EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 16, No. 3, 2023, 1980-1990
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Note on Generalized Neighborhoods Structures in Fuzzy Bitopological Spaces

Ahlam Ahmed Alharbi[1,2], Adem Kilicman[2,*]
[1] *Department of Mathematics, Faculty of Science, Taibah University, Madinah 42353, Kingdom of Saudi Arabia*
[2] *Department of Mathematics and Statistics, Faculty of Science, Universiti Putra Malaysia, 43400 UPM Serdang, Selangor, Malaysia*

Abstract. This article's main aim is to study the concepts of the generalized neighborhood and generalized quasi-neighborhood in fuzzy bitopological spaces. It also introduces fundamental theorems for determining the relationships between them. Additionally, some significant examples were examined to demonstrate the significance of the interconnections, some theorems were also introduced to study some main properties of neighborhood structures. Finally, we also studied the concepts of closure, interior, and each of their critical theories and properties by generalized neighborhood systems in fuzzy bitopological spaces.

2020 Mathematics Subject Classifications: 03B52, 03E72, 54A40, 54E55, 57N40, 94D05
Key Words and Phrases: Fuzzy bitopological spaces ($fbts$), fuzzy generalized closed sets ($g-closed$), fuzzy closure operator (cl), fuzzy interior operator (int), fuzzy generalized neigborhood ($N^{g\varphi}$), and fuzzy generalized quasi neigborhood ($N^{g\varphi Q}$)

1. Introduction

In this project, we have prioritized our study on fuzzy bitopology, which derived from fuzzy topology that was first introduced in 1965 by the scientist Zadeh [8]. Following this, many researchers applied fundamental ideas on fuzzy settings from general topology and improved the concept of fuzzy topology. Such as Chang, in 1968 introduced some fuzzy concepts in fuzzy topology [4]. In addition, in 1989 Kandil introduced fuzzy bitopological spaces [1]. Also, generalized fuzzy closed groups were established in fuzzy topology in 1997 by Balasubramanian and Sundaram [6]. After that, many scientists applied the notion of a generalized closed set in fuzzy space and in 2005 El-Shafei introduced some applications of it [9]. Also, in 2009 Xuzhu Wang et al presented a book that contains all the basic operations in fuzzy science [12]. As Zahran and El-Maghrabi studied in 2011 some operations on it in fuzzy space [13]. Then in 2017 Benchalli et al studied delta

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v16i3.4808

Email addresses: aasehli@taibahu.edu.sa (Ahlam Ahmed Alharbi), akilic@upm.edu.my (Adem Kilicman)

generalized beta closed in topological spaces [3]. After one year, Kandil et al defined the concept of locally pairwise closed sets and studied some of their properties [7]. In 2019 Ramaboopathi and Dharmalingam introduced a new class of generalized closed sets in bitopological spaces [11], as in the same year Andal and Thiripurasundari introduced a new concept of fuzzy generalized *pi* closed in fuzzy bitopological spaces [2]. Finally, in 2021 Das et al introduced the idea of γ generalized fuzzy quasi neighborhood of a fuzzy point [5].

2. Preliminaries

In the following part, we go over important antecedent notions that are essential to the development of this paper.

Definition 1. *[10] Suppose the set X is not empty and the I sign represents the unit period $[0, 1]$, then the following defined as:*

(1) *an operator with X domain and I range is known as a fuzzy set E, where $E(x) \in (0, 1]$ when $x \in E$, and $E(x) = 0$ in case $x \notin E$.*

(2) *a set D is including E indicated via $E \subseteq D$ if $E(x) \leq D(x)$, whenever $x \in X$*

(3) *E and D combination indicated by $E \vee D$ if $(E \vee D)(x) = max\{E(x), D(x)\}$ $\forall$ $x \in X$.*

(4) *the intersection of E, D indicated by $E \wedge D$ if $(E \wedge D)(x) = min\{E(x), D(x)\}$ $\forall$ $x \in X$.*

(5) *the completeness of E denoted via E^c as $(E(x))^c = 1 - E(x)$, $\forall$ $x \in X$.*

The following definitions explain the meaning of fuzzy topology and fuzzy bitopological spaces.

Definition 2. *[10] A fuzzy topology of X is a class of fuzzy groups $\delta \in I$ which holds the coming three conditions:*

1. 0 *and* 1 *contained in δ, where $0(x) = 0$, $1(x) = 1$, whenever $x \in X$.*

2. *For any $E, D \in \delta$, $E \wedge D \in \delta$.*

3. *For any $(E_{i\in I}) \in \delta$, $\vee_{i\in I} E_i \in \delta$.*

The term "fuzzy topological space," or "fts," refers to the pair (X, δ).
The components of δ are named fuzzy open sets. If $F^c \in \delta$, then F is mean as fuzzy closed.
The collection including whole fuzzy closed groups in fuzzy topology δ denote by $\mathcal{F}_\delta$.

Definition 3. *[1] A fuzzy bitopological spaces, or fbts for short, (X, δ_1, δ_2) since X is not empty, δ_1, and δ_2 are fuzzy topological spaces on X. Over this dissertation X perform fuzzy bitopology (X, δ_1, δ_2), and Y to (Y, σ_1, σ_2), where $i \neq j$, and $i, j \in \{1, 2\}$.*

In the section which follows, the definitions of fuzzy set interiors and closings are covered.

