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New Improved Bounds for Signless Laplacian Spectral Radius and Nordhaus-Gaddum Type Inequalities for Agave Class of Graphs

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Abstract. Core-satellite graphs $\Theta(c, s, \eta) \cong K_c \bigtriangledown (\eta K_s)$ are graphs consisting of a central clique K_c (the core) and η copies of K_s (the satellites) meeting in a common clique. They belong to the class of graphs of diameter two. Agave graphs $\Theta(2, 1, \eta) \cong K_2 \bigtriangledown (\eta K_1)$ belong to the general class of complete split graphs, where the graphs consist of a central clique K_2 and η copies of K_1 which are connected to all the nodes of the clique. They are the subclass of Core-satellite graphs. Let $\mu(G)$ be the spectral radius of the signless Laplacian matrix $Q(G)$. In this paper, we have obtained the greatest lower bound and the least upper bound of signless Laplacian spectral radius of Agave graphs. These bounds have been expressed in terms of graph invariants like m the number of edges, n the number of vertices, δ the minimum degree, Δ the maximum degree, and η copies of the satellite. We have made use of the approximation technique to derive these bounds. This unique approach can be utilized to determine the bounds for the signless Laplacian spectral radius of any general family of graphs. We have also obtained Nordhaus-Gaddum type inequality using the derived bounds.

2020 Mathematics Subject Classifications: 05C50

Key Words and Phrases: Spectral radius, signless Laplacian spectral radius, Core-satellite graphs, Agave graphs, Complete split graph

1. Introduction

Hierarchical products of graphs, developed through iterative hierarchical products of complete graphs, are used as models for real-world networks. Core-satellite graphs can be redesigned to resemble some important characteristics of a complex network to exhibit a hierarchical structure [14, 18]. Estrada [6] demonstrated how certain classes of real-world networks can be modeled using core-satellite graphs. This family of graphs has topological features of both small world and scale-free networks. Using heuristic analysis, Xu and Zhang [20] investigated the effects of cover time. They showed that the network topology

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discussed by them in their work, having both small world and scale-free features, exhibits the most minimal cover time.

Agave graphs are formed by connecting η disjoint vertices to both vertices of a complete graph K_2. They belong to the general class of complete split graphs which are formed by the join of central clique K_r and a set of $(n-r)$ independent vertices [6, 7]. Nair et al. [1] distinctly picturised the complete split graphs which belong to the family of generalized core-satellite graphs. The generalized core-satellite graphs belong to the larger family of graphs of diameter two.

The significant features of Agave graphs are identified from the article by Ernesto and Eusebio [6]. The distance-sum heterogeneity index $\varphi(G)$, plays a key role in the structural analysis of complex networks. This allows an interpretation of the Wiener and Balaban index. It is found to be useful in the analysis of molecular graphs. Furthermore, it is found that there is proximity with a node's closeness centrality and average path length in the analysis of complex networks. This index highlights more of the structural properties of a network which helps us to understand the functional and dynamic processes in complex systems. The authors have conjectured that among graphs with a specific number of nodes $\varphi(G)$ is the maximum for graphs with structures resembling the Agave graph.

Due to its enhanced effectiveness, simplicity, and better performance, the signless Laplacian spectral radius is utilized to examine graph properties. When compared to spectra of other graph matrices, the spectrum of the signless Laplacian matrix is thoroughly investigated and studied extensively [4, 11, 19].

Through different methods, the bounds are obtained for $\mu(Q(G))$. Y. Chen and L.Wang [3] provided two sharp upper bounds for $\mu(Q(G))$ in terms of the maximum degree and the minimum degree of the graph G and employed a new technique to obtain another sharp upper bound. In [5], Xing and Zhou provided a generalized theorem for the upper bound and lower bound for the spectral radius of a non-negative matrix using its row sums. They have applied these bounds to various non-negative matrices associated with graphs which includes the adjacency matrix, the signless Laplacian matrix, the distance matrix, the distance signless Laplacian matrix, and the reciprocal distance matrix. In [21], Zhang et al. determined the largest signless Laplacian spectral radius among the bicyclic graphs with perfect matchings. In [10], the authors determined sharp upper bounds for the spectral radius and signless Laplacian spectral radius of a uniform hypergraph in terms of the average 2 degrees or degrees of vertices, degree and the minimum degree of the vertices of G. Moreover, they proposed a new proving technique to obtain a sharp upper bound for $\mu(Q(G))$. In [13], Feng and Yu studied the family of connected graphs with prescribed order and diameter and then determined the extremal graph that has the maximal signless Laplacian spectral radius.

