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2-Locating Sets in a Graph

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Abstract. Let G be an undirected graph with vertex-set $V(G)$ and edge-set $E(G)$, respectively. A set $S \subseteq V(G)$ is a 2-locating set of G if $\left|\left[\left(N_G(x)\backslash N_G(y)\right) \cap S\right] \cup \left[\left(N_G(y)\backslash N_G(x)\right) \cap S\right]\right| \geq 2$, for all $x, y \in V(G)\backslash S$ with $x \neq y$, and for all $v \in S$ and $w \in V(G)\backslash S$, $\left(N_G(v)\backslash N_G(w)\right) \cap S \neq \varnothing$ or $\left(N_G(w)\backslash N_G[v]\right) \cap S \neq \varnothing$. In this paper, we investigate the concept and study 2-locating sets in graphs resulting from some binary operations. Specifically, we characterize the 2-locating sets in the join, corona, edge corona and lexicographic product of graphs, and determine bounds or exact values of the 2-locating number of each of these graphs.

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Key Words and Phrases: 2-locating set, 2-locating number, join, corona, edge corona, lexicographic product

1. Introduction

Resolving sets and metric basis are emphasized for their application in computer science, medical sciences and chemistry. The locating set in graphs can be viewed as the set of monitors that can determine the exact location of an intruder. The concept of 2-locating set is obtained from the concept of locating set. Requiring such a set to be 2-locating implies that every pair of vertices where there is no monitor must be connected to at least two monitoring devices that are connected to other monitors. Also, for every vertex and monitoring device there exists at least one monitor that is connected to it. Hence, 2-locating set can be viewed as the set of monitors that can determine the presence of an intruder.

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In 1975, Slater [19] introduced the concept of locating sets and its minimum cardinality as locating number. Harary and Melter also utilized a similar idea, although they referred to the locating set and the locating number, respectively, using the terms resolving set and metric dimension. Resolving sets and locating sets, however, are defined differently in more recent studies. In 2013, Bailey et al. [1] defined a resolving set as a set of vertices S in a graph G such that for any two vertices u, v, there exists $x \in S$ such that the distance $d(u, x) \neq d(v, x)$. On the other hand, Canoy and Malacas [8] defined a locating set as a set $S \subseteq V(G)$ of G such that for every two distinct vertices u and v of $V(G) \setminus S$, $N_G(u) \cap S \neq N_G(v) \cap S$. Other variations of locating sets are studied in [16], [6], [11], [13], [14], [15] and [7].

In 2021, J. Cabaro and H. Rara [5] studied the idea of the 2-resolving sets in the join and corona of graphs wherein they introduced the idea of 2-locating sets. This work is therefore motivated by the recent studies on these variations of 2-resolving set and 2-metric dimension that utilize the concepts of 2-locating set and 2-locating number. Other studies that deal with the concept of 2-locating sets are located in [6], [9], [10], [12] and [18].

2. Terminology and Notation

In this study, we consider finite, simple, connected, undirected graphs. For basic graph-theoretic concepts, we then refer readers to [3] and [4]. The following concepts are found in [2], [3], [5] and [17] respectively.

The *open neighborhood* of a vertex v in a graph G is defined as the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$, while the *closed neighborhood* of a vertex v in G is defined as $N_G[v] = N_G(v) \cup \{v\}$. The *open neighborhood* of a set $S \subseteq V(G)$ is defined as $N_G(S) = \bigcup_{v \in X} N_G(v)$, while its *closed neighborhood* is $N_G[S] = N_G(S) \cup S$. A connected graph G of order $n \geq 3$ is *point distinguishing* if for any two distinct vertices u and v of G, $N_G[u] \neq N_G[v]$. It is *totally point determining* if for any two distinct vertices u and v of G, $N_G(u) \neq N_G(v)$ and $N_G[u] \neq N_G[v]$.