Definition 4. *[10] Closing and internal of any fuzzy set M of (X, δ) are indicated also defined as follows:*

$$cl(M) = \wedge \{F : M \leq F, F^c \in \delta\}$$
$$int(M) = \vee \{O : O \leq M, O \in \delta\}, \quad respectively.$$

The closing, internal, and complements of M of X are indicated by $\delta_i - cl(M)$, $\delta_i - int(M)$, and M_i^c, respectively, with regard to fuzzy topology δ_i. Additionally, we designate the class of all fuzzy δ_j-closed by the mathematical symbol $\mathcal{F}_{\delta_j}$.

One of the work's core tenets is the definition of the fuzzy generalized closed set, which as following:

Definition 5. *[6] Any fuzzy group E of X is termed fuzzy generalised closed when closure E is subset of W, wherever E is subset of W, W is fuzzy open.*
i.e., E is fuzzy generalised closed in case of $cl(E) \leq W$, wherever $E \leq W$, W is fuzzy open.

Definition 6. *[10]*

(1) *A fuzzy point x_λ is claimed that quasi-coincident with E, shown by $x_\lambda \, q \, E$ if $\lambda > E^c(x)$, or $\lambda + E(x) > 1$ and x_λ is claimed does not quasi-coincident with E if $\lambda + E(x) \leq 1$ and we write $E \, \overline{q} \, x_\lambda$.*

(2) *E is claimed quasi-coincident with C indicated as $E \, q \, C$ if there exists $x \in X$ so that $E(x) > C^c(x)$ or $E(x) + C(x) > 1$, and E is claimed does not quasi-coincident with C if there exists $x \in X$ so that $E(x) + C(x) \leq 1$ and we write $E \, \overline{q} \, C$.*
If $E \, q \, C$ (resp, $E \, \overline{q} \, C$) is true, then E and C are quasi-coincident (resp, not quasi-coincident) with each other at x.

3. Generalized Neighborhoods Structures at Fuzzy Bitopological Spaces

This section introduces the idea of generalized neighborhoods concepts by using $(\in)$ relationship and quasi coincident concept (q) in fuzzy bitopological spaces and characterize it in terms of important theorems and some properties.

Definition 7. *A fuzzy subgroup E of fbts (X, δ_1, δ_2) is known as:*

(1) *Fuzzy $(i, j)-$generalized $\varphi-$closed (in sum, $(i, j) - g\varphi - closed$) if $\delta_j - \varphi - cl(E) \leq U$ where $E \leq U$, $U \in \delta_i$, and φ including the types (alpha (α), semi (s), pre (p), and beta (β)).*

(2) *The supplement of the fuzzy $(i, j) - g\varphi - closed$ set is referred to $(i, j) - g\varphi - open$ set in X.*

Remark 1. (1) *The universal set of all fuzzy $(i, j) - g\varphi-$open, and $(i, j) - g\varphi-$closed sets of fbts (X, δ_1, δ_2) is represented by $\mathcal{O}^{fg\varphi}_{(i,j)}$, $\mathcal{F}^{fg\varphi}_{(i,j)}$, and so forth.*

(2) *Also, the family of all $g\varphi-$open,and $g\varphi-$closed subsets of X pertaining to the fuzzy topology δ_i is indicated $\mathcal{O}_i^{fg\varphi}$, and $\mathcal{F}_i^{fg\varphi}$, $i = 1, 2$.*

Proposition 1. *A fuzzy group E in fbts (X, δ_1, δ_2) is fuzzy $(i, j) - g\varphi$-open $\iff$ $F \leq \delta_j - \varphi - int(E)$ wherever $F^c \in \delta_i$, $F \leq E$.*

Proof. Assume E is fuzzy $(i, j) - g\varphi-$open, $F^c \in \delta_i$, when $F \leq E$. Then $E^c \leq F^c$. As E^c is fuzzy $(i, j) - g\varphi-$closed, thus $\delta_j - \varphi - cl(E^c) = (\delta_j - \varphi - int(E))^c \leq F^c$ that indicates $F \leq \delta_j - \varphi - int(E)$.
Conversely, assume E is fuzzy set of X, $F \in \mathcal{F}_i$ so $F \leq \delta_j - \varphi - int(E)$, $F \leq E$. After adding the supplement to both sides, we find $(\delta_j - \varphi - int(E))^c \leq F^c$ so $E^c \leq F^c$ and F^c is fuzzy open in δ_i, thus E^c is fuzzy $(i, j) - g\varphi-$closed (Defention7). Hence E is fuzzy $(i, j) - g\varphi-$open.

Definition 8. *A fuzzy group E in fbts (X, δ_1, δ_2) is known as:*

(1) *Fuzzy $(i, j)-$generalized $\varphi-$neighborhood (shortly, $(i, j)-g\varphi-nbd$) of fuzzy singleton set x_r if $\exists$ fuzzy $(i, j) - g\varphi - open$ set C so $x_r \in C \leq E$.*
The family of all fuzzy $(i, j) - g\varphi - nbds$ of fuzzy singleton set x_r, will be denoted by $N^{g\varphi}_{(i,j)}(x_r)$.