Y. Hong and Shu [8] obtained a sharp upper bound of the Nordhaus-Gaddum type for the chromatic numbers of G and G^c, respectively. Huiqing Liu et al. [11] obtained a sharp upper bound for the Nordhaus-Gaddam type relation of the Laplacian spectral radius $L(G)$ by making use of the relation $\mu(A(G)) \leq \mu(Q(G))$. Shuchao and Tian [9] and Shi [19], have derived sharp bounds for Nordhaus-Gaddam type of relation in terms n, m, $\delta(G)$, and $\Delta(G)$. Nikiforov [15–17], conjectured and obtained improved bounds on Nordhaus-

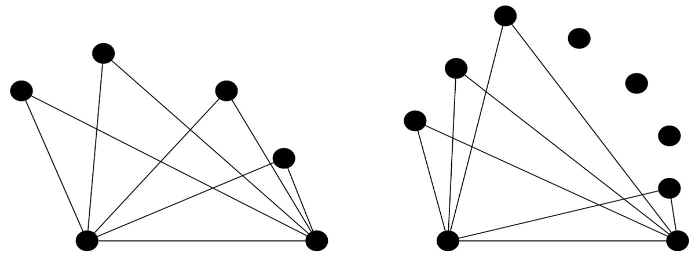

Figure 1: Agave graph $K_2 \bigtriangledown 4K_1$ and $K_2 \bigtriangledown \eta K_1$

Gaddum type inequality. In [2], Mustapha Aouchiche and Pierre Hansen mentioned many results on lower and upper bounds in their survey article on Nordhaus-Gaddum type inequality on the sum and the product of many other graph invariants of a graph and its complements.
Motivated by the above-mentioned results, we have implemented some unique techniques to obtain tight upper and lower bounds for the signless Laplacian spectral radius for the Agave class of graphs. We have utilized these bounds to obtain the corresponding tight bounds for Nordhaus-Gaddam type inequality.

2. Preliminaries

Here we mention some preliminaries, definitions, and results which will be used to derive our main results.

Let $G = (V, E)$ be a simple, connected graph with n vertices and m edges. Let $\delta(G) = \delta$ and $\Delta(G) = \Delta$ be the minimum and the maximum degree of vertices of G, respectively. Let G^c be the complement of graph G. Let $A(G)$ be the adjacency matrix of the graph G and the degree diagonal matrix $D(G) = \text{diag } (d(v_1), d(v_2), ..., d(v_n))$ is the diagonal matrix of vertex degrees. For any $n \times n$ real symmetric matrix M, by Geršgorin's Theorem, its eigenvalues are non-negative real numbers. Let $\rho(G)$ be the spectral radius of adjacency matrix $A(G)$, with non-increasing sequence of real eigenvalues $\rho(G) = \rho_1(G) \geq \rho_2(G) \geq ... \geq \rho_n(G)$. Let $\mu(G)$ be the signless Laplacian spectral radius of the matrix $Q(G) = D(G) + A(G)$, which is irreducible, symmetric, and non-negative. Thus the matrix $Q(G)$ is positive and semi-definite. It has non-increasing sequence of real eigenvalues $\mu(Q(G)) = \mu_1(Q(G)) \geq \mu_2(Q(G)) \geq ... \geq \mu_n(Q(G)) = 0$.

Definition 1. [6] The join (or complete product) $G_1 \bigtriangledown G_2$ of graphs G_1 and G_2 is the graph obtained from $G_1 \cup G_2$ by joining every vertex of G_1 with every vertex of G_2.

Definition 2. [6] Core-satellite graphs $\Theta(c, s, \eta) \cong K_c \bigtriangledown (\eta K_s)$ are the graphs consisting of η copies of K_s (the satellites) meeting in a common clique K_c (the core), where $c \geq 1$, $s \geq 1$ and $\eta \geq 2$.

Core-satellite graphs have central nodes in a network that are connected among themselves. Also, there are a few cliques of the same size which are connected to the central core but not connected among themselves. For $v \in V$, the degree of v, written as $d(v)$, is the number of edges incident on v. The vertices in the satellite group have minimum degree δ and the vertices of the central clique (core) have the maximum degree Δ. Agave graphs are graphs with K_c as K_2, K_s as K_1 and η is the number of copies of K_s. This class belongs to the general class of complete split graphs [6].

The following results are useful in proving the main results.

Lemma 1. [19] Let B be a real symmetric $n \times n$ matrix and μ be an eigenvalue of B with an eigenvector $\mathbf{x}$ all of whose entries are non-negative. Denote the i^{th} row sum of B by $R_i(B)$. Then

$$\min_{1 \leq i \leq n} R_i(B) \leq \mu(B) \leq \max_{1 \leq i \leq n} R_i(B). \tag{1}$$

Moreover, if all entries of $\mathbf{x}$ are positive then either of equalities holds if and only if row sums of B are all equal.

