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Approximation of Generalized Biaxisymmetric Potentials in L^β-Norm

Devendra Kumar[1,2]

[1] *Department of Mathematics, Faculty of Sciences Al-Baha University, P.O.Box-7738 Alaqiq, Al-Baha-65799, Saudi Arabia*
[2] *Research and Post Graduate Studies, Department of Mathematics, M. M. H. College, Model Town, Ghaziabad-201001, U.P., India*

Abstract. Let F be a real valued generalized biaxisymmetric potential (GBASP) in L^β on S_R, the open sphere of radius R about the origin. In this paper we have obtained the necessary and sufficient conditions on the rate of decrease of a sequence of best harmonic polynomial approximates to F such that F is harmonically continues as an entire function GBASP and determine their (p,q)-order and generalized (p,q)-type with respect to proximate order $\rho(r)$.

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1. Introduction

Let $F = F(x,y)$ be a real-valued regular solution of the generalized biaxisymmetric potential (GBASP) equation

$$\frac{\partial^2 F}{\partial x^2} + \frac{2\mu}{y}\frac{\partial F}{\partial y} + \frac{\partial^2 F}{\partial y^2} + \frac{2\nu}{x}\frac{\partial F}{\partial x} = 0, \mu, \nu > 0, \tag{1.1}$$

which are even in x and y. A polynomial of degree n which is even in x and y is said to be a GBASP polynomial of degree n if it satisfies (1.1). A GBASP F, regular about origin, have local expansions of the form

$$F(x,y) = \sum_{n=0}^{\infty} a_n R_n^{(\mu-\frac{1}{2},\nu-\frac{1}{2})}(x,y), \tag{1.2}$$

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Email address: d_kumar001@rediffmail.com (D. Kumar)

$R_n^{(\mu-\frac{1}{2},\nu-\frac{1}{2})}(x,y) = (x^2+y^2)^n P_n^{(\mu-\frac{1}{2},\nu-\frac{1}{2})}(\frac{(x^2-y^2)}{(x^2+y^2)})/P_n^{(\mu-\frac{1}{2},\nu-\frac{1}{2})}(1)$, where $x = r\cos\theta, y = r\sin\theta$ and $P_n^{(\mu-\frac{1}{2},\nu-\frac{1}{2})}(t)$ are Jacobi polynomials [1, 17]. The series (1.2) can be represented in (r,θ) by

$$F \equiv F(r,\theta) = \sum_{n=0}^{\infty} a_n r^{2n} P_n^{(\mu-\frac{1}{2},\nu-\frac{1}{2})}(\cos 2\theta).$$

Let $S_R = \{(x,y) : x^2+y^2 < R^2\}, 0 < R \leq \infty$, be the open sphere of radius R about the origin and $\overline{S_R}$ be the closure of S_R. In this paper we consider those GBASP $F \in L^\beta(S_R), 1 \leq \beta < \infty$, that harmonically continue as an entire function GBASP. The characteristic feature follows from the rate of convergence of a sequence of best GBASP polynomial approximates to F in $L^\beta(S_R)$. The concepts of index-pair $(p,q), p \geq q \geq 1$, (p,q)-order and (p,q)-type were introduced by Juneja et al. [15, 16]. Following the Juneja et al. [15, 16] the (p,q)-order of an entire GBASP function is defined as

$$\limsup_{r\to\infty} \frac{\log^{[p]} M(r,F)}{\log^{[q]} r} = \rho(p,q) \equiv \rho,$$

and, the function having (p,q)-order $\rho(b < \rho(p,q) < \infty)$ is said to be of (p,q)-type T if

$$\limsup_{r\to\infty} \frac{\log^{[p-1]} M(r,F)}{(\log^{[q-1]} r)^\rho} = T(p,q) \equiv T,$$

where $M(r,F) = \max_{x^2+y^2<r^2}\{|F(x,y)|\}$, b=1 if $p = q$ and b=0 otherwise.