(2) *Fuzzy $(i, j) -$ generalized $\varphi - Q -$ neighborhood (shortly, $(i, j) - g\varphi Q - nbd$) of fuzzy singleton x_r if $\exists$ fuzzy $(i, j) - g\varphi - open$ set C so $x_r \, q \, C \leq E$.*
The family of all fuzzy $(i, j) - g\varphi Q - nbds$ of fuzzy singleton x_r, will be denoted by $N^{g\varphi Q}_{(i,j)}(x_r)$.

Remark 2. *In general, every fuzzy $\delta - Q-$neighborhood of a fuzzy point does not include the point itself. The coming example show that:*

Example 1. *Assume $x_{0.7}$ is fuzzy point of $X = \{a, b, c\}$ and E is fuzzy set of X defined as $E(a) = 0.4$, $E(b) = 0.5$, $E(c) = 0.3$. Let $\delta = \{0, 1, E\}$ on X. Then $E \in N^Q_\delta(x_{0.7})$ but $0.7 \nleq E(x)$, and hence $x_{0.7} \notin E$ but $x_{1-0.7=0.3} \in E$.*

Corollary 1. *In fbts (X, δ_1, δ_2) every fuzzy $(i, j) - g\varphi Q - nbd$ of fuzzy point x_r of X is equivlant to $(i, j) - g\varphi - nbd$ of fuzzy point x_{1-r}.*

Theorem 1. (1) *Every fuzzy $\delta_j - nbd$ of fuzzy point x_r is fuzzy $(i, j) - g - nbd$ of x_r.*

(2) *Every fuzzy $(i, j) - g - nbd$ of fuzzy point x_r is fuzzy $(i, j) - g\alpha - nbd$ of x_r.*

(3) *Every fuzzy $(i, j) - g\alpha - nbd$ of x_r is fuzzy $(i, j) - gs - nbd$ and $(i, j) - gp - nbd$ of x_r.*

(4) *Every fuzzy $(i, j) - gs - nb$ or $(i, j) - gp - nbd$ of x_r is fuzzy $(i, j) - g\beta - nbd$ of x_r.*

Proof.

(1) Suppose that $E \in N_j(x_r)$, thus $\exists C \in \delta_j$, so $x_r \in C \leq E$. As every $\delta_j - open$ set is $(i, j) - g - open$ set, then $\exists C$ is fuzzy $(i, j) - g - open$ set, so $x_r \in C \leq E$, and hence $E \in N^g_{(i,j)}(x_r)$.

(2) Suppose that $E \in N^{g}_{(i,j)}(x_r)$, thus $\exists C$ is fuzzy $(i,j) - g - open$ set, so $x_r \in C \leq E$, and since every fuzzy $(i,j) - g - open$ is fuzzy $(i,j) - g\alpha - open$, then $\exists C$ is fuzzy $(i,j) - g\alpha - open$ set, so $x_r \in C \leq E$, and hence $E \in N^{g\alpha}_{(i,j)}(x_r)$.

(3) Suppose that $E \in N^{g\alpha}_{(i,j)}(x_r)$, thus $\exists C$ is fuzzy $(i,j) - g\alpha - open$ set, so $x_r \in C \leq E$, and since every fuzzy $(i,j) - g\alpha - open$ is fuzzy $(i,j) - gs - open$ and $(i,j) - gp - open$, then $\exists C$ is fuzzy $(i,j) - gs - open$ and $(i,j) - gp - open$ set, so $x_r \in C \leq E$, and hence $E \in N^{gs}_{(i,j)}(x_r)$, and $E \in N^{gp}_{(i,j)}(x_r)$.

(4) Suppose that $E \in N^{gs}_{(i,j)}(x_r)$, or $E \in N^{gp}_{(i,j)}(x_r)$, thus $\exists C$ is fuzzy $(i,j) - gs - open$ or $(i,j) - gp - open$ set, so $x_r \in C \leq E$, and since every fuzzy $(i,j) - gs - open$ or $(i,j) - gp - open$ is fuzzy $(i,j) - g\beta - open$, then $\exists C$ is fuzzy $(i,j) - g\beta - open$ set, so $x_r \in C \leq E$, and hence $E \in N^{g\beta}_{(i,j)}(x_r)$.