For an ordered set of vertices $W = \{w_1, w_2, ..., w_k\} \subseteq V(G)$ and a vertex v in G, we refer to the k-vector (ordered k-tuple)

$$r_G(v/W) = (d_G(v, w_1), d_G(v, w_2), ..., d_G(v, w_k))$$

as the *(metric) representation of v with respect to W*. The set W is called a *resolving set* for G if distinct vertices have distinct representations with respect to W. Hence, if W is a resolving set of cardinality k for a graph G of order n, then the set $\{r_G(v/W) : v \in V(G)\}$ consists of n distinct k-vectors. A resolving set of minimum cardinality is called a *minimum resolving set* or a *basis*, and the cardinality of a basis for G is the *dimension* $\dim(G)$ of G. An ordered set of vertices $W = \{w_1, ..., w_k\}$ is a *k-resolving set* for G if, for any distinct vertices $u, v \in V(G)$, the (metric) representations $r_G(u/W)$ and $r_G(v/W)$ of u and v, respectively, differ in at least k positions. If $k = 1$, then the k-resolving set is called a *resolving set* for G. If $k = 2$, then the k-resolving set is called a 2-resolving set for G. If G has a k-resolving set, the minimum cardinality $\dim_k(G)$ of a k-resolving set is called the *k-metric dimension* of G. If G has a 2-resolving set, we denote the least size of a 2-resolving set by $dim_2(G)$ is called a 2-*metric dimension* of G. A resolving set of size $dim_2(G)$ is called a 2-*metric basis* for G.

Let G be any nontrivial connected graph and $S \subseteq V(G)$. A set $S \subseteq V(G)$ is a 2-*locating set* of G if it satisfies the following conditions:

(i) $\left|[\big(N_G(x)\backslash N_G(y)\big)\cap S]\cup[\big(N_G(y)\backslash N_G(x)\big)\cap S]\right| \geq 2$, for all $x, y \in V(G)\backslash S$ with $x \neq y$.

(ii) $\big(N_G(v)\backslash N_G(w)\big) \cap S \neq \varnothing$ or $\big(N_G(w)\backslash N_G[v]\big) \cap S \neq \varnothing$, for all $v \in S$ and for all $w \in V(G)\backslash S$.

The 2-*locating number* of G, denoted by $ln_2(G)$, is the smallest cardinality of a 2-locating set of G. A 2-locating set of G of cardinality $ln_2(G)$ is referred to as an ln_2-set of G.

A set $S \subseteq V(G)$ is a $(2,2)$-*locating* ($(2,1)$-*locating*, respectively) *set* in G if S is 2-locating and $|N_G(y)\cap S| \leq |S|-2$ ($|N_G(y)\cap S| \leq |S|-1$, respectively), for all $y \in V(G)$. The $(2,2)$-*locating* ($(2,1)$-*locating*, respectively) *number* of G, denoted by $ln_{(2,2)}(G)$ ($ln_{(2,1)}(G)$, respectively), is the smallest cardinality of a $(2,2)$-locating ($(2,1)$-locating, respectively) set in G. A $(2,2)$-locating ($(2,1)$-locating, respectively) set in G of cardinality $ln_{(2,2)}(G)$ ($ln_{(2,1)}(G)$, respectively) is referred to as an $ln_{(2,2)}$-*set* ($ln_{(2,1)}$-*set*, respectively) in G.

3. Known Results

The following known results are taken from [5].

Remark 1. For any connected nontrivial graph G of order $n \geq 2$, $2 \leq ln_2(G) \leq n$. Moreover, $ln_2(K_n) = n$, for $n \geq 2$.

Theorem 1. Let G be a connected nontrivial graph. Then $ln_2(G) = 2$ if and only if $G \cong P_2$ or $G \cong P_3$.

Remark 2. Let $S \subseteq V(G)$ For any pair of vertices $x, y \in S$, $r(x/S)$ and $r(y/S)$ differ in at least 2 positions. Hence, to prove that S is a 2-resolving set in G, we only need to show that for every pair of vertices $x, y \in V(G)$ where $x \in S$ and $y \in V(G) \setminus S$ or both $x, y \in V(G) \setminus S$, $r(x/S)$ and $r(y/S)$ differ in at least 2 positions.

Remark 3. Every 2-locating set in G is a 2-resolving set in G. However, a 2-resolving set in G need not be a 2-locating set in G. Thus,

$$dim_2(G) \leq ln_2(G).$$

4. Preliminary Results

Every nontrivial connected graph G admits a 2-locating set. Indeed, the vertex-set of G is a 2-locating set.