Lemma 2. [19] Let M be a real symmetric $n \times n$ matrix and λ be an eigenvalue of M with an eigenvector $\mathbf{x}$ whose entries are all non-negative. Let P be any polynomial. Then

$$\min_{1 \leq i \leq n} S_i(P(M)) \leq P(\lambda) \leq \max_{1 \leq i \leq n} S_i(P(M))$$

where $S_i(P(M))$ is the row sum of the i^{th} row of the matrix $P(M)$. Moreover, if all entries of $\mathbf{x}$ are positive then either of the equalities holds if and only if the row sums of $P(M)$ are all equal.

Lemma 3. [9] Let $G = (V, E)$ be a simple graph. Then

$$\sqrt{2} \min_{v \in V(G)} \sqrt{d^2(v) + \sum_{uv \in E(G)} d(u)} \leq \mu(G) \leq \sqrt{2} \max_{v \in V(G)} \sqrt{d^2(v) + \sum_{uv \in E(G)} d(u)} \tag{2}$$

Moreover, if G is connected, both the equalities hold if and only if $2\left(d^2(v) + \sum_{uv \in E(G)} d(u)\right)$ is the same for all $v \in V(G)$.

Lemma 4. [12] The signless Laplacian spectrum of $\overline{K_a} \vee K_b$ is

$$\left\{ \frac{(3b + a - 2) \pm \sqrt{(b + a - 2)^2 + 4ab}}{2}, b^{a-1}, (a - b - 2)^{b-1} \right\}$$

where an exponent indicates the multiplicity of the corresponding signless Laplacian eigenvalue.

3. Main Results

Theorem 1. Let G_A be an Agave graph with n vertices, m edges, and η copies of the satellite graph. Let Δ and δ be the maximum degree and minimum degree of G_A. Also

$$\left(2\delta^2 + 2(2m - \eta\delta)\right)^{1/2} \leq \mu\left(G_A\right) \leq \left(2\Delta^2 + 2(2m - \Delta)\right)^{1/2} \tag{3}$$

Moreover, both the equalities hold if and only if $n = 3$ and $\eta = 1$.

Proof. Using Lemma 3, the lower bound is attained when the vertex v is in the satellite graph.

$$\sqrt{2} \min_{v \in V(G)} \sqrt{d^2(v) + \sum_{uv \in E(G_A)} d(u)}$$

$$= \sqrt{2} \sqrt{\delta^2 + \left(2m - d(u) - \sum_{uv \notin E(G_A)} d(u)\right)}$$

$$= \sqrt{2\delta^2 + 2\left(2m - \delta - (\eta - 1)\delta\right)}$$

$$= \sqrt{2\delta^2 + 2(2m - \eta\delta)} \leq \mu(G_A).$$

Similarly, the upper bound is attained when v is the vertex in the core graph. Also for vertices in the core graph, that is $v \in C$, we have $d(v) = \Delta = (n-1)$ and

$$\sum_{uv \notin E(G)} d\left(u\right) = 0$$

since $v \in C$ is connected to all the vertices of the graph G_A. Therefore

$$\sqrt{2} \max_{u \in V(G)} \sqrt{d^2(v) + \sum_{uv \in E(G_A)} d(u)} = \sqrt{2} \sqrt{\Delta^2 + (2m - d\left(v\right) - \sum_{uv \notin E(G)} d\left(u\right))}$$

$$= \sqrt{2\Delta^2 + 2(2m - \Delta)} \geq \mu\left(G_A\right)$$

Hence,

$$\sqrt{2\delta^2 + 2(2m - \eta\delta)} \leq \mu(G_A) \leq \sqrt{2\Delta^2 + 2(2m - \Delta)}$$

Moreover, both the equalities hold for $n = 3$ and $\eta = 1$, that is when $\delta = \Delta$.

Example 1. According to the result derived in Theorem 1, for $n = 7$, $m = 11$, $\eta = 5$, $\Delta = 6$, $\delta = 2$ and $\mu(G_A) = 8.5311$, we have

$$5.6568 \leq \mu(G_A) \leq 10.198 \tag{4}$$

Remark 1. The following relationships between the parameters in the graph G_A hold:

$$(i)\ m = (n + \eta - 1) = (2n - 3) \quad (ii)\ \Delta = (n - 1) \quad (iii)\ (\delta + \Delta) = (n + 1)$$

$$(iv)\ (\Delta - \delta) = (\eta - 1) = (n - 3) \quad (v)\ (n - \eta) = \delta = 2.$$

We prove the following Lemmas, which are used to prove the theorems.

Lemma 5. For the graph G_A,

$$\mu(G_A) > \left(2\delta^2 + 2(2m - \eta\delta) + (n + 1)\eta\right)^{1/2} > \left(2\delta^2 + 2(2m - \eta\delta) + n\eta\right)^{1/2}$$

for all $n > 6$.