Remark 3. *In fbts (X, δ_1, δ_2) every fuzzy $N^{gs}_{(i,j)}(x_r)$, and $N^{gp}_{(i,j)}(x_r)$ are independents.* *The following example show that if $X = \{a, b, c\}$, $\delta_1 = \{0, 1, E\}$, and $\delta_2 = \{0, 1, C, D\}$.* *As $E_{a,b,c} = \{0.7, 0.5, 0.6\}, C_{a,b,c} = \{0.5, 0.4, 0.3\}$, and $D_{a,b,c} = \{0.4, 0.3, 0.2\}$, then $\exists$ $S_{a,b,c} = \{0.5, 0.5, 0.6\} \in N^{gs}_{(i,j)}(x_r)$, but $S \notin N^{gp}_{(i,j)}(x_r)$, as $E^c \leq S$, but $E^c \nleq \delta_2 - p - int(S) = C$.* *On other hand for the same topologies above if $E_{a,b,c} = \{0.3, 0.5, 0.4\}, C_{a,b,c} = \{0.6, 0.8, 0.9\}$, and $D_{a,b,c} = \{0.4, 0.3, 0.2\}$, then $\exists$ $S_{a,b,c} = \{0.8, 0.6, 0.5\} \in N^{gp}_{(i,j)}(x_r)$, but $S \notin N^{gs}_{(i,j)}(x_r)$, as $E^c \leq S$, but $E^c \nleq \delta_2 - s - int(S) = C^c$.*

The following Figure explaining the relation between nbds structures of all cases.

$$\begin{array}{ccccccccccccccc}
N_j(x_{1-r}) & \Leftrightarrow & N^{Q}_{j}(x_r) & \rightarrow & N^{gQ}_{(i,j)}(x_r) & \Leftrightarrow & N^{g}_{(i,j)}(x_{1-r}) & \rightarrow & N^{g\alpha}_{(i,j)}(x_{1-r}) & \Leftrightarrow & N^{g\alpha Q}_{(i,j)}(x_r) & \rightarrow & N^{gpQ}_{(i,j)}(x_r) & \Leftrightarrow & N^{gp}_{(i,j)}(x_{1-r}) \\
 & & & & & & & & & \downarrow & & & & \downarrow & \\
 & & & & & & & & N^{gs}_{(i,j)}(x_{1-r}) & \Leftrightarrow & N^{gsQ}_{(i,j)}(x_r) & \rightarrow & N^{g\beta Q}_{(i,j)}(x_r) & \Leftrightarrow & N^{g\beta}_{(i,j)}(x_{1-r})
\end{array}$$

Figure 1: Explain the relations between all types of fuzzy $N^{g\varphi}_{(i,j)}(x_r)$, and all types of $N^{g\varphi Q}_{(i,j)}(x_r)$.

The reversal of the prior relationships in Figure (1) is wrong in fbts (X, δ_1, δ_2), as demonstrated by the instances that follow: Suppose $X = \{a, b, c\}$, $\delta_1 = \{0, 1, E\}$, and $\delta_2 = \{0, 1, H, R\}$.

Example 2. *If $E_{a,b,c} = \{0.7, 0.5, 0.4\}$, $H_{a,b,c} = \{0.7, 0.6, 0.5\}$, $R_{a,b,c} = \{0.2, 0.4, 0.3\}$, $S_{a,b,c} = \{0.6, 0.6, 0.5\}$. The conclusion is $S \in N^{g}_{(1,2)}(x_r)$, but never $S \notin N_2(x_r)$.*

The next example clear that $N^{g\alpha}_{(1,2)}(x_r) \nRightarrow N^{g}_{(1,2)}(x_r)$.

Example 3. *Suppose $E_{a,b,c} = \{0.5, 0.4, 0.3\}$, $H_{a,b,c} = \{0.7, 0.5, 0.4\}, R_{a,b,c} = \{0.4, 0.3, 0.2\}$, $S_{a,b,c} = \{0.7, 0.6, 0.8\}$. The conclusion is $S \in N^{g\alpha}_{(1,2)}(x_r)$, but never $S \notin N^{g}_{(1,2)}(x_r)$.*

In the coming example we show that $N^{gs}_{(1,2)}(x_r) \nRightarrow N^{g\alpha}_{(1,2)}(x_r)$.

Example 4. *Suppose* $E_{a,b,c} = \{0.7, 0.5, 0.5\}$, $H_{a,b,c} = \{0.5, 0.4, 0.3\}, R_{a,b,c} = \{0.4, 0.3, 0.2\}$, $S_{a,b,c} = \{0.5, 0.5, 0.5\}$. *The conclusion is* $S \in N^{gs}_{(1,2)}(x_r)$, *but never* $S \notin N^{g\alpha}_{(1,2)}(x_r)$.

The following example clear that $N^{gp}_{(1,2)}(x_r) \nRightarrow N^{g\alpha}_{(1,2)}(x_r)$.

Example 5. *Suppose* $E_{a,b,c} = \{0.7, 0.5, 0.4\}, H_{a,b,c} = \{0.6, 0.8, 0.8\}, R_{a,b,c} = \{0.4, 0.3, 0.2\}$, $S_{a,b,c} = \{0.8, 0.6, 0.7\}$. *The conclusion is* $S \in N^{gp}_{(1,2)}(x_r)$, *but never* $S \notin N^{g\alpha}_{(1,2)}(x_r)$.

The example follow indicates that $N^{g\beta}_{(1,2)} \nRightarrow N^{gs}_{(1,2)}(x_r)$.

Example 6. *Suppose* $E_{a,b,c} = \{0.5, 0.7, 0.6\}$, $H_{a,b,c} = \{0.6, 0.5, 0.4\}, R_{a,b,c} = \{0.4, 0.3, 0.2\}$, $S_{a,b,c} = \{0.5, 0.5, 0.6\}$. *The conclusion is* $S \in N^{g\beta}_{(1,2)}(x_r)$, *but never* $S \notin N^{gs}_{(1,2)}(x_r)$.