For entire function GBASP the growth of this sequence is used to calculate the (p,q)-order and generalized (p,q)-type with respect to proximate order $\rho_{p,q}(r)$. The function in the class $L^\beta(S_\infty)$ are called entire GBASP. The growth parameters (p,q)-order $\rho(p,q)$ and (p,q)-type $T(p,q)$ of entire function GBASP F for $(p,q) = (2,1)$ have been studied in $L^\beta(S_R)$ by McCoy [14], but these concepts are inadequate to compare the growth of those entire function GBASP which are of the same order but of infinite type. Hence, for a refinement of the above scale one may utilize the concept of proximate order cf. [4, 11].

A positive function $\rho(r)$ defined on $[r_0,\infty), r_0 > \exp^{[q-1]} 1$, is said to be a proximate order of an entire function with index-pair (p,q) if
(i) $\rho(r) \to \rho(p,q) \equiv \rho \;\; as \;\; r \to \infty, b < \rho < \infty$;
(ii) $\bigwedge_{[q]}(r)\rho'(r) \to 0 \;\; r \to \infty$,
where $\rho'(r)$ denotes the derivative of $\rho(r)$, and $\bigwedge_{[q]}(r) = \prod_{i=0}^{q} \log^{[i]} r$.
The (p,q)-type T^* of F with respect to a given proximate order $\rho(r)$ is defined as

$$\limsup_{r\to\infty} \frac{\log^{[p-1]} M(r,F)}{(\log^{[q-1]} r)^{\rho(r)}} = T^*(p,q) \equiv T^*.$$

If the quantity T^* is different from zero and infinity then $\rho(r)$ is said to be the proximate order of a given GBASP function F with index-pair (p,q).

P.A. McCoy [14] obtained the results by using integral operator method [2, 4, 6–8], but our method is different from McCoy [14] and the results are the extension of those of McCoy [14].

For the purpose of motivation, it is significant to mention that the Euler-Poisson Darboux equation, arising in gas dynamics, is viewed in terms of equation (1.1) after a transformation and has a variety of physical interpretations. The solution of equation (1.1) which satisfies a suitable radiation condition, corresponding to scattered waves, and their singularities are related to the quantum states of the scattered particles. The GBASP play an important role in many aspects of mathematical physics, in particular, in an understanding of compressible flow in the transonic region (see [14]).

The limit $\mu \downarrow \nu$ produces the generalized axisymmetric potential equation. Reduction of the GBASP equation to the harmonic function follows from the limit $\mu \downarrow 0$ that also reduces the zonal harmonics to the circular harmonics. These functions form complete sets for even harmonic, respectively analytic functions, regular at the origin. The GBASP functions, then, are natural extensions of harmonic or analytic functions.

Let $A_\beta(S_R)$ denote the space of GBASP that is regular and analytic in S_R with finite norm

$$\| F \|_{\beta,R} = [\int\int_{\overline{S_R}} |F|^p dxdy]^{\frac{1}{\beta}}, 1 \leq \beta < \infty,$$

where $\| \, . \, \|_{\beta,R}$ denotes the L^β-norm.

The best polynomial approximation error for the GBASP is defined by

$$E_n^\beta(F,R) = \inf_{g_{R,n} \in P_{R,n}} \{||F - g_{R,n}||_{\beta,R}\}, n = 0, 1, \ldots, \tag{1.3}$$

with $P_{R,n} = P_{R,n}(z) = P_n(\frac{z}{R})$; where P_n denotes the set of all GBASP polynomials of degree no higher than n. For $\beta = \infty$, the above norm is sup norm. For each n there is an extremal GBASP polynomial $g^*_{R,n} \in P_{R,n}$ for which $\| F - g^*_{R,n} \|_{\beta,R} = E_n^\beta(F,R)$.

For GBASP functions there is a large literature concerning the growth and approximation of this topic. Kasana and Kumar [10] studied the growth and approximation of solutions (not necessarily entire) of certain elliptic partial differential equations. They obtained the characterization of q-type and lower q-type ($q \geq 2$) of a GBASP having fast rates of growth in terms of ratio of approximation errors in L^β- norm. In [12], Kumar obtained some results for GBASP and the polynomial approximation of pseudo analytic functions, while in [13] Kumar obtained the characterization of growth parameters in terms of axially symmetric harmonic polynomial and Lagrange polynomials approximation errors in n-dimensions. In the present paper, using a different technique, we derive formulae for the (p,q)-order and generalized (p,q)-type with respect to a proximate order, of entire GBASP functions in terms of GBASP polynomials approximation errors in L^β-norm. Our results extend and improve the results obtained by McCoy [14].