Proposition 1. For any connected graph G of order $n \geq 2$, $2 \leq ln_2(G) \leq n$. Moreover, (*i*) $ln_2(G) = 2$ if and only if $G = K_2$ or $G = P_3$;

(*ii*) if $G = K_n$, then $ln_2(G) = n$;

(*iii*) if $n = 3$, then $ln_2(G) = 3$ if and only if $G = K_3$; and

(*iv*) if $n = 4$, then $ln_2(G) = 4$ if and only if $G \in \{C_4, K_4, T\}$. Otherwise, $ln_2(G) = 3$ if and only if $G \in \{P_4, K_{1,3}, T'\}$ where T and T' are graphs shown in Figure 1.

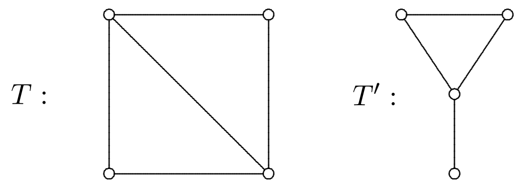

Figure 1: Graphs T and T'

Proof. From Theorem 1 and Remark 1, (*i*) and (*ii*) hold.
From (*ii*), (*iii*) holds.
(*iv*) Suppose $n = 4$, then the possible connected isomorphic graphs are K_4, P_4, C_4, $K_{1,3}$, T and T'. Thus, it can be verified that $ln_2(G) = 4$ if and only if $G \in \{K_4, C_4, T\}$ and $ln_2(G) = 3$ if and only if $G \in \{P_4, K_3, T'\}$. □

Remark 4. [5] Every 2-locating set of a connected graph G is 2- resolving. Thus, $dim_2(G) \leq ln_2(G)$.

Theorem 2. Let a and b be any positive integers such that $2 \leq a \leq b$. Then there exists a connected graph G such that $dim_2(G) = a$ and $ln_2(G) = b$.

Proof. Suppose that $a = b$. Consider the graph $G = K_a$. Then $dim_2(G) = ln_2(G) = a$. Next, suppose that $a < b$. Consider the following cases:
Case 1: $a = 2$
Let $m = b - a$ and consider the graph G in Figure 2. Let $S_1 = \{x_1, x_2\}$ and $S_2 = S_1 \cup \{v_1, v_2, \ldots, v_m\}$. Then S_1 and S_2 are $dim_2 - set$ and $ln_2 - set$ of G, respectively. Hence, $dim_2(G) = a$ and $ln_2(G) = a + m = b$.

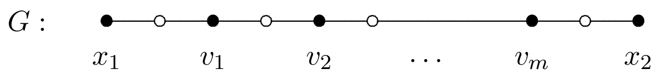

Figure 2: A graph G

Case 2: $a \geq 3$.
Let $m = b - a$ and consider the graph G' in Figure 3. Let $S_1 = \{y_1, y_2, \ldots, y_a\}$ and $S_2 = S_1 \cup \{u_1, u_2, \ldots, u_m\}$. Then S_1 and S_2 are dim_2-set and ln_2-set of G', respectively. Hence, $dim_2(G') = a$ and $ln_2(G') = a + m = b$.

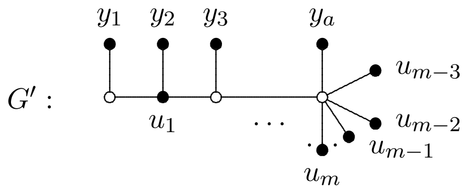

Figure 3: A graph G' .

Corollary 1. For each positive integer n, there exists a connected graph G such that $ln_2(G) - dim_2(G) = n$, that is, $ln_2 - dim_2$ can be made arbitrarily large.

We now characterize the 2-locating sets in some graphs under some binary operations.

5. Join of Graphs

This section presents the characterizations on the 2-locating sets in the join of graphs.

Theorem 3. [6] Let G and H be nontrivial connected graphs. A proper subset S of $V(G+H)$ is a 2-resolving set in $G+H$ if and only if $S_G = V(G) \cap S$ and $S_H = V(H) \cap S$ are 2-locating sets in G and H, respectively, where S_G or S_H is (2,2)-locating set or S_G and S_H are (2,1)-locating sets.

Theorem 4. [6] Let G be a connected graph of order greater than 3 and let $K_1 = \langle v \rangle$. Then $S \subseteq V(K_1 + G)$ is a 2-resolving set of $K_1 + G$ if and only if either $v \notin S$ and S is a (2,2)-locating set in G or $S = \{v\} \cup T$ is (2,1)-locating set in G.