As well, the coming example demonstrates that $N^{g\beta}_{(1,2)} \nRightarrow N^{gp}_{(1,2)}(x_r)$.

Example 7. *Suppose* $E_{a,b,c} = \{0.5, 0.7, 0.6\}, H_{a,b,c} = \{0.4, 0.6, 0.7\}, R_{a,b,c} = \{0.3, 0.4, 0.5\}$, $S_{a,b,c} = \{0.5, 0.5, 0.6\}$. *The conclusion is* $S \in N^{g\beta}_{(1,2)}(x_r)$, *but never* $S \notin N^{gp}_{(1,2)}(x_r)$.

Definition 9. *A fuzzy singleton set* x_r *is named fuzzy* $(i,j)-$ *generalized* $\varphi -$ *cluster point of fuzzy subset* E *in fbts* (X, δ_1, δ_2) *if and only if all* $H \in N^{g\varphi Q}_{(i,j)}$ *of* $x_r, H\, q\, E$.

The following theorem examines some of the generalized neighborhood characteristics.

Theorem 2. *If* (X, δ_1, δ_2) *is fbts. Next, we find:*

(1) $\forall\, x_r \in X, N^{g\varphi}_{(i,j)}(x_r) \neq \phi$.

(2) $\forall H \in N^{g\varphi}_{(i,j)}(x_r), x_r \in H$.

(3) *when* $H, R \in N^{g\varphi}_{(i,j)}(x_r)$, *then* $H \wedge R \in N^{g\varphi}_{(i,j)}(x_r)$.

(4) *when* $H \in N^{g\varphi}_{(i,j)}(x_r)$ *and* $H \leq R$, *then* $R \in N^{g\varphi}_{(i,j)}(x_r)$.

(5) *when* $H \in N^{g\varphi}_{(i,j)}(x_r)$, *then there exists* $R \in N^{g\varphi}_{(i,j)}(x_r)$ *such that* $R \leq H$ *and* $R \in N^{g\varphi}_{(i,j)}(x_h)$, $\forall\, x_h \in R$.

Proof.

From Definition 8 we conclude the prove of (1) and (2).

(3) Assume $H, R \in N^{g\varphi}_{(i,j)}(x_r)$. Thus $\exists\, S, T$ are fuzzy $(i,j) - g\varphi-$open, so $x_r \in S, x_r \in T$, then
$x_r \in S \wedge T$. As S, T are fuzzy $(i,j) - g\varphi-$open, and hence we find $S \wedge T$ is fuzzy $(i,j) - g\varphi-$open, $S \wedge T \leq H \wedge R$. As a result of that, $A \wedge B \in N^{g\varphi}_{(i,j)}(x_r)$.

(4) Assume $H \in N^{g\varphi}_{(i,j)}(x_r)$. So $\exists\, S$ is fuzzy $(i,j)-g\varphi-$open and $x_r \in S \leq H$ but $H \leq R$, then $x_r \in S \leq R$. As a result of that, $R \in N^{g\varphi}_{(i,j)}(x_r)$.

(5) Assume $H \in N^{g\varphi}_{(i,j)}(x_r)$, so there exists fuzzy $(i,j)-g\varphi-$open set R, $x_r \in R \leq H$. As R is fuzzy $(i,j)-g\varphi-$open, then $R \in N^{g\varphi}_{(i,j)}(x_r)$. Since it is an $(i,j)-g\varphi-nbd$ of each of it is point. As a result of that, $B \in N^{g\varphi}_{(i,j)}(x_h), \forall x_h \in B$.

The following theorem examines some of the generalized Q-neighborhood properties.

Theorem 3. *When* (X, δ_1, δ_2) *is fbts. Then we have:*

(1) $\forall\, x_r \, q\, or \in X, N^{g\varphi Q}_{(i,j)}(x_r) \neq \phi$.

(2) $\forall E \in N^{g\varphi Q}_{(i,j)}(x_r), x_r \, q\, E$.

(3) *when* $E, T \in N^{g\varphi Q}_{(i,j)}(x_r)$, *then* $E \wedge T \in N^{g\varphi Q}_{(i,j)}(x_r)$.

(4) *when* $E \in N^{g\varphi Q}_{(i,j)}(x_r)$, *and* $E \leq T$, *then* $T \in N^{g\varphi Q}_{(i,j)}(x_r)$.

(5) *when* $E \in N^{g\varphi Q}_{(i,j)}(x_r)$, *then* $\exists\; T \in N^{g\varphi Q}_{(i,j)}(x_r)$ *so* $T \leq E$, *and* $T \in N^{g\varphi Q}_{(i,j)}(x_h)\,\forall\, x_h \in T$.

Proof. It resembles the earlier Theorem 2 proof.

Using the above-mentioned novel notion of fuzzy neighbourhood and quasi-neighborhood structure, we introduced the study of the degree of affiliation of a fuzzy element to fuzzy generalised closure in the subsequent theorem.