Proof. To prove this we make use of Lemma 4. We consider the signless Laplacian spectral radius for $\overline{K_a} \vee K_b$

$$\mu\left(G_A\right) = \frac{(3b + a - 2) + \sqrt{(b + a - 2)^2 + 4ab}}{2} \tag{5}$$

Substituting $b = 2$ and $a = \eta$, where $\eta = n - 2$ and making use of Remark 1, we reduce equation (5) in terms of n as

$$\mu\left(G_A\right) = \left\{\frac{(n + 2) + \sqrt{(n^2 + 4n - 12)}}{2}\right\}$$

Squaring the above expression we have

$$\left(\mu\left(G_A\right)\right)^2 = \left\{\frac{(n + 2)^2 + (n^2 + 4n - 12) + 2(n + 2)\sqrt{(n^2 + 4n - 12)}}{4}\right\}$$

In the above expression, using the following binomial approximation

$$\sqrt{(n^2 + 4n - 12)} \simeq \left(1 + \frac{n}{2}\left(\frac{4}{n} - \frac{12}{n^2}\right)\right)$$

when

$$\frac{1}{2}\left|\frac{4}{n} - \frac{12}{n^2}\right| < 1$$

for $n \geq 3$, we get

$$\left(\mu\left(G_A\right)\right)^2 \simeq \left\{\frac{(n + 2)^2 + (n^2 + 4n - 12) + 2(n + 2)\left(1 + \frac{n}{2}\left(\frac{4}{n} - \frac{12}{n^2}\right)\right)}{4}\right\}$$

$$= \left(n^2 + 4n - \frac{6}{n} - 3\right).$$

To prove

$$\mu(G_A) > \left(2\delta^2 + 2(2m - \eta\delta) + (n+1)\eta\right)^{1/2}$$

consider

$$(\mu(G_A))^2 - \left(\left(2\delta^2 + 2(2m - \eta\delta) + (n+1)\eta\right)^{1/2}\right)^2$$

Expressing the second term in the above expression in terms of n and using the approximation

$$(\mu(G_A))^2 \simeq \left(n^2 + 4n - \frac{6}{n} - 3\right).$$

we obtain

$$(\mu\,(G_A))^2 - \left(n^2 + 3n + 2\right) \simeq \left(n - \frac{6}{n} - 5\right) > 0$$

for $n > 6$. Hence proved.

Lemma 6. For the graph G_A,

$$\mu\,(G_A) > \left(2\Delta^2 + 2(2m - \Delta) - n\eta\right)^{1/2}$$

for $n \geq 3$.

Proof. We follow the procedure adopted to prove Lemma 5. The above inequality is proved by reducing the RHS expression in terms of n and on using the approximation for $(\mu\,(G_A))^2$, we obtain

$$(\mu\,(G_A))^2 - \left(n^2 + 4n - 8\right) \simeq \left(5 - \frac{6}{n}\right) > 0$$

for $n \geq 3$.

Lemma 7. For the graph G_A,

$$\mu\,(G_A) < \left(2\delta^2 + 2(2m - \eta\delta) + (n+2)\eta\right)^{1/2}$$

for $n \geq 3$.

Proof. The proof is on the same lines as that of Lemma 5 and we have

$$(\mu\,(G_A))^2 - \left(n^2 + 4n\right) \simeq \left(-\frac{6}{n} - 3\right) < 0$$

for $n \geq 3$.

Corollary 1. Using Lemma 5, we obtain the approximation for $\mu\left(G_A\right)$ as

$$\mu\left(G_A\right) \simeq \left(n^2+4n-4.2\right)^{1/2} \tag{6}$$

for all $n > 6$. We now consider

$$\mu(G_A) \simeq \left(2\delta^2+2(2m-\eta\delta)+(n+1)\eta+\gamma\right)^{1/2}$$

Substituting $\delta = 2$, $m = (2n-3)$ and $\eta = (n-2)$ in the above equation, we get

$$\mu(G_A) \simeq (8+2(2(2n-3)-2(n-2))+(n+1)(n-2)+\gamma)^{1/2}$$

$$\mu(G_A) \simeq \left(n^2+3n+2+\gamma\right)^{1/2}$$

For $n = 7$, we have

$$\mu(G_A) = 8.5311 = (72+\gamma)^{1/2}$$

For $n = 8$,

$$\mu(G_A) = 9.5826 = (90+\gamma)^{1/2}$$

For $n = 9$,

$$\mu(G_A) = 10.6235 = (110+\gamma)^{1/2}$$

For $n = 10$,

$$\mu(G_A) = 11.6569 = (132+\gamma)^{1/2}$$

For $n = 11$,

$$\mu(G_A) = 12.6847 = (156+\gamma)^{1/2}$$

By observing the pattern, we find that $\gamma \simeq (n-6.2)$ for all $n > 6$.

$$\mu(G_A) \simeq \left(n^2+3n+2+(n-6.2)\right)^{1/2}$$

$$= \left(n^2+4n-4.2\right)^{1/2}$$

for $n > 6$.