Theorem 5. Let G and H be connected graphs. Then $S \subseteq V(G + H)$ is a 2-locating set in $G + H$ if and only if S is a 2-resolving set in $G + H$ where $S = S_G \cup S_H$, $S_G \subseteq V(G)$ and $S_H \subseteq V(H)$.

Proof. Suppose S is a 2-locating set in $G + H$. Let $p, q \in V(G + H)$. Consider $p, q \in V(G + H) \setminus S$ or $[p \in V(G + H) \setminus S$ or $q \in S]$. Since S is a 2-locating set, then $r_{G+H}(p/S)$ and $r_{G+H}(q/S)$ differ in at least 2 positions. By Definition of 2-resolving set and Remark 2, S is a 2-resolving set.

For the converse, suppose S is a 2-resolving set in $G + H$. Let $S = S_G \cup S_H$ where $S_G \subseteq V(G)$ and $S_H \subseteq V(H)$. Let $p, q \in V(G + H) \setminus S$. Consider the following cases.
Case 1 $p, q \in V(G) \setminus S_G$.
Since S is a 2-resolving set, $r_{G+H}(p/S)$ and $r_{G+H}(q/S)$ differ in at least 2 positions. By definition of $G+H$, $r_G(p/S_G)$ and $r_G(q/S_G)$ differ in at least 2 positions. Since $d_{G+H}(p, u)$ and $d_{G+H}(q, u)$ is either 0,1 or 2 for each $u \in V(G + H)$, there exist at least two vertices $x, y \in S_G$ such that $x, y \in N_G(p) \setminus N_G(q)$ or $x, y \in N_G(q) \setminus N_G(p)$ or $x \in N_G(p) \setminus N_G(q)$ and $y \in N_G(q) \setminus N_G(p)$. Hence,

$$\left|\left[\left(N_{G+H}(p)\setminus N_{G+H}(q)\right) \cap S\right] \cup \left[\left(N_{G+H}(q)\setminus N_{G+H}(p)\right) \cap S\right]\right| \geq 2. \quad (1)$$

Case 2. $p, q \in V(H) \setminus S_H$
Proof is similar to Case 1.

Case 3. $p \in V(G) \setminus S_G$ and $q \in V(H) \setminus S_H$
Note that $r_{G+H}(p/S) = (2, 2, 2, \ldots, 1, 1, \ldots, 1)$ and $r_{G+H}(q/S) = (1, 1, 1, \ldots, 2, 2, \ldots, 2)$. Then there exist $x \in S_G \setminus N_G(p)$ and $y \in S_H \setminus N_H(q)$ or $\exists w, r \in S_G \setminus N_G(p)$ or $w, r \in S_H \setminus N_H(q)$. Hence, inequality (1) holds.

Suppose $p \in S$ and $q \in V(G + H) \setminus S$. Consider the following cases.
Case 1 $p \in S_G$ and $q \in V(G) \setminus S_G$.
Since $r_{G+H}(p/S)$ and $r_{G+H}(q/S)$ differ in at least 2 positions, by definition of $G + H$, $r_G(p/S_G)$ and $r_G(q/S_G)$ differ in at least 2 positions, This implies that $\big(N_G(p)\backslash N_G(q)\big) \cap S_G \neq \varnothing$ or $\big(N_G(q)\backslash N_G(p)\big) \cap S_G \neq \varnothing$.
Hence, $\big(N_{G+H}(p)\backslash N_{G+H}(q)\big) \cap S \neq \varnothing$ or $\big(N_{G+H}(q)\backslash N_{G+H}(p)\big) \cap S \neq \varnothing$.
Case 2. $p \in S_G$ and $q \in V(H) \setminus S_H$
Note that $r_{G+H}(p/S) = (\ldots, 0, \ldots, 1, 1, \ldots, 1)$ and $r_{G+H}(q/S) = (1, 1, \ldots, 0, \ldots)$. Hence, there exist at least one vertex $x \in S_G \setminus N_G(p)$ or there exist at least one vertex $y \in S_H \setminus N_G(q)$.
Thus,
$\big(N_{G+H}(p)\backslash N_{G+H}(q)\big) \cap S \neq \varnothing$ or $\big(N_{G+H}(q)\backslash N_{G+H}(p)\big) \cap S \neq \varnothing$.
The proof that $p \in V(G + H) \setminus S$ and $q \in S$ is similar.
Therefore, S is a 2-locating set of $G + H$. □

The following corollaries follow immediately from Theorem 5, Theorem 4, and Theorem 3.