Theorem 4. *If* E *is fuzzy set and* x_r *is fuzzy point of fbts* (X, δ_1, δ_2), *then the following propositions are correct:*

(1) $x_r \in (i,j)-g\varphi-cl(E) \iff \forall\, T \in N^{g\varphi Q}_{(i,j)}(x_r), T\, q\, E$.

(2) $x_r \in (i,j)-g\varphi-cl(E) \iff \forall\, T \in N^{g\varphi}_{(i,j)}(x_{1-r}), T\, q\, E$.

(3) *If* E *is fuzzy* $(i,j)-g\varphi-$*closed, and hence* $\delta_i - cl(x_r)\, q\, E$ *holds* $\forall\, x_r\, q\, \delta_j - \varphi - cl(E)$.

Proof.

(1)

$$
\begin{aligned}
Let\ x_r \in (i,j)-g\varphi-cl(E) \ &\Leftrightarrow \forall\, F\, is\, fuzzy\, (i,j)-g\varphi-closed, E \leq F, r \leq F(x)\\
&\Leftrightarrow \forall\, F^c\, is\, fuzzy\, (i,j)-g\varphi-open,\ F^c \leq E^c, F^c(x) \leq 1-r\\
&\Leftrightarrow \forall\, T\, is\, fuzzy\, (i,j)-g\varphi-open,\ T \leq E^c, T(x) \leq 1-r\\
&\Leftrightarrow \forall\, T\, is\, fuzzy\, (i,j)-g\varphi-open,\ 1-r < T(x) \Rightarrow T \nleq E^c\\
&\Leftrightarrow \forall\, T\, is\, fuzzy\, (i,j)-g\varphi-open,\ x_r\, q\, T, T\, q\, E\\
&\Leftrightarrow \forall\, T \in N^{g\varphi Q}_{(i,j)}(x_r), T\, q\, E.
\end{aligned}
$$

(2) From (1) we have $x_r \in (i,j) - g\varphi - cl(E) \Leftrightarrow \forall T \in N^{g\varphi Q}_{(i,j)}(x_r), T\, q\, E$. So, we need to show that $T \in N^{g\varphi}_{(i,j)}(x_{1-r}) \Leftrightarrow T \in N^{g\varphi Q}_{(i,j)}(x_r)$. Assume $T \in N^{g\varphi}_{(i,j)}(x_{1-r})$. After that, $\exists$ fuzzy $(i,j) - g\varphi - open$ set V so $x_{1-r} \in V \leq T$, then $x_r\, q\, V \leq T$. As a result of that, $T \in N^{g\varphi Q}_{(i,j)}(x_r)$.

In the opposite direction, assume $T \in N^{g\varphi Q}_{(i,j)}(x_r)$. Thus $\exists$ fuzzy $(i,j) - g\varphi - open$ set V so $x_r\, q\, V \leq T$, and hence $x_{1-r} \in V \leq T$. As a result of that, $T \in N^{g\varphi}_{(i,j)}(x_{1-r})$.

(3) Assume E be fuzzy $(i,j) - g\varphi-$closed. Suppose there $\exists$ fuzzy point x_r so $x_r\, q\, \delta_j - \varphi - cl(E)$, but $\delta_i - cl(x_r)\, \overline{q}\, E$. Thus $E \leq (\delta_i - cl(x_r))^c$. As E is fuzzy $(i,j) - g\varphi-$closed, thus

$\delta_j - \varphi - cl(E) \leq (\delta_i - cl(x_r))^c$, hence $\delta_j - \varphi - cl(E)\, \overline{q}\, \delta_i - cl(x_r)$. As $x_r \in \delta_i - cl(x_r)$, thus $x_r\, \overline{q}\, \delta_j - \varphi - cl(E)$ that is a contradiction. As a result of that, $\delta_i - cl(x_r)\, q\, E$ holds $\forall\, x_r\, q\, \delta_j - \varphi - cl(E)$.

Corollary 2. *If E is fuzzy $(i,j) - g\varphi-$closed, and x_r is fuzzy point in fbts (X, δ_1, δ_2), then $\delta_i - cl(x_r)\, q\, E$ holds $\forall\, x_r\, q\, \delta_j - \beta - cl(E)$.*

In the theory that follows, we studied the most fundamental generalized closure characteristics and demonstrated them using new neighborhood structure notions.

Theorem 5. *If E, and T are fuzzy subsets of fbts (X, δ_1, δ_2), thus the following arguments are correct:*

(1) *0, and 1 are fuzzy $(i,j) - g\varphi - closed$.*

(2) *when $E \leq T$, then $(i,j) - g\varphi - cl(E) \leq (i,j) - g\varphi - cl(T)$.*

(3) *$E \leq (i,j) - g\varphi - cl(E)$, $\forall$ fuzzy set $E \in I^X$.*

(4) *when E is fuzzy $(i,j) - g\varphi - closed$, then $(i,j) - g\varphi - cl(E) = E$. The converse is false, as the intersection of fuzzy $(i,j) - g\varphi-$closed sets need not be fuzzy $(i,j) - g\varphi-$closed.*

(5) *$(i,j) - g\varphi - cl((i,j) - g\varphi - cl(E)) = (i,j) - g\varphi - cl(E)$.*

(6) *when v is $(i,j) - g\varphi - open$, then $v\, q\, E \iff v\, q\, (i,j) - g\varphi - cl(E)$.*

(7) *$(i,j) - g\varphi - cl(E) \vee (i,j) - g\varphi - cl(T) \leq (i,j) - g\varphi - cl(E \vee T)$.*

Proof. By using Definition 7 and Theorem 4 we can easily proved (1), (2), (3), and (4).

