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Codimension one foliation and the prime spectrum of a ring

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Abstract. Let $\mathcal{F}$ be a transversally oriented codimension-one foliation of class C^r, $r \geq 0$, on a closed manifold M. A leaf class of a leaf F is the union of all leaves having the same closure as F. Let X be the leaf classes space and X_0 be the union of all open subsets of X homeomorphic to $\mathbb{R}$ or $\mathbb{S}^1$. In [3, Theorem 3.15] it is shown that if a codimension one foliation has a finite height, then the singular part of the space of leaf classes is homeomorphic to the prime spectrum (or simply the spectrum) of unitary commutative ring. In this paper we prove that the singular part of the space of leaf classes is homeomorphic to the spectrum of unitary commutative ring if and only if every family of totaly ordered leaves is bounded below.

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1. Introduction

A space X is a *spectral space* [5] if it is (1) sober (i.e., every nonempty irreducible closed subset of Y is the closure of a unique point), (2) quasi-compact, (3) the quasi-compact open subsets of X form a basis and (4) the family of quasi-compact open subsets of X is closed under finite intersections. In particular a finite T_0-space is a spectral space.

(1), (2), (3) and (4) are called *spectral properties*.

A foliation of codimension one on a smooth manifold M of dimension m is an open equivalence relation $\mathcal{F}$ on M with each equivalence class (called a leaf) is a weakly embedded $m-1$ sub-manifold such as the canonical projection of M on the space of leaves $M/\mathcal{F}$ is a locally submersion. In that event for each $x \in M$, there is a chart (U, φ) such that $\varphi(U) = \mathbb{R}^m$ and each equivalence class of the restriction of $\mathcal{F}$ to U is homeomorphic to $\mathbb{R}^p \times \{y\}$ where $y \in \mathbb{R}$. Such chart (U, φ) is a distinguished chart.

The notions of proper leaf, minimal set, local minimal set are introduced in [4, chapter 4.4].

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Recently, we study the relationships between graphs and the prime spectrum of unitary commutative rings [1].

In [3] the authors studied relationships between foliations and spectral space. In particular, if a codimension one foliation has a finite height, then the singular part of the space of leaf classes is homeomorphic to the spectrum of unitary commutative ring.

In this paper we prove that the singular part of the space of leaf classes is homeomorphic to the spectrum of unitary commutative ring if and only if every family of totaly ordered leaves is bounded below.

2. Useful notions

A topological space X is a T_0-*space* (or Kolmogorov space) if for every $x \neq y$, there is a neighborhood containing one of them but not the other; which is equivalent to the following implication ($\overline{\{x\}} = \overline{\{y\}} \Rightarrow x = y$).

Let $(X, \leq)$ be an ordered set and $\mathcal{T}$ be a topology on X. We say that $\mathcal{T}$ is *compatible with* $\leq$ if, for each element $x \in X$, $\overline{\{x\}} = \{y \in X : x \leq y\} = [x, \rightarrow[$ ($\overline{\{x\}}$ is the closure of $\{x\}$).

Proposition 2.1. *Let $(X, \leq)$ be an ordered set and $\mathcal{T}$ be a topology on X. If $\mathcal{T}$ is compatible with $\leq$, then $(X, \mathcal{T})$ is a T_0-space.*

Proof. Let x and y be two points of X.

- If $x < y$, then $x \in X - [y, \rightarrow[$ and so the open set $X - [y, \rightarrow[$ contains x and not contains y.
- If x and y are not comparable, then the open set $X - [x, \rightarrow[$ contains y and not contains x.

Remark 2.2. If $(X, \mathcal{T})$ is a T_0-space, then X is an ordered set by the order defined by $x \leq_{\mathcal{T}} y$ if and only if $x \in \overline{\{y\}}$.

According to [2] we have the following proposition:

Proposition 2.3. *If $(X, \mathcal{T})$ and $(X', \mathcal{T}')$ are two homeomorphic T_0-spaces, then the ordered sets $(X, \leq_{\mathcal{T}})$ and $(X', \leq_{\mathcal{T}'})$ are isomorphic.*

Proof. Let h be a homeomorphism between $(X, \mathcal{T})$ and $(X', \mathcal{T}')$. If $x \leq_{\mathcal{T}} y$, then $x \in \overline{\{y\}}^{\mathcal{T}}$. Since h is continuous, $h(x) \in \overline{\{h(y)\}}^{\mathcal{T}'}$ and so $h(x) \leq_{\mathcal{T}'} h(y)$. If now $h(x) \leq_{\mathcal{T}'} h(y)$, then, by the continuity of h^{-1}, $x \leq_{\mathcal{T}} y$. Therefore h is an isomorphism.

The converse of the Proposition 2.3 is false. Indeed, all the compatible topologies with an order $\leq$ induce the same order $\leq$ but are not necessarily homeomorphic.