Corollary 2. Let G be a nontrivial connected graph and $K_1 = \langle v \rangle$. Then $S \subseteq V(K_1 + G)$ is a 2-locating set in $K_1 + G$ if and only if it satisfies the following conditions:
(*i*) $v \notin S$ and S is (2,2)-locating set of G.
(*ii*) $S = \{v\} \cup T$ and T is (2,1)-locating set in G.

Corollary 3. Let G be any nontrivial connected graph. Then

$$ln_2(K_1 + G) = \min\{ln_{(2,2)}(G), ln_{(2,1)}(G) + 1\}.$$

Corollary 4. Let G and H be nontrivial connected graphs. A set $S \subseteq V(G + H)$ is a 2-locating set in $G+H$ if and only if $S = S_G \cup S_H$ where $S_G = V(G) \cap S$ and $S_H = V(H) \cap S$ are 2-locating sets of G and H, respectively, where S_G or S_H is a $(2, 2)$-locating set or S_G and S_H are $(2, 1)$-locating sets.

Corollary 5. Let G and H be nontrivial connected graphs. Then

$$\begin{aligned} ln_2(G + H) = \min\{&ln_{(2,2)}(G) + ln_2(H), ln_2(G) + ln_{(2,2)}(H), ln_{(2,1)}(G) \\ &+ ln_{(2,1)}(H)\}. \end{aligned}$$

6. Corona of Graphs

This section presents the characterizations on the 2-locating sets in the corona of graphs.

Theorem 6. Let G and H be nontrivial connected graphs with $\Delta(H) \leq |V(H)| - 3$. A set $S \subseteq V(G \circ H)$ is a 2-locating set of $G \circ H$ if and only if $S = A \cup \left(\bigcup_{v \in V(G)} S_v\right)$ where $A \subseteq V(G)$ and $V(H^v) \cap S \neq \varnothing$ for each $v \in V(G)$ and the following are satisfied

(*i*) S_v is a 2-locating set of H^v for each $v \in V(G)$ and S_u or S_v is total 2-dominating for $u, v \in V(G) \setminus A$ or otherwise, S_u and S_v are total dominating;

(*ii*) for each $v \in V(G) \setminus A$, S_v is a $(2,2)$-locating set of H^v with $N_G(v) \cap A = \varnothing$ and S_v is (2,1)-locating set, otherwise; and

(*iii*) for each $v \in A$, S_v is a $(2,1)$-locating set of H^v if $N_G(v) \cap A = \varnothing$.

Proof. Suppose $S \subseteq V(G \circ H)$ is a 2-locating set in $G \circ H$. Let $A = V(G) \cap S$, $S_v = S \cap V(H^v)$ for all $v \in V(G)$. Then $S = A \cup \left(\bigcup_{v \in V(G)} S_v\right)$ where $A \subseteq V(G)$ and $S_v \subseteq V(H^v)$. Now, suppose $S_v = \varnothing$ for some $v \in V(G)$. Let $x, y \in V(H^v)\backslash S_v$. Then $\big|[\big(N_{H^v}(x)\backslash N_{H^v}(y)\big) \cap S_v] \cup [\big(N_{H^v}(y)\backslash N_{H^v}(x)\big) \cap S_v]\big| = 0$, for all $x, y \in V(H^v)\backslash S_v$ with $x \neq y$, a contradiction to the assumption of S. Thus, $S_v \neq \varnothing$ for all $v \in V(G \circ H)$.

To prove (*i*), let $x, y \in V(H^v)$ where $v \in V(G)$. Then $x, y \in V(G \circ H)$. Since $N_{H^v}(x) = N_{G \circ H}(x)\backslash\{v\}$ and $N_{H^v}(y) = N_{G \circ H}(y)\backslash\{v\}$, and S is a 2-locating set, this implies that S_v is also 2-locating set in H^v. Next, suppose S_u or S_v is not a total dominating, say S_v is not a total dominating set for some $v \in V(G) \setminus A$. Let $x \in V(H^u)\backslash S_u$ and $y \in V(H^v)\backslash S_v$. Since S is a 2-locating set, there exist $w, z \in (N_{H^v}(x)\backslash N_{H^v}(y)) \cap S_u$ implying that S_u is a total 2-dominating set.