(5) Assume x_r is fuzzy point with $x_r \notin (i,j) - g\varphi - cl(E)$. After that, $\exists V \in N^{g\varphi Q}_{(i,j)}(x_r)$ so $x_r\, q\, V$, $V\, \overline{q}\, E$, then $\exists U$ is fuzzy $(i,j) - g\varphi - open$ so $x_r\, q\, U \leq V$ and $U\, \overline{q}\, E$. Thus from

(6) $U\, \overline{q}\, (i,j) - g\varphi - cl(E)$. As $\exists U$ is fuzzy $(i,j) - g\varphi - open$ so $x_r\, q\, U$ and $U\, \overline{q}\, (i,j) - g\varphi - cl(E)$. Then $x_r \notin (i,j) - g\varphi - cl((i,j) - g\varphi - cl(E))$, after that

$(i,j) - g\varphi - cl((i,j) - g\varphi - cl(E)) \leq (i,j) - g\varphi - cl(E)$. But

$(i,j) - g\varphi - cl(E) \leq (i,j) - g\varphi - cl((i,j) - g\varphi - cl(E))$. As a result of that,

$(i,j) - g\varphi - cl(E) = (i,j) - g\varphi - cl((i,j) - g\varphi - cl(E))$.

(6) Sufficiency, assume $V\,q\,E$. After that, $E \leq V^c, V^c$ is fuzzy $(i,j)-g\varphi-closed$, then by applying $(i,j)-g\varphi-clouser$ for all sides and from (5) we find $V\,\overline{q}\,(i,j)-g\varphi-cl(E)$. As a result of that, $V\,q\,E \iff V\,q\,(i,j)-g\varphi-cl(E)$.

(7) As $E \leq (E \vee T)$, and $T \leq (E \vee T)$, then $(i,j)-g\varphi-cl(E) \vee (i,j)-g\varphi-cl(T) \leq (i,j)-g\varphi-cl(E \vee T)$.

From the relationship between closure, interior, complement, and Theorem 5 we conclude the following:

Theorem 6. *If E and T are fuzzy subsets of fbts (X, δ_1, δ_2), then the coming statements are correct:*

(1) *0, and 1 are fuzzy $(i,j)-g\varphi-open$.*

(2) *when $E \leq T$, then $(i,j)-g\varphi-int(E) \leq (i,j)-g\varphi-int(T)$.*

(3) *$(i,j)-g\varphi-int(E) \leq E$, $\forall$ fuzzy set $E \in I^X$.*

(4) *when E is fuzzy $(i,j)-g\varphi-open$, then $(i,j)-g\varphi-int(E) = E$. The converse is false, as the combination of fuzzy $(i,j)-g\varphi-open$ sets not necessary to be fuzzy $(i,j)-g\varphi-open$.*

(5) *$(i,j)-g\varphi-int((i,j)-g\varphi-int(E)) = (i,j)-g\varphi-int(E)$.*

(6) *when v is $(i,j)-g\varphi-closed$, then $v\,\overline{q}\,E \iff v\,\overline{q}\,(i,j)-g\varphi-int(E)$.*

(7) *$(i,j)-g\varphi-int(E \wedge T) \leq (i,j)-g\varphi-int(E) \wedge (i,j)-g\varphi-int(T)$.*

Theorem 7. *If x_r is fuzzy point, and E is fuzzy subset of fbts (X, δ_1, δ_2), then $x_r \in (i,j)-g\varphi-int(E) \iff \exists$ fuzzy $(i,j)-g\varphi-open$ set G, so $x_r \in G \leq E$.*

Theorem 8. *Suppose E is fuzzy set in fbts (X, δ_1, δ_2). If E is fuzzy $(i,j)-g\varphi-open$, then $E \in N^{g\varphi}_{(i,j)}(x_r)$ for each $x_r \in E$.*

Theorem 9. *If (X, δ_1, δ_2) is fbts, E is fuzzy $(i,j)-g\varphi-closed$, and $E \leq T \leq \delta_j-\varphi-cl(E)$, then T is fuzzy $(i,j)-g\varphi-closed$.*

Proof. Assume $T \leq U$, and U is fuzzy open of δ_i. As $E \leq T$, thus $E \leq U$, after that $\delta_j-\varphi-cl(E) = \delta_j-\varphi-cl(T)$, which implies $\delta_j-\varphi-cl(T) \leq U$. As a result of that, T is fuzzy $(i,j)-g\varphi-closed$.