In the following Theorem, we determine the values of κ and κ' in order to obtain the greatest lower bound and the least upper bound for $\mu\left(G_A\right)$.

Theorem 2. For the Agave graph G_A,

$$\mu\left(G_A\right) > \left(2\delta^2+2(2m-\eta\delta)+(n+2)\eta-\kappa\right)^{1/2} \tag{7}$$

where $5 \leq \kappa \leq (n^2-4)$ and the greatest lower bound is attained when $\kappa = 5$ also

$$\mu\left(G_A\right) < \left(2\Delta^2+2(2m-\Delta)-n\eta+\kappa'\right)^{1/2} \tag{8}$$

where $4 \leq \kappa' \leq n(n-2)$ and the least upper bound is attained when $\kappa' = 4$.

Proof. Consider Lemma 7,

$$\mu(G_A) < \left(2\delta^2 + 2(2m - \eta\delta) + (n+2)\eta\right)^{1/2}$$

Reducing this expression in terms of n, we have

$$\mu(G_A) \simeq \left(n^2 + 4n - 4.2\right)^{1/2} < \left(n^2 + 4n\right)^{1/2}$$

where this is true for all $n > 6$. We determine a positive number κ such that

$$\mu(G_A) \simeq \left(n^2 + 4n - 4.2\right)^{1/2} > \left(2\delta^2 + 2(2m - \eta\delta) + (n+2)\eta - \kappa\right)^{1/2}$$

From the above inequality, we have

$$\left(n^2 + 4n - 4.2\right)^{1/2} > \left(n^2 + 4n - \kappa\right)^{1/2}$$

This implies that

$$\kappa > 4.2 \qquad \Rightarrow \kappa = 5$$

We can easily verify that the expression $\left(2\delta^2 + 2(2m - \eta\delta) + (n+2)\eta - 5\right)^{1/2}$ is the greatest lower bound. Consider

$$\mu(G_A) \simeq \left(2\delta^2 + 2(2m - \eta\delta) + (n+2)\eta - 5 + 1\right)^{1/2}$$

to be the greatest lower bound of $\mu(G_A)$. That is

$$\mu(G_A) \simeq \left(n^2 + 4n - 4.2\right)^{1/2} < \left(n^2 + 4n - 4\right)^{1/2}.$$

This is a contradiction for all $n > 6$.

Hence $\left(2\delta^2 + 2(2m - \eta\delta) + (n+2)\eta - 5\right)^{1/2}$ is the greatest lower bound. We observe that the greatest lower bound occurs when $\kappa = 5$ and the loose lower bound is attained when $\kappa = (n^2 - 4)$.

Similarly, in order to determine the least upper bound for $\mu(G_A)$, we consider the upper bound given in equation (3) of Theorem 1. Consider Lemma 6,

$$\mu(G_A) > \left(2\Delta^2 + 2(2m - \Delta) - n\eta\right)^{1/2}$$

Reducing the above expression in terms of n, we have

$$\mu(G_A) \simeq \left(n^2 + 4n - 4.2\right)^{1/2} > \left(n^2 + 4n - 8\right)^{1/2}$$

In order to determine the least upper bound, we need to find the value κ' such that

$$\mu(G_A) < \left(2\Delta^2 + 2(2m - \Delta) - n\eta + \kappa'\right)^{1/2} = \left(n^2 + 4n - 8 + \kappa'\right)^{1/2}$$

$$\mu(G_A) \simeq \left(n^2 + 4n - 4.2\right)^{1/2} < \left(n^2 + 4n - 8 + \kappa'\right)^{1/2}$$

This implies that

$$\kappa' > 3.8$$

$$\Rightarrow \kappa' = 4$$

Therefore

$$\mu(G_A) < \left(2\Delta^2 + 2(2m - \Delta) - n\eta + 4\right)^{1/2}$$

It can easily be shown that the above expression is the least upper bound.

Consider

$$\mu(G_A) < \left(2\Delta^2 + 2(2m - \Delta) - n\eta + 3\right)^{1/2} = \left(n^2 + 4n - 5\right)^{1/2}$$

to be the least upper bound. From this, we notice that

$$\mu(G_A) \simeq \left(n^2 + 4n - 4.2\right)^{1/2} < \left(n^2 + 4n - 5\right)^{1/2}$$

This is a contradiction. Therefore

$$\mu(G_A) < \left(2\Delta^2 + 2(2m - \Delta) - n\eta + 4\right)^{1/2}. \tag{9}$$

is the least upper bound.

Clearly, we observe that the least upper bound occurs when $\kappa' = 4$ and the loose upper bound is attained when $\kappa' = n(n-2)$.

We prove that the lower bound and the upper bound are the greatest lower bound and the least upper bound for $\mu(G_A)$ by making use of the relation between the parameters given in Remark 1.