2. Lemmas and Results

To prove our main results the following lemmas are required.

Lemma 2.1. Let $F \in A_\beta(S_R)$, then for all $n \in \mathbb{N}$ the following inequality holds:

$$|a_n|R_0^{2n+2} \leq \frac{(\pi R_0^2)^{\frac{1}{\eta}}(2n+2)((2n+\mu+\nu)C(n,\mu,\nu))\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+1)}E_{n-1}^{\beta}(F,R_0)$$

where

$$C(n,\mu,\nu) = \frac{\Gamma(n+1)\Gamma(n+\mu+\nu)}{\Gamma(n+\mu+\frac{1}{2})\Gamma(n+\nu+\frac{1}{2})},$$

$\alpha = \max(\mu - \frac{1}{2}, \nu - \frac{1}{2})$ and $\frac{1}{\eta} + \frac{1}{\beta} = 1$.

Proof. From the orthogonality property of Jacobi polynomials and uniform convergence of the series (1.2) on $\overline{S_R}$, we have

$$\begin{aligned} a_n\tau^{2n} =& 2(2n+\mu+\nu)C(n,\mu,\nu)\int_0^{\frac{\pi}{2}}(F(\tau,\theta) - g^*_{\tau,n-1}(\tau,\theta))\times \\ & \times P_n^{(\mu-\frac{1}{2},\nu-\frac{1}{2})}(\cos 2\theta)\sin^{2\mu}\theta\cos^{2\nu}\theta d\theta, \end{aligned} \tag{2.1}$$

where $g^*_{\tau,n-1} \in P_{\tau,n-1}, 0 < \tau < R_0$. Using [3, p.168]

$$\max_{-1\leq t\leq 1}|P^{(\mu-\frac{1}{2},\nu-\frac{1}{2})}(t)| = \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+1)} \tag{2.2}$$

in (2.1), we obtain

$$|a_n|\tau^{2n} = \frac{(2n+\mu+\nu)C(n,\mu,\nu)\Gamma(n+\alpha+1)}{2\Gamma(\alpha+1)\Gamma(n+1)}\int_0^{2\pi}|(F(\tau,\theta) - g^*_{\tau,n-1}(\tau,\theta))|d\theta,$$

since F and $g^*_{\tau,n-1}$ are even in x and y. Multiplying both sides of the above inequality by $\tau d\tau$ and integrating from 0 to R_0, we get

$$\begin{aligned} |a_n|{R_0}^{2n+2} =& \frac{2(n+1)(2n+\mu+\nu)C(n,\mu,\nu)\Gamma(n+\alpha+1)}{2\Gamma(\alpha+1)\Gamma(n+1)}\times \\ & \times \int\int_{\overline{S_{R_0}}}|(F(x,y) - g^*_{R_0,n-1}(x,y))|dxdy. \end{aligned} \tag{2.3}$$

For $F \in A_\beta(\overline{S_{R_0}})$, there exists $g^*_{R_0,n-1} \in P_{R_0,n-1}$ such that

$$\begin{aligned} 2E_{n-1}^{\beta}(F,R_0) &\geq \| F - g^*_{R_0,n-1} \|_{\beta,R_0} \\ &\geq (\int\int_{\overline{S_{R_0}}}|(F(x,y) - g^*_{R_0,n-1}(x,y))|^{\beta}dxdy)^{\frac{1}{\beta}} \\ &\geq \frac{1}{(\pi R_0^2)^{\frac{1}{\eta}}}\int\int_{\overline{S_{R_0}}}|(F(x,y) - g^*_{R_0,n-1}(x,y))|dxdy. \end{aligned} \tag{2.4}$$

Now combining (2.3) and (2.4) we get the required result.