From the above we conclude the following:

Corollary 3. *Assume (X, δ_1, δ_2) is fbts, E is fuzzy $(i,j)-g\varphi-closed$, and $E \leq T \leq \delta_j-\beta-cl(E)$. Then T is fuzzy $(i,j)-g\varphi-closed$.*

Corollary 4. *Assume (X, δ_1, δ_2) is fbts, E is fuzzy $(i,j)-g\varphi-open$, and $\delta_j-\varphi-int(E) \leq T \leq E$. Then T is fuzzy $(i,j)-g\varphi-open$.*

Corollary 5. *Assume (X, δ_1, δ_2) is fbts, E is fuzzy $(i, j) - g\varphi - open$, and $\delta_j - \beta - int(E) \leq T \leq E$. Then T is fuzzy $(i, j) - g\varphi - open$.*

The following important study demonstrates when equivalence between the types of generalized closed sets in fuzzy bitopology and types of fuzzy sets from one topology is attained.

Theorem 10. *In fbts (X, δ_1, δ_2) the following statements are equivalents:*

(i) $\delta_i \subseteq \mathcal{F}^{f\varphi}_{(X,\delta_j)}$

(ii) *All fuzzy groups of X are fuzzy $(i, j) - g\varphi - closed$.*

Proof.

$(i) \to (ii)$ Assume E is fuzzy subset of X, so $E \leq U \in \delta_i$. Then from (i) we find $U \in \mathcal{F}^{f\varphi}_{(X,\delta_j)}$, after that $\delta_j - \varphi - cl(E) \leq U$. As a result of that, E is fuzzy $(i, j) - g\varphi - closed$.

$(ii) \to (i)$ Let E be fuzzy $(i, j) - g\varphi - closed$, $E \in \delta_i$. Since $E \leq E$, then $\delta_j - \varphi - cl(E) \leq E$, thus E is fuzzy $\delta_j - \varphi - closed$. Therefore $\delta_i \subseteq \mathcal{F}^{f\varphi}_{(X,\delta_j)}$.

From the above and the complenent relation we conclude the following:

Corollary 6. *In fbts (X, δ_1, δ_2) the following statements are equivalents:*

(1) $\mathcal{F}_{\delta_i} \subseteq \mathcal{O}^{f\varphi}_{(X,\delta_j)}$

(2) *All fuzzy subset of X is fuzzy $(i, j) - g\varphi - open$.*

Corollary 7. *Assume E, and T are fuzzy $(i, j) - g\varphi - closed$ sets in fbts (X, δ_1, δ_2) with $E \vee \delta_i - int(T) = T \vee \delta_i - int(E) = 1$, then $E \wedge T$ is fuzzy $(i, j) - g\varphi - closed$.*

4. Conclusion

In this study, we introduced and studied the definition of some types of generalized neighborhood and generalized quasi-neighborhood ideas fuzzy bitopology space, and we prove some relations and inclusion relation between them by listing some examples, then applied them to, closure, interior, and studied some key properties of them.

Acknowledgements

The authors would like to thank the referee(s) for their valuable comments.

References

[1] A. Kandil, Biproximities and fuzzy bitopological spaces. *Simon Stevin*, (1989), pp. 45-66.

[2] Andal, M., and Thiripurasundari, V. Fuzzy Generalized π Closed Set in Fuzzy Topological Spaces. *Journal of Information and Computational Science*, (2019), pp. 1548–7741.

[3] Benchalli, S. S., Patil, P. G., Toranagatti, J. B., and Vighneshi, S. R. A New Class of Generalized Closed Sets in Topological Spaces. *Global Journal of Pure and Applied Mathematics*, (2017), pp. 331–345.

[4] C. Chang, Fuzzy topological spaces. *J. Math. Anal Appl*, (1968), pp. 182–190.

[5] Das, B., Bhattacharya, B., Chakraborty, J., and Tripathy, B. C. Generalized fuzzy closed sets in a fuzzy bitopological space via γ-open sets. *Afrika Matematika*, (2021), pp. 333–345.

[6] G. Balasubramanian and P. Sundaram, On some generalizations of fuzzy continuous functions. *Fuzzy Sets and Systems*, (1997), pp. 93–100.

[7] Kandil, A., Tantawy, O., El-Sheikh, S., and Shalaby, E. Generalized Locally Pairwise Closed Sets on Bitopological Spaces and Some of Its Properties. *Journal of the Egyptian Mathematical Society*, (2018), 116–126.

[8] L. Zadeh, Fuzzy sets, *Information and Control*, (1965), pp. 338–353.

[9] M. El-Shafei, Some applications of generalized closed sets in fuzzy topological space. *Kyngpook Math*, (2005), pp. 13–19.

[10] N. Palaniappan, Fuzzy topology. *Alpha Science International Ltd*, (2002), pp. 1–177.

[11] Ramaboopathi, M., and Dharmalingam, K. M. On $(1,2)^*$-$\check{}$g-closed sets in bitopological spaces. *Malaya Journal of Matematik*, (2019), pp. 463–467.

[12] Xuzhu Wang, Da Ruan, and Etienn E. Kerre. Mathematics of Fuzziness-Basic Issues. *Springer Nature*, (2009).

[13] Zahran, A. M and El-Maghrabi, A. I. Generalized-Operations on Fuzzy Topological Spaces. *Abstract and Applied Analysis, Hindawi*, (2011), pp. 1–12.