To prove (*ii*), let $v \in V(G)\backslash A$. Suppose $N_G(v) \cap A = \varnothing$. Since $S_v \subseteq N_{G \circ H}(v)$ and S is 2-locating, there exist at least two vertices $x, y \in S_v \setminus N_{H^v}(p)$ for each $p \in V(H^v)$. Thus, S_v is $(2,2)$- locating set. On the other hand, if $N_G(v) \cap A \neq \varnothing$, there exists at least one vertex $z \in S_v \setminus N_{H^v}(p)$. This implies that S_v is $(2,1)$ -locating.

To prove (*iii*), let $v \in A$ and $N_G(v) \cap A = \varnothing$. Since S_v is a 2-locating set, there exists $r \in S_v \setminus N_{H^v}(p)$ for every $p \in V(H^v)$. Thus, S_v is a (2,1)-locating set in H^v.

For the converse, suppose S is a set as described and satisfies the given conditions. Let $p, q \in V(G \circ H)$ with $p \neq q$ and let $u, v \in V(G)$ such that $p \in V(u + H^u)$ and $q \in V(v + H^v)$. Suppose $p, q \in V(G \circ H)\backslash S$. Consider the following cases:

Case 1. $u = v$

Subcase 1.1 $p, q \in V(H^u) \setminus S_u$

Since S_u is a 2-locating set of H^u, $N_{H^u}(p) = N_{G \circ H}(p)$ and $N_{H^u}(q) = N_{G \circ H}(q)$. Then

$$|[(N_{G \circ H}(q)\backslash N_{G \circ H}(p)) \cap S] \cup [(N_{G \circ H}(p)\backslash N_{G \circ H}(q)) \cap S]| \geq 2$$

and for all $r \in S_u$, $(N_{G\circ H}(r) \backslash N_{G\circ H}(q)) \cap S \neq \varnothing$. Thus, S is a 2-locating set.
Subcase 1.2 $p = v$ and $q \in V(H^v) \setminus S_v$
If $N_G(v) \cap A = \varnothing$, by (ii) S_v is a (2,2)-locating set. Hence, there exist at least two distinct vertices $x, y \in V(H^v) \setminus N_{H^v}(q)$. Thus, $x, y \in N_{G\circ H(p)} \setminus N_{G\circ H}(q)$. If $N_G(v) \cap A \neq \varnothing$, then there exists $z \in (N_{G\circ H}(v) \cap A) \setminus N_{G\circ H}(q)$. Since $\gamma(H) \neq 1$, there exists $w \in S_v \setminus N_{H^v}(q)$. Hence, $w, z \in N_{G\circ H}(p) \setminus N_{G\circ H}(q) \cap S$. Thus, $|(N_{G\circ H}(p) \setminus N_{G\circ H}(q) \cap S)| \geq 2$.
Subcase 1.3 $q = v$ and $p \in V(H^u) \setminus S_u$
The proof is similar to the proof of Subcase 1.2.

Case 2. $u \neq v$
Subcase 2.1 $p \in V(H^u) \setminus S_u$ and $q \in V(H^v) \setminus S_v$
If $u, v \in A$, then we are done. Suppose $u, v \notin A$. Since S_u and S_v are total dominating, there exist $x \in (N_{H^u}(p) \cap S_u) \setminus N_{H^v}(q)$ and $y \in (N_{H^v}(q) \cap S_v) \setminus N_{H^u}(p)$.
Subcase 2.2 $p = u$ and $q \in V(H^v) \setminus S_v$
Since $p \notin A$, S_u is a total dominating set of H^u. Hence, $|S_u| \geq 2$. Thus, $|(N_{G\circ H}(p) \setminus N_{G\circ H}(q)) \cap S| \geq 2$.