A topological space X is quasi-compact if it satisfies the property of Borel-Lebesgue but it is not necessarily a Hausdorff space. A subset A of X is quasi-compact if it is a quasi-compact space equipped with the relative topology of X. We have the following properties:

1. The quasi-compactness is invariant under continuous map.
2. Closed subsets of a quasi-compact space are quasi-compact.
3. The union of finitely many quasi-compact subsets is quasi-compact.

The intersection of tow quasi-compact open subsets is not necessarily quasi-compact. The following example confirm this result:

Example 2.4. In the two Euclidean space we consider the following points: $C(0,1)$, $A(-1,0)$, $B(1,0)$, $A_n(-1, \frac{-1}{n}))$ and $B_n(1, \frac{-1}{n}))$. Let X be the set $\{C, A, B, A_n, B_n : n \geq 1\}$ equipped with the following topology: $\{\emptyset, X, U = \{A, A_n : n \geq 1\}, V = \{B, B_n : n \geq 1\}, U_n = \{A_p : n \geq p \geq 1\}, V_n = \{B_p : n \geq p \geq 1\}\}$. Note that U and V are quasi-compact. Since $U \cap V = \bigcup_n U_n \cup V_n$ and $U_n \cup V_n$ is an increasing sequence of open subsets, $U \cap V$ is not a quasi-compact subset.

A closed subset C is irreducible if it is not the union of two proper closed subsets or if the intersection of two nonempty open subsets is nonempty. An element x of C is called a generic point if the closure of the singleton $\{x\}$ is equal to C: $\overline{\{x\}} = C$.

Let A be a commutative and unitary ring and $Spec(A)$ be the set of prime ideals of A. If P is an ideal of A, the family $\{V(P) = \{Q \in Spec(A) : P \subset Q\}\}$ defines the closed subsets of the Zariski topology in $Spec(A)$. Equipped with this topology, $Spec(A)$ satisfies the following properties:

1. $Spec(A)$ is Kolmogorov.
2. Every irreducible closed subset of $Spec(A)$ has a generic point.
3. $Spec(A)$ is a quasi-compact space.
4. There is a basis of quasi-compact open subsets of $Spec(A)$.
5. The family of quasi-compact open subsets of $Spec(A)$ is stable under finite intersection.

If a topological space X satisfies the above five properties, then there exists a commutative and unitary ring A such that X is homeomorphic to $Spec(A)$ equipped with the Zariski topology [5].

Let $\mathcal{F}$ be a codimension-one transversally oriented foliation of class C^r, $r \geq 0$, on a closed m-manifold M. Dippolito [4, chapter 4.4] defined the boundary δU of a nonempty saturated connected open subset distinct of M. The boundary δU is equal to the set of points $x \in M - U$ such as there is a curve $c : [0,1] \to M$ such that $c(0) = x$ and $c(]0,1]) \subset U$. Dippolito proved that δU is a union of a finitely many leaves ([4, chapter 4.4]). We have $\overline{\delta U} = \overline{U} - U$.

Note that an attracting proper leaf from one side is introduced in [4, chapter 4.4].

Recall that if $A \subset X$, the saturation $Sat_{\mathcal{R}}(A)$ of A is the union of all equivalence classes meeting A. The subset A is called invariant (or saturated) if $A = Sat_{\mathcal{R}}(A)$. Note that, the interior, the closure, the boundary of each saturated subset is also saturated; indeed the relation $\mathcal{R}$ is open.

We denote by T_s the invariant topology on X formed by the invariant open subsets of X. An invariant open subset $U \subset X$ is called *compact by saturation* if it is quasi-compact for the invariant topology T_s. That is, every covering (U_i) of U by invariant open subsets U_i contains a finite sub-cover.

Lemma 2.5. *Let $\mathcal{R}$ be an open equivalence relation on a topological space X. An open subset V of the quotient space $X/\mathcal{R}$ is quasi-compact if and only if the open subset $U = q^{-1}(V)$ is compact by saturation.*

Proof. Suppose that V is quasi-compact, and let $(U_i, i \in I)$ be a covering of $U = q^{-1}(V)$ by saturated open subsets. Thus the open subsets $(q(U_i))$ cover V, and some finite number of these, $q(U_{i_1}), ..., q(U_{i_n})$, covers V. Because every U_i is saturated, $q^{-1}(q(U_i)) = U_i$ and hence $U = U_{i_1} \cup ... \cup U_{i_n}$.

Conversely, let $(V_i, i \in I)$ be a family of open subsets of $X/\mathcal{R}$ such that $V = \bigcup_i V_i$. Since $q^{-1}(V_i)$ is a saturated open subset and U is compact by saturation, it follows that $U = q^{-1}(V_{i_1}) \cup ... \cup q^{-1}(V_{i_n})$ which implies that $V = V_{i_1} \cup ... \cup V_{i_n}$.

Lemma 2.6. *[3] Let U be a connected nonempty invariant open subset of M. Then the following properties are equivalent:*

a) The following two properties hold:

i) Each leaf $L \subset \delta^{\epsilon}U$, $\epsilon = \pm$, is attracting from the side ϵ (i.e L is attracting from the side of U).

ii) For each leaf $F \subset U$, the intersection $\overline{F} \cap U$ contains a local minimal set in U.

b) U is compact by saturation.