Let $w = \psi(z)$ be the univalent function mapping the complement of $\overline{S_R}$ on $|w| > 1$ such that $\psi(\infty) = \infty$ and $\psi'(\infty) > 0$. Set $S_R = \{z : \psi(z) = r, r > 1\}$. Then

Lemma 2.2. Let $F \in A_\beta(S_R)$ be an entire GBASP function of (p, q)-order ρ and generalized (p, q)-type T^* with respect to $\rho(r)$. Then

$$\limsup_{r\to\infty} \frac{\log^{[p]} \overline{M(r, F)}}{\log^{[q]} r} = \rho,$$

$$\limsup_{r\to\infty} \frac{\log^{[p-1]} \overline{M(r, F)}}{(\log^{[q-1]} r)^{\rho(r)}} = \frac{T^*}{\gamma},$$

where $\overline{M(r, F)} = \max_{z\in S_R} |F|, \gamma = R^{-\rho}$ for $q = 1$ and $\gamma = 1$, otherwise.

This lemma is an immediate consequence of [18, Lemma 3.1].

Lemma 2.3. Let $F \in A_\beta(S_R), r' > 1$, be an entire GBASP function. Then, for all sufficiently large values of n, we have

$$E_n^\beta(F, R) \leq K\overline{M(r, F)}(n + 1)^{\alpha+\frac{1}{2}} (\frac{r'R}{r})^{2(n+1)}, \tag{2.5}$$

where K is a constant independent of n and r and $r > 2r'R$.

Proof. Let us consider the GBASP polynomial

$$g_{n,r} = \sum_{k=0}^{\infty} a_k r^{2k} P_k^{(\mu-\frac{1}{2},\nu-\frac{1}{2})}(\cos 2\theta).$$

Then $g_{n,r} \in P_{n,r}$. Using the definition of approximation error $E_n^\beta(F, R)$ for all $r, 0 < r < R$, we get

$$\begin{aligned} E_n^\beta(F, R) \leq & |F - g_{n,r}|_{\beta,R} \\ & \leq \sum_{k=n+1}^{\infty} |a_k| R^{2k} |P_k^{(\mu-\frac{1}{2},\nu-\frac{1}{2})}(\cos 2\theta)| \\ & \leq \frac{1}{\Gamma(\alpha + 1)} \sum_{k=n+1}^{\infty} |a_k| R^{2k} \frac{\Gamma(k + \alpha + 1)}{\Gamma(k + 1)}. \end{aligned} \tag{2.6}$$

For $F \in A_\beta(S_R)$, we have [5]

$$|a_k| \leq \frac{\overline{M(r, F)}}{r^{2k}} [(2k + \mu + \nu)C(k, \mu, \nu)C(\mu, \nu)]^{\frac{1}{2}} \tag{2.7}$$

for every $r < R$.
Combining (2.6) and (2.7) we get

$$E_n^\beta(F,R) \leq \frac{\overline{M(r,F)}}{\Gamma(\alpha+1)}(C(\mu,\nu))^{\frac{1}{2}} \sum_{k=n+1}^{\infty} \frac{\Gamma(k+\alpha+1)}{\Gamma(k+1)}[(2k+\mu+\nu)C(k,\mu,\nu)]^{\frac{1}{2}}(\frac{R}{r})^{2k}. \quad (2.8)$$

Since $\frac{\Gamma(x+a)}{\Gamma(x)} \sim x^a$ as $x \to \infty$, we have

$$\frac{\Gamma(k+\alpha+1)}{\Gamma(k+1)}[(2k+\mu+\nu)C(k,\mu,\nu)]^{\frac{1}{2}} \sim \sqrt{2}k^{\alpha+\frac{1}{2}} \quad as \quad k \to \infty.$$

Hence

$$\frac{\Gamma(k+\alpha+1)}{\Gamma(k+1)}[(2k+\mu+\nu)C(k,\mu,\nu)]^{\frac{1}{2}} < 2\sqrt{2}k^{\alpha+\frac{1}{2}} \quad for \quad all \quad k > k_0.$$

