$$

From here we can see that adjacency matrix $A(\lambda(\Sigma))$ is a Kronecker product of the matrices M and $A(\Sigma)$, where $M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Easily we can see that $\{\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\}$ are the eigenvalues of the matrix M.

So the *adjacency spectrum* of the splitting signed graph $\lambda(\Sigma)$ is given as $\{(\frac{1+\sqrt{5}}{2})\lambda_i, (\frac{1-\sqrt{5}}{2})\lambda_i\}$, where $1 \leq i \leq n$.

Now consider $L(\Sigma)$ be the laplacian matrix of the signed graph Σ and $\psi_1, \psi_2, ..., \psi_i$ are the eigenvalues of the laplacian matrix $L(\Sigma)$. By some easy calculations we can find that the laplacian matrix $L(\Gamma(\Sigma))$ of the splitting signed graph is a Kronecker product of the matrices M and $L(\Sigma)$, where $M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

So the *laplacian spectrum* of the splitting signed graph $\lambda(\Sigma)$ is given as $\{(\frac{1+\sqrt{5}}{2})\psi_i, (\frac{1-\sqrt{5}}{2})\psi_i\}$, where $1 \leq i \leq n$.

From Equation 2, we can see that spectral radius of Σ will always be greater than equal to the index , i.e.

$$\rho(\Sigma) \geq \lambda_1(\Sigma)$$

In [1] Acharya provided the spectral criterion for balance in Σ as,

Theorem 4. *[1] A signed graph Σ is balanced if and only if it is cospectral to it's underlying graph.*

It provides that the balanced signed graphs have the spectral radius equal to the index. So, here in this section we characterize the balanced splitting signed graphs.
Sampathkumar provided an important characterization of balanced signed graphs based on marking:

Theorem 5. *[?] The balance of a signed graph $\Sigma = (\Sigma^u, \sigma)$ can be determined if and only if there is a marking μ of its vertices such that the sign of each edge vu in Σ satisfies the condition $\sigma(vu) = \mu(v)\mu(u)$.*

The operation of changing the sign of every edge in a signed graph Σ to its opposite, based on the marking μ of its vertices, is called *switching* Σ with respect to μ. This operation is performed whenever the end vertices of an edge have opposite signs in Σ_μ. The concept of switching signed graphs is closely connected to the concept of balance, as indicated by the following theorem:

Theorem 6. *[20] A signed graph $\Sigma = (\Sigma^u, \sigma)$ is considered balanced if and only if it is equivalent under switching to its underlying graph Σ^u.*

Theorem 7. *[13] The splitting signed graph $\Gamma(\Sigma)$ is balanced if and only if the signed graph Σ is balanced.*

Remark 1. *The splitting signed graph $\Gamma(\Sigma)$ is cospectral to it's underlying graph $\Gamma(\Sigma^u)$ if and only if Σ is balanced.*

5. Energy of splitting signed graph

In this section, we explore the connection between the energy of a signed graph Σ and its splitting signed graph $\Gamma(\Sigma)$.

Theorem 8. *Let $E(\Sigma)$ be the energy of signed graph Σ and $E(\Gamma(\Sigma))$ be the energy of splitting signed graph $\Gamma(\Sigma)$, then $E(\Gamma(\Sigma)) = \sqrt{5}E(\Sigma)$.*

Proof. Let $V = \{v_1, v_2, ..., v_n\}$ be the vertex set of the signed graph Σ. Then adjacency matrix of Σ is given by,

$$A(\Sigma) = \begin{bmatrix} 0 & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & 0 & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & 0 \end{bmatrix}$$

Let v'_i be the vertex corresponding to v_i, $1 \leq i \leq n$, which is added in Σ to construct $\Gamma(\Sigma)$, such that $N(v'_i) = N(v_i)$, for $1 \leq i \leq n$. Then the adjacency matrix of $\Gamma(\Sigma)$, $A(\Gamma(\Sigma))$, can be expressed as a block matrix with blocks as follows

$$A(\Gamma(\Sigma)) = \begin{bmatrix} 0 & a_{1,2} & \cdots & a_{1,n} & 0 & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & 0 & \cdots & a_{2,n} & a_{2,1} & 0 & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & 0 & a_{n,1} & a_{n,2} & \cdots & 0 \\ 0 & a_{1,2} & \cdots & a_{1,n} & 0 & 0 & \cdots & 0 \\ a_{2,1} & 0 & \cdots & a_{2,n} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

We can write it as,

$$A(\Gamma(\Sigma)) = \begin{bmatrix} A(\Sigma) & A(\Sigma) \\ A(\Sigma) & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \otimes A(\Sigma)$$

Let $\lambda_1, \lambda_2, ..., \lambda_n$ are the eigenvalues of the signed graph Σ and we can observe that the eigenvalues of $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ are $\{\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\}$.
Therefore,

$$spec(\Gamma(\Sigma)) = \begin{pmatrix} (\frac{1+\sqrt{5}}{2})\lambda_i & (\frac{1-\sqrt{5}}{2})\lambda_i \\ n & n \end{pmatrix}$$

$$E(\Gamma(\Sigma)) = \sum_{i=1}^{n} |(\frac{1 \pm \sqrt{5}}{2})\lambda_i| = \sum_{i=1}^{n} |\lambda_i|[\frac{1+\sqrt{5}}{2} + \frac{1-\sqrt{5}}{2}] = \sqrt{5}\sum_{i=1}^{n} |\lambda_i|$$

Hence,

$$E(\Gamma(\Sigma)) = \sqrt{5}E(\Sigma)$$

6. Conclusion and Scope

In conclusion, this research gives an algorithm for generating a splitting signed graph and a splitting root signed graph from a given signed graph, provided it exists. Additionally, a spectral analysis of the resulting graph is conducted by studying its eigenvalues and eigenvectors through the adjacency and laplacian matrices. The research also establishes a relationship between the energy of the original signed graph Σ and the energy of the splitting signed graph $\Gamma(\Sigma)$. The scope of this research could be extended by exploring further applications of the proposed algorithm and studying the properties of splitting signed graphs in more detail.

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