Lemma 8. For the graph G_A, the lower bound and upper bound are improved as

$$(2\delta^2 + 2(2m - \eta\delta) + (n+2)\eta - 5)^{1/2} < \mu\left(G_A\right) \tag{10}$$

and

$$\left(2\Delta^2 + 2(2m - \Delta) - (n\eta - 4)\right)^{1/2} > \mu\left(G_A\right) \tag{11}$$

We verify that when $((n+2)\eta - 5)$ is added to the expression in the lower bound in equation (3) of Theorem 1, we get the greatest lower bound and when the term $(n\eta - 4)$ is subtracted from the expression in the upper bound in equation (3) of Theorem 1, we obtain the least upper bound for the signless Laplacian spectral radius $\mu\left(G_A\right)$.

Proof. Let us assume

$$L.B^* = \left(2\delta^2 + 2(2m - \eta\delta) + (\epsilon_1 + 1)\right)^{1/2} = (2\delta^2 + 2(2m - \eta\delta) + ((n+2)\eta - 5 + 1))^{1/2}$$

to be the greatest lower bound of $\mu\left(G_A\right)$ and let

$$U.B = (2\Delta^2 + 2(2m - \Delta) - (n\eta - 4))^{1/2}.$$

By making use of the relationship between the parameters given in Remark 1, we observe that $(L.B^*)^2 - (U.B)^2 =$

$$(2\delta^2 + 2(2m - \eta\delta) + (n+2)\eta - 4) - (2\Delta^2 + 4m - 2\Delta - (n\eta - 4))$$

$$= 2(\delta + \Delta)(\delta - \Delta) - 2\eta\delta + n\eta + 2\eta - 4 + 2(\eta + 1) + n\eta - 4$$

$$= 2(n+1)(1-\eta) - 4\eta + n\eta + 2\eta + 2\eta - 4 + 2 + n\eta - 4$$

$$= 2(n - \eta) - 2n\eta + 4 + 2n\eta - 8 = 0$$

This implies $L.B^* = U.B$ which is a contradiction.

Therefore the following expression is the greatest lower bound

$$(2\delta^2 + 2(2m - \eta\delta) + (n+2)\eta - 5)^{1/2} < \mu\left(G_A\right).$$

Similarly, we prove that $\epsilon_2 = (n\eta - 4)$ is the maximum integral value required to obtain the least upper bound.

Let us assume that

$$U.B^* = \left((2\Delta^2 + 2(2m - \Delta) - (\epsilon_2 + 1)\right)^{1/2} = \left((2\Delta^2 + 2(2m - \Delta) - (n\eta - 3)\right)^{1/2}$$

is the least upper bound and let $L.B = (2\delta^2 + 2(2m - \eta\delta) + (n+2)\eta - 5)^{1/2}$.

Consider $(U.B^*)^2 - (L.B)^2 =$

$$(2\Delta^2 + 4m - 2\Delta - (n\eta - 3)) - (2\delta^2 + 2(2m - \eta\delta) + (n+2)\eta - 5)$$

$$= 2(\delta + \Delta)(\delta - \Delta) + 2\eta\delta - 2\Delta - (n\eta - 3) + (n+2)\eta - 5 = 0$$

Using the relationship between the parameters given in Remark 1, we observe that $U.B^* - L.B = 0$ which implies $U.B^* = L.B$, which is a contradiction. Therefore, we conclude that $U.B$ is the greatest upper bound. Therefore we have

$$L.B = \left(2\delta^2 + 2(2m - \eta\delta) + (n+2)\eta - 5\right)^{1/2} < \mu\left(G_A\right) <$$

$$\left(2\Delta^2 + 2(2m - \Delta) - (n\eta - 4)\right)^{1/2} = U.B. \tag{12}$$

Remark 2. By using the improved bounds derived in Theorem 2, for the graph $K_2 \bigtriangledown 5K_1$, with $n = 7$, $m = 11$, $\eta = 5$, $\Delta = 6$, $\delta = 2$ and $\mu(G_A) = 8.5311$, we have

$$8.4852 < \mu(G_A) < 8.544$$

4. Bounds of the Nordhaus-Gaddam type

In this section, we obtain the upper and the lower bounds on $\mu(G_A)+\mu(G_A^c)$ (Nordhaus-Gaddam type inequality) for the signless Laplacian spectral radius of the Agave graphs in terms of the maximum degree Δ, the minimum degree δ, the order n, and the size m of the graph G.

Theorem 3. Let G_A be an Agave graph with n vertices, m edges and η copies of the satellite graph K_1. Let Δ and δ be the maximum degree and minimum degree of G_A, respectively.

$$2\left(\frac{(4n^2-20n+40)+((n+2)\eta-\kappa)}{2}\right)^{1/2}$$
$$<\mu(G_A)+\mu(G_A^c)<2\left(\frac{(6n^2-22n+28)-(n\eta-\kappa')}{2}\right)^{1/2} \tag{13}$$

for $5\le\kappa\le(n^2-4)$ and $4\le\kappa'\le(n(n-2))$.

