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Atomic Solution of Certain Inverse Problems

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Abstract. In this note we find atomic solution for certain degenerate and non-degenerate inverse problems. The main idea of the proofs are based on theory of tensor product of Banach spaces.

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1. Introduction

Let X be a Banach space, and $I = [0,1]$. The Banach space of continuous functions from I into X, is denoted by $C(I,X)$. It is well known [4], that $C(I,X)$ is isometrically isomorphic to the injective tensor product of $C(I)$ with X, where $C(I)$ is the space of all real (or complex) valued continuous functions on I.

One of the classical differential equations in Banach spaces is the so called Abstract Cauchy Problem. The general form of such a problem, which we will denote by (P_1) is

$$Bu'(t) = Au(t) + f(t)z, \; u(0) = y, \tag{1}$$

where A, B are densely defined linear operators on the codomain of the function u , where u is continuously differentiable on $I = [0,1]$ or $[0,\infty)$ with values in the Banach space X. If B^{-1}exists, then the equation is called degenerate, otherwise, it is called non-degenerate. If $f = 0$ or $z = 0$, then the equation is homogeneous, otherwise it is called nonhomogeneous. In such equation, only u is the unknown.

However, if u and f are both unknowns, but additional conditions are added to be able to determine u and f then problem $P1$ is called an inverse problem.

The theory of inverse problems for differential equations is being extensively developed with the frame work of mathematical physics. To determine the solution of an inverse problem additional conditions are needed. Almost all researchers ,[1], [2], and [3] studied such type of problems using a semigroup approach. For more references and results on inverse

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problems we refer to [3]. In this note, we use a new method that involves tensor product techniques to solve inverse problems for what we call atomic functions.

We refer to [4] for the basic theory of tensor products of Banach spaces.

2. Degenerate Inverse Problem

Consider the problem which we will denote by P_1:

$$u'(t)x + u(t)Ax = f(t)z \tag{2}$$

with the conditions

(i) There is $x^* \in X^*$ and $g \in C(I, R)$ such that $< u(t)x, x^* >= g(t)$

(ii) $\ell n(g(1)/g(0)) \in \rho(A)$, the resolvent set of A.

Theorem 1. *Problem P_1 has a unique solution.*

Proof. Using tensor product notation, equation (2) can be written in the form

$$u' \otimes x + u \otimes Ax = f \otimes z \tag{3}$$

Since an atom $x \otimes y$ has infinite number of representations: $ax \otimes \frac{1}{a}y$, then without loss of generality we can assume that $u(0) = f(0) = 1$.

Further, [5], since in equation (3) the sum of two atoms is an atom, then we have two cases.

(1) $u' = \lambda u$, and (2) $Ax = \alpha x$.

Now, assume $u' = \lambda u$. Then $u(t) = e^{\lambda t}$, noting that $u(0) = 1$. Using condition (i) in P_1, we get $< x, x^* >= g(0)$. So $e^{\lambda t} g(0) = g(t)$. This implies that $\lambda = \ell n(\frac{g(1)}{g(0)})$, and u is determined.

Substitute such values in equation (2) and apply x^* to both sides of (4) we get $\lambda e^{\lambda t} < x, x^* > + e^{\lambda t} < Ax, x^* >= f(t) < z, x^* >$. Hence

$$g'(t) + e^{\lambda t} < Ax, x^* >= f(t) < z, x^* > . \tag{4}$$

Consequently, for $t = 0$ we get $< Ax, x^* >=< z, x^* > - g'(0)$. But this together with (4) determines f uniquely. Finally we have to determine x.

In (2) put $t = 0$ to get $\lambda x + Ax = z$, and so $(\lambda + A)x = z$. However, the value of λ and condition (ii) in problem P_1 determines x uniquely. This ends the proof for case (1)

(2) $Ax = \eta x$. Then from (2) we have

$$u' \otimes x + \eta u \otimes x = f \otimes z. \tag{5}$$

Apply x^* to both sides of (5) to get

$$g'(t) + \eta g(t) = f(t) < z, x^* > . \tag{6}$$

Use condition (i) in P_1 and put $t = 0$ to get $\eta = \frac{<z,x^*>-g'(0)}{g(0)}$ and so η is determined.

In (6), since η and $g(t)$ are known, then $f(t)$ is determined uniquely. So u and x are what is left to be determined.

From (5), we have $(u' + \eta u) \otimes x = f \otimes z$. So we have equality of two atoms. Hence from theory of tensor product we have

$$u' + \eta u = \gamma f, \text{ and} \tag{7}$$

$$x = \frac{1}{\gamma} z \tag{8}$$

The first equation is a linear differential equation of order one. So
$u(t) = e^{-\eta t}[\int \gamma f(t) e^{\eta t} dt] + c\, e^{-\eta t}$.