Thus, for $n > k_0$ and $r > 2r'R$, using (2.8) with above inequality, we obtain

$$\begin{aligned} E_n^\beta(F,R) \leq & \frac{\overline{M(r,F)}}{\Gamma(\alpha+1)} 2(2C(\mu,\nu))^{\frac{1}{2}} \sum_{k=n+1}^{\infty} k^{\alpha+\frac{1}{2}}(\frac{r'R}{r})^{2k} \\ & \leq \frac{\overline{M(r,F)}}{\Gamma(\alpha+1)} 2(2C(\mu,\nu))^{\frac{1}{2}}(n+)^{\alpha+\frac{1}{2}}(\frac{r'R}{r})^{2(n+1)} \sum_{k=0}^{\infty}(1+\frac{k}{k_0+1})^{\alpha+\frac{1}{2}}(\frac{r'R}{r})^{2k}. \end{aligned}$$

Hence the proof is completed from the above inequality.

Lemma 2.4. Let $F \in A_\beta(S_R), R > R_*$, be an entire GBASP function. Then

$$h(z) = \sum_{n=1}^{\infty}[\frac{2(n+1)(2n+\mu+\nu)C(n,\mu,\nu)(n+1)^\alpha}{\Gamma(n+1)}]^2 E_{n-1}^\beta(F,R)(\frac{z}{R_*})^{2n} \quad (2.9)$$

is entire. Further, $\rho(F) = \rho(h)$ and for $b < \rho(F) = \rho(h) < \infty, T^*(F) = \gamma T^*(h)$.

Proof. Since

$$[\frac{2(n+1)(2n+\mu+\nu)C(n,\mu,\nu)(n+1)^\alpha}{\Gamma(n+1)}]^{\frac{1}{2n}} \sim (\sqrt{2(n+1)}\sqrt{2}n^{\alpha+\frac{1}{2}})^{\frac{1}{n}} \to 1 \quad as \quad n \to \infty,$$

it follows from Lemma 2.2 that $h(z)$ is entire and

$$E_n^\beta(F,R) \leq K\overline{M(r+1,F)}(\frac{r'R}{r+1})^{2n},$$

we have

$$h(z) = \sum_{n=1}^{\infty}[\frac{2(n+1)(2n+\mu+\nu)C(n,\mu,\nu)(n+1)^\alpha}{\Gamma(n+1)}]^2 E_{n-1}^\beta(F,R)(\frac{z}{R_*})^{2n},$$

so we get

$$M(\frac{r}{Rr'},h) \leq Q(r) + K\overline{M(r+1,F)}\sum_{n=0}^{\infty}[\frac{r}{R_*(r+1)}]^{2n}$$
$$= Q(r) + K\frac{{R_*}^2(r+1)^2\overline{M(r+1,F)}}{(r+1)^2{R_*}^2 - r^2}, r' > 1, \tag{2.10}$$

where $Q(r)$ is a polynomial for all sufficiently large value of r.

On the other hand, using (1.2), (2.2) and Lemma 2.1, we get

$$|\sum_{n=0}^{\infty} a_n r^{2n} P_n^{(\mu-\frac{1}{2},\nu-\frac{1}{2})}(\cos 2\theta)|$$
$$\leq |a_0| + \frac{1}{\Gamma(\alpha+1)}\sum_{n=1}^{\infty}|a_k|R^{2n}\frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)} \leq |a_0| +$$
$$+ KK_0\sum_{n=1}^{\infty}[\frac{2(n+1)(2n+\mu+\nu)C(n,\mu,\nu)(n+1)^{\alpha}}{\Gamma(n+1)}]^2 \times$$
$$\times E_{n-1}^{\beta}(F,R)(\frac{r}{R_0})^{2n+2}, z \in S_R, R_0 < R.$$

or

$$M(r,F) \leq M(\frac{r}{R_0}, |a_0| + KK_0 h(z)). \tag{2.11}$$

Now the proof follows from (2.10) and (2.11).

3. Main Results

In this section we will prove our main results.