Proof. Consider the family of Agave graphs G_A. Let η be the number of satellites K_1 join with K_2 and let G_A^c be the complement having m^c edges, where

$$m^c=nC_2-m,\qquad \Delta^c=\delta^c=(n-3)$$

From equation (7) of Theorem 2, we consider the lower bound

$$\mu(G_A)>\left(2\delta^2+2(2m-\eta\delta)+(n+2)\eta-\kappa\right)^{1/2}$$

and the signless Laplacian spectral radius of complement of G_A

$$\mu(G_A^c)=\mu(K_{\eta-1})=(2(\Delta-\delta)^2+2(n(n-1)-2m-(n-3))^{1/2}$$

We have

$$\mu(G_A)+\mu(G_A^c)>\left(2\delta^2+2(2m-\eta\delta)+(n+2)\eta-\kappa\right)^{1/2}$$
$$+\left(2(\Delta-\delta)^2+2(n(n-1)-2m-(n-3)\right)^{1/2}=g(m) \tag{14}$$

Then

$$\frac{dg}{dm}<0$$

if and only if

$$\left\{\frac{2}{\sqrt{(2\delta^2+4m-2\eta\delta+(n+2)\eta-\kappa)}}\right\}>\left\{\frac{(-2)}{\sqrt{(2(\Delta-\delta)^2+2n(n-1)-4m-2(n-3))}}\right\}$$

By reducing Δ and δ in terms of n and solving for m, we have

$$m > \left(\frac{(4n^2 - 12n + 8) - ((n+2)\eta - \kappa)}{8}\right)$$

Substituting in (14), we get

$$\mu(G_A) + \mu(G_A^c) >$$

$$\left(2\delta^2 + 4\left(\frac{(4n^2 - 12n + 8) - ((n+2)\eta - \kappa)}{8}\right) - 2\eta\delta) + (n+2)\eta - \kappa\right)^{1/2}$$

$$+\left(2(\Delta - \delta)^2 + 2n(n-1) - 4\left(\frac{(4n^2 - 12n + 8) - ((n+2)\eta - \kappa)}{8}\right) - 2(n-3)\right)^{1/2}$$

Therefore

$$\mu(G_A) + \mu(G_A^c) > 2\left(\frac{(4n^2 - 20n + 40) + ((n+2)\eta - \kappa)}{2}\right)^{1/2}$$

for $5 \le \kappa \le (n^2 - 4)$.

(15)

Similarly, we derive the condition for the existence of the upper bound. We consider the upper bound given in equation (8) of Theorem 2

$$\mu(G_A) < \left(2\Delta^2 + 2(2m - \Delta) - (n\eta - \kappa')\right)^{1/2} \tag{16}$$

$$\mu(G_A^c) = \left(2(\Delta - \delta)^2 + 2(2m^c - \Delta^c)\right)^{1/2}$$

where $m^c = nC_2 - m$

$$\mu(G_A^c) = \left(2(\Delta - \delta)^2 + 2(n(n-1) - 2m - (n-3)\right)^{1/2} \tag{17}$$

$$\mu(G_A) + \mu(G_A^c) < \left(2\Delta^2 + 4m - 2\Delta - (n\eta - \kappa')\right)^{1/2}$$

$$+ \left(2(\Delta - \delta)^2 + 2n(n-1) - 4m - 2(n-3)\right)^{1/2} = g(m) \tag{18}$$

Then

$$\frac{dg}{dm} > 0$$

if and only if

$$\left\{\frac{2}{\sqrt{(2\Delta^2 + 4m - 2\Delta - (n\eta - \kappa'))}}\right\} < \left\{\frac{(-2)}{\sqrt{(2(\Delta - \delta)^2 + 2n(n-1) - 4m - 2(n-3))}}\right\}$$

Reducing Δ and δ in terms of n and solving for m, we get

$$m < \left(\frac{(2n^2 - 10n + 20) + (n\eta - \kappa')}{8}\right)$$

Substituting the value of m in (18), we get

$$\mu(G_A) + \mu(G_A^c) < 2\left(\frac{(6n^2 - 22n + 28) - (n\eta - \kappa')}{2}\right)^{1/2}$$

for $4 \leq \kappa' \leq (n(n-2))$.

Therefore

$$2\left(\frac{(4n^2 - 20n + 40) + ((n+2)\eta - \kappa)}{2}\right)^{1/2}$$
$$< \mu(G_A) + \mu(G_A^c) < 2\left(\frac{(6n^2 - 22n + 28) - (n\eta - \kappa')}{2}\right)^{1/2}$$

for $5 \leq \kappa \leq (n^2 - 4)$ and $4 \leq \kappa' \leq n(n-2)$.