Now, u will be determined completely if γ and c are determined. To do that we take the tensor product of x with both sides of $u' + \eta u = \gamma f$ to get $(u' + \eta u) \otimes x = \gamma f \otimes x$. Again, we use condition (i) in P_1 to conclude $g'(t) + \eta g(t) = \gamma f(t) < x, x^* >$. But $< x, x^* >= g(0)$. Hence $\gamma = \frac{g'(0)+\eta g(0)}{f(0)}$ and γ is determined uniquely. Hence in equation (7) u can be determined uniquely, noting that $u(0) = 1$ which determines c.

As for x, from (5) we have $u'(t)x + \eta u(t)x = f(t)z$. Since this is true for all t, we get $x = \frac{z}{u'(0)+\eta}$. This ends the proof of the theorem.

3. Non-Degenerate Inverse Problem

Let A and B be two closed linear operators on the Banach space X. An element $w \in X$ is called uniquely imaged by an operator J on X if there is a unique $y \in X$ such that $Jy = w$. Note that for injective operators, every element in the range is uniquely imaged.

Consider the problem

$$u'(t)Bx + u(t)Ax = f(t)z \tag{9}$$

with the conditions

(i) There is $x^* \in X^*$ and $g \in C(I, R)$ such that $< u(t)x, x^* >= g(t)$.

(ii) z is uniquely imaged for the operators A and $\ell n(\frac{g(1)}{g(0)})B + A$

We call such problem $P_{2.}$

Theorem 2. *Problem P_2 has a unique solution.*

Proof. We solve the problem if we can determine u, x, and f uniquely. As in Theorem 1, we can assume that $u(0) = f(0) = 1$. So condition (i) implies that $< x, x^* >= g(0)$, and so from condition (i) we get $u(t) = \frac{g(t)}{g(0)}$ and u is determined uniquely.

Now we write equation (9) in tensor product form to get $u' \otimes Bx + u \otimes Ax = f \otimes z$. Since we have the sum of two atoms is an atom, then we have two cases

(1) $u' = \lambda u$ and (2) $Bx = \gamma Ax$.

Let us consider case (1). Since $u(0) = 1$, we get $u(t) = e^{\lambda t}$, where λ to be determined. Further, condition (i) gives $u(t) = \frac{g(t)}{g(0)}$. Hence $\lambda = \ln \frac{g(t)}{g(0)}$. So λ is determined.

Equation (9) now reads $\lambda e^{\lambda t} Bx + e^{\lambda t} Ax = f(t)z$. That is $e^{\lambda t} \otimes (\lambda Bx + Ax) = f \otimes z$. So two atoms are equal. Consequently,

(a) $e^{\lambda t} = af(t)$, and

(b) $(\lambda Bx + Ax) = \frac{1}{a} z$.

From (a) we get $1 = af(0)$, and a is determined, so f is determined. Remains to determine x. From (b), we have $(\lambda B + A)x = z$. Since we assumed x to be uniquely imaged for $\lambda B + A$, then x is uniquely determined.

(2) Now we consider case (2). Equation (9) now reads

$$u'(t)\gamma Ax + u(t)Ax = f(t)z. \tag{10}$$

Since condition (i) together with $u(0) = 1$ gives $u(t) = \frac{g(t)}{g(0)}$, then in equation (10) only x, f, and γ are to be determined. Equation (10) can be written in the form $(\gamma u' + u) \otimes Ax = f \otimes z$. Since we have two equal atoms, then

(c) $\gamma u' + u = \eta f$ and

(d) $Ax = \frac{1}{\eta} z$.

From (c) we get

$$\gamma u'(0) + 1 = \eta. \tag{11}$$

From (d) and the condition z is uniquely imaged under A we get $x = \frac{1}{\eta} A^{-1} z$. Substitute this in (10) and apply x^* to both sides of the resulting equation, together with $< x, x^* >= g(0)$ implies

$$(\gamma u'(0) + 1)g(0) = \frac{1}{\eta} < A^{-1}z, x^* > . \tag{12}$$

Equations (11) and (12) determine γ and η uniquely. Hence from $x = \frac{1}{\eta} A^{-1} z$, x is determined uniquely, and finally (5) determines f uniquely.

This ends the proof.

As an application one can consider the following example:

Let

$$B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},\ A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ u = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix},\ z = \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ u(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ x^* = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

and $g(t) = 2 + t^2$. Then the system

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = f(t) \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

with the conditions: $< x^*, u(t) >= x + y = 2 + t^2$, and $x(0) = y(0) = 1$ has a unique solution, noting that uniquely imaged condition is satisfied.

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