Theorem 3.1. Let the GBASP $F \in A_\beta(S_1), \beta \geq 1$. Then F harmonically continues as an entire function GBASP if and only if

$$\lim_{n\to\infty}[E_n^{\beta}(F,R)]^{\frac{1}{n}} = 0. \tag{3.1}$$

Proof. Let $F \in A_\beta(S_1)$, then for $0 < R < 1, F \in A_\beta(S_R)$. First suppose that F is entire. Then it follows from Lemma 2.3 that

$$\limsup_{n\to\infty}[E_n^{\beta}(F,R)]^{\frac{1}{n}} \leq (\frac{r'R}{r}), r > 2r'R.$$

Thus, for all sufficiently large r, we have

$$\limsup_{n\to\infty}[E_n^{\beta}(F,R)]^{\frac{1}{n}} = 0.$$

To prove only if part, suppose that (3.1) holds, then it follows from (2.11) that series on the right hand side of (1.2) converges uniformly on every compact subset of S_∞ and GBASP F is entire.

Theorem 3.2. Let the GBASP $F \in A_\beta(S_R), r > 2r'R$. Then F harmonically continues as an entire function GBASP of finite (p, q)-order ρ if and only if

$$\rho(p, q) = P(L^*(p, q)),$$

where

$$L^*(p, q) = \limsup_{n\to\infty} \frac{\log^{[p-1]} n}{\log^{[q]}[E_n^\beta(F, R)]^{-\frac{1}{n}}},$$

and $P(L^*(p, q)) = \{L^*(p, q) \quad if \quad q < p < \infty,\ 1 + L^*(p, q) \quad if \quad p = q = 2,\ \max(1 + L^*(p, q)) \ \ if \ \ 3 \le p = q,\ \infty \quad if \ \ p = q = \infty\}$.

Proof. Using Theorem 3.1, we have $F \in A_\beta(S_R)$ is harmonically continues as an entire function GBASP if and only if $h(z)$ is an entire function. Using Lemma 2.4, F and $h(z)$ have same (p, q)-order. The remaining part of the proof can be obtain easily.

Theorem 3.3. Let the GBASP $F \in A_\beta(S_R), r > 2r'R$. Then F harmonically continues as an entire function GBASP of finite (p, q)-order $\rho(b < \rho < \infty)$ and generalized (p, q)-type T^* of F with respect to a proximate order $\rho(r)$ if and only if

$$\frac{T^*(p, q)}{M\gamma} = \limsup_{n\to\infty} [\frac{\phi(\log^{[p-2]} n)}{\log^{[q-1]}[E_n^\beta(F, R)]^{-\frac{1}{n}}}]^{\rho-A},$$

where $A = 1$ if $q = 2$, $A = 0$ if $q \neq 2$ and $M \equiv M(p, q) = \{\frac{(\rho-1)^{(\rho-1)}}{\rho^\rho} \quad if \quad (p, q) = (2, 2)$, $\frac{1}{e\rho} \quad if \quad (p, q) = (2, 1),\ 1 \quad otherwise\}$.

The function $\phi(x)$ be the unique solution of the equation

$$x = (\log^{[q-1]} r)^{\rho(r)-A} \Leftrightarrow \phi(x) = \log^{[q-1]} r.$$

Proof. Applying Theorem 3 of Nandan et al. [9] to the function $h(z)$ and resulting characterization of $T^* = \gamma T^*(h)$, with Lemma 2.4, taking together completes the proof.

Remark 3.1. For $(p, q) = (2, 1)$, Theorem 3.2 gives the Theorem 2 of P.A. McCoy [14] .

Remark 3.2. For $(p, q) = (2, 1)$ and $x = \phi(n)$ is the function inverse to $n = x^{\rho(r)}$, Theorem 3.3 gives the Theorem 3 of P.A. McCoy [14] .

4. Conclusions

We estimate formulae for the (p, q)-order and generalized (p, q)-type with respect to a proximate order of entire GBASP functions in terms of GBASP polynomial approximation errors in L^β-norm, which made it possible to obtain the necessary and sufficient conditions under which a GBASP function harmonically continues to entire GBASP. Our results improve and extends the results of McCoy [14]. The relevance of our study is due to the fact that GBASP play and important role not only in theoretical mathematical research, but are used in gas dynamics in order to describe different stationary processes. Thus, the special interest are global properties characterising solutions to the partial differential equation that are determined from local properties.

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