Hence proved.

Remark 3. It can be easily verified from equation (13) of Theorem 3, for the graph $K_2 \bigtriangledown 5K_1$, where $n = 7$, $m = 11$, $\Delta = 6$, $\delta = 2$, $\eta = 5$, $\kappa = 5$ and $\kappa' = 4$, the bounds are

$$16.4924 < \mu(G_A) + \mu(G_A^c) < 16.553$$

5. Discussion

In this paper, we have utilized a new technique to improve the bounds and obtain the tight upper and lower bounds for the signless Laplacian spectral radius of Agave class of graphs. These bounds are in terms of the maximum degree Δ, the minimum degree δ, number of satellites η, n the number of vertices and m the number of edges of the graph G_A. They are exceptionally proximate to $\mu(G_A)$. Also, we have derived new improved upper and lower bounds for the Nordhaus-Gaddum type inequality. We anticipate that the technique used to obtain these bounds will be helpful in determining the tight lower and the upper bounds for the signless Laplacian spectral radius of any general class of graphs.

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References

[1] N Abreu, Justel C.M, and L. Markenzon. Integer Laplacian eigenvalues of chordal graphs. *Linear Algebra and its Applications*, 614:68–81, 2021.

[2] M Aouchiche and Hansen. P. A survey of Nordhaus–Gaddum type relations. *Discrete Applied Mathematics*, 161:466 – 546, 2013.

[3] Yanqing Chen and Ligong Wang. Sharp bounds for the largest eigenvalue of the signless Laplacian of a graph. *Linear Algebra and its Applications*, 433:908–913, 2010.

[4] Dragoš Cvetkovic, Peter Rowlinson, and Slobodan K. Simic. Signless Laplacians of finite graphs. *Linear Algebra and its Applications*, 423:155–171, 2007.

[5] X Duan and Zhou. Sharp bounds on the spectral radius of a nonnegative matrix. *Linear Algebra and its Applications*, 439(10):2961–2970, 2013.

[6] Ernesto Estrada and Michele Benzi. Core–satellite graphs: Clustering, assortativity, and spectral properties . *Linear Algebra and its Applications*, 517:30–52, 2017.

[7] Ernesto Estrada and Eusebio Vargas-Estrada. Distance-sum heterogeneity in graphs and complex networks. *Applied Mathematics and Computation*, 218:10393–10405, 2012.

[8] Yuan Hong and Jin-Long Shu. A sharp upper bound for the spectral radius of the Nordhaus-Gaddum type. *Discrete Mathematics*, 211:229–232, 2000.

[9] Shuchao Li and Yi Tian. Some bounds on the largest eigenvalues of graphs. *Applied Mathematics Letters*, 25:326–332, 2012.

[10] Hongying Lin, Biao Mo, Bo Zhou, and Weiming Weng. Sharp bounds for ordinary and signless Laplacian spectral radii of uniform hypergraphs. *Applied Mathematics and Computation*, 285:217–227, 2016.

[11] Huiqing Liu, Mei Lu, and Feng Tian. On the Laplacian spectral radius of a graph. *Linear Algebra and its Applications*, 376:135–141, 2004.

[12] X. Liu and P. Lu. Signless Laplacian spectral characterization of some joins. *The Electronic Journal of Linear Algebra*, 30:443–454, 2015.

[13] Feng Li-Hua and Yu Gui-Hai. The signless Laplacian spectral radius of graphs with given diameter. *Utilitas Mathematica*, 83:265-276, 2010.

[14] M. E. J. Newman. The structure and function of complex networks. *SIAM Rev*, 45:167–256, 2003.

[15] V. Nikiforov and X. Yuan. More eigenvalue problems of Nordhaus–Gaddum type. *Linear Algebra and its Applications*, 451:231 – 245, 2014.

[16] Vladimir Nikiforov. Eigenvalue problems of Nordhaus-Gaddum type. *Discrete Mathematics*, 307:774 –780, 2007.

[17] Csikvári. P. On a conjecture of V. Nikiforov. *Discrete Mathematics*, 309(13):4522–4526, 2009.

[18] Erzsebet Ravasz and Albert-Laszlo' Barabasi. Hierarchical organization in complex networks. *Physical Review*, E67:026–112, 2003.

[19] Lingsheng Shi. Bounds on the (Laplacian) spectral radius of graphs. *Linear Algebra and its Applications*, 422:755–770, 2007.

[20] Xu Wanyue and Zhongzhi Zhang. Optimal Scale-Free Small-World Graphs with Minimum Scaling of Cover Time. *ACM Transactions on Knowledge Discovery from Data*, 17:1–19, 2023.

[21] Zhang, Jing-Ming, Huang, Ting-Zhu, and Ji-Ming Guo. On the signless Laplacian spectral radius of bicyclic graphs with perfect matchings. *The Scientific World Journal*, 2014, 2014.