Lemma 2.7. *[4, chapter 4.4] For each leaf F of a nonempty invariant open subset $U \subset M$, the intersection $\overline{F} \cap U$ contains at most finitely many local minimal sets in U.*

Let $X = M/\widetilde{\mathcal{F}}$ be the leaf classes space. Consider X_0 the union of all open subsets of X homeomorphic to $\mathcal{R}$ or S^1.

Proposition 2.8. *[3] The inverse image of X_0 by the canonical projection p is the union of all stable proper leaves.*

Bouacida et all showed, in [3], that if $\mathcal{F}$ has a well defined height, then the singular part $X - X_0$ of the leaf classes space is homeomorphic to the spectrum of a unitary commutative ring equipped with the Zariski topology.

Note that, the height of a foliation is well defined [4, chapter 4.4] if and only if every totally ordered family of leaves is well-ordered (*i.e.* it has a minimal element). Recall that a family of leaves is ordered by inclusion of their closures.

Precisely, the authors of [3] showed that $X - X_0$ verifies the following properties:

(1) $X - X_0$ is a sober space.

(2) $X - X_0$ is a quasi-compact space.

(3) $X - X_0$ has a basis of quasi-compact open subsets.

(4) If $\mathcal{F}$ has a well defined height, then the family of quasi-compact open subsets of $X - X_0$ is closed under finite intersections.

In this paper we prove that the singular part of the space of leaf classes is homeomorphic to the spectrum of unitary commutative ring if and only if every family of totaly ordered leaves is bounded below (Theorem 3.1).

3. Main result

Theorem 3.1. *Let $\mathcal{F}$ be a codimension-one transversally oriented foliation of class C^r, $r \geq 0$, on a closed manifold M. Consider X the leaf classes space and let X_0 be the union of all open subsets of X homeomorphic to $\mathbb{R}$ or $\mathbb{S}^1$. Then, the space $X - X_0$ is homeomorphic to the spectrum of unitary commutative ring if and only if every family of totaly ordered leaves is bounded below.*

We need the following lemmas.

Lemma 3.2. *[3] Let $\mathcal{F}$ be a codimension-one transversally oriented foliation of class C^r, $r \geq 0$, on a closed manifold M. Consider X the leaf classes space and let X_0 be the union of all open subsets of X homeomorphic to $\mathbb{R}$ or $\mathbb{S}^1$. Then, e get the following properties:*

(1) The space $X - X_0$ is sober.

(2) The space $X - X_0$ is quasi-compact.

(3) The space $X - X_0$ has a basis of quasi-compact open subsets.

Proof. of Theorem 3.1. If $X - X_0$ is homeomorphic to the prime spectrum of unitary commutative ring, then it satisfies the condition (K_1) of Kaplansky and so every totally ordered family of orbits has an infimum.

By Lemma 3.2, $X - X_0$ satisfies three spectral properties (1), (2) and (3). It suffices to show the fourth spectral property, that is, if every family of totaly ordered leaves is bounded below, then the family of quasi-compact open subsets of $X - X_0$ is closed under finite intersections.

According to Lemma 2.6, it suffices to show that the intersection $W = U \cap V$ of two compact by saturation open sets is also compact by saturation. According to that fact that $\delta^\epsilon W \subset \delta^\epsilon U \cup \delta^\epsilon V$, $\epsilon = \pm$, we prove that W verifies the property $a - ii)$ of Lemma 2.6. We can suppose that W is a connected set, differently we can take the connected component of W containing a leaf $F \subset W$.

Consider $\{F_i\}$ a maximal totally ordered family of leaves such that $F_i \subset \overline{F} \cap W$, for every i, and we denote by L the greatest lower bound leaf of this family (this leaf L exists from the hypothesis). According to Lemmas 2.7 and 2.6-a-ii), there exist two local minimal sets E_1 and E_2 of $\mathcal{F}$ restricted to U and V respectively which are subsets of the closure $\overline{F_i}$, for every i. Consider L_1 and L_2 two leaves such as $L_1 \subset E_1$ and $L_2 \subset E_2$. Therefore $\overline{L_1} = \overline{E_1} \subset \overline{L}$ and $\overline{L_2} = \overline{E_2} \subset \overline{L}$. Consequently, $L \subset U$ and $L \subset V$, thus $L \subset W$ and $\overline{L} \cap W$ is a local minimal set of $\mathcal{F}$ restricted to W. Differently, there is a leaf S such that $\overline{S} \neq \overline{L}$ and $S \subset \overline{L} \cap W$. Thus the family $\{F_i\}$ is not maximal which leads to a contradiction. We deduce that the open set W verifies the two items of the property $a)$ in Lemma 2.6. Therefore W it is a compact by saturation open set. This ends the proof of Theorem 3.1.

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