Suppose $p \in S$ and $q \in V(G \circ H) \setminus S$. Consider the following cases

Case 1 $u = v$
Subcase 1.1 $p \in S_v$ and $q \in V(H^v) \setminus S_v$
Since S_v is a 2-locating, then $(N_{G\circ H}(p) \backslash N_{G\circ H}(q)) \cap S \neq \varnothing$.
Subcase 1.2 $u = p$ and $q \in V(H^v) \setminus S_v$. Then $u \in A$.
If $N_G(p) \cap A \neq \varnothing$, then we are done. Suppose $N_G(p) \cap A = \varnothing$. Then by (iii), S_v is a (2,1)-locating. Thus, $(N_{G\circ H}(p) \setminus N_{G\circ H}(q)) \cap S \neq \varnothing$.

Case 2 $u \neq v$
Subcase 2.1 $p \in S_u$ and $q \in V(H^v) \setminus S_v$
If $u \in A$ or $v \in A$, then we are done. If $u, v \notin A$, then by (i) S_u and S_v are total dominating. Hence, there exist $x \in (N_{G\circ H}(p) \cap S) \setminus N_{G\circ H}(q)$ and $y \in (N_{G\circ H}(q) \cap S) \setminus N_{G\circ H}(p)$.
Subcase 2.2 $p = u$ and $q \in V(H^v) \setminus S_v$
Since $S_u \neq \varnothing$, $(N_{G\circ H}(p) \cap S) \setminus N_{G\circ H}(q) \neq \varnothing$.
Subcase 2.3 $p \in S_u$ and $q = v$
Similar to the proof of Subcase 2.2.

Accordingly, S is a 2-locating set of $G \circ H$. □

Corollary 6. Let G of order n and H be nontrivial connected graphs with $\gamma(H) \neq 1$. Then

(i) $ln_2(G \circ H) \leq \gamma_t(G) + n \cdot ln_2(H)$; and

(ii) If $ln_2(H) = ln_{(2,1)}(H) = ln_{(2,2)}(H)$. Then $ln_2(G \circ H) = n \cdot ln_2(H)$.

Proof. $(i.)$ Let $S = V(G \circ H)$ be a 2-locating set in $G \circ H$. Let A be a γ_t-set of G and S_v be an ln_2-set of H^v. Then $S = A \cup \left(\bigcup_{v \in V(G)} S_v \right)$ is a 2-locating set of $G \circ H$. Thus,

$$\begin{aligned} ln_2(G \circ H) &\leq |S| \\ &= |A| + \sum_{v \in V(G)} |S_v| \\ &= \gamma_t(G) + |V(G)|(ln_2(H)) = \gamma_t(G) + n \cdot ln_2(H). \end{aligned}$$

$(ii.)$ Let $A = \varnothing$ and S_v be an ln_2-set of H^v. Then $S = A \cup \left(\bigcup_{v \in V(G)} S_v \right)$ is a 2-locating set of $G \circ H$. Thus,

$$\begin{aligned} ln_2(G \circ H) &\leq |S| \\ &= \sum_{v \in V(G)} |S_v| \\ &= |V(G)| ln_2(H) = n \cdot ln_2(H). \end{aligned}$$

Next, let S_0 be an ln_2-set in $G \circ H$. Then by Theorem 6, $S_0 = A_0 \cup \left(\bigcup_{v \in V(G)} S_v \right)$ where $A_0 \subseteq V(G)$ and S_v is a 2-locating set of H^v, for all $v \in V(G)$. Thus,

$$\begin{aligned} ln_2(G \circ H) &= |S_0| \\ &= |A_0| + |\bigcup_{v \in V(G)} S_v| \\ &\geq \sum_{v \in V(G)} |S_v| \\ &\geq n \cdot ln_2(H) \end{aligned}$$

Thus, equality holds. □

7. Edge Corona of Graphs

This section presents characterizations on the 2-locating sets in the edge corona of graphs.

Theorem 7. Let G and H be nontrivial connected graphs where $G \neq P_2$ and $\Delta(H) \leq |V(H)| - 3$. A set $C \subseteq V(G \diamond H)$ is a 2-locating set of $G \diamond H$ if and only if C is a 2-resolving set of $G \diamond H$.

Proof. Let C be a 2-locating set of $G \diamond H$. By Remark 3, C is a 2-resolving set of $G \diamond H$.

Conversely, suppose C is a 2-resolving set of $G \diamond H$. Let $a, b \in V(H^{uv}) \backslash S_{uv}$ where $a \neq b$ or $[a \in S_{uv}$ and $b \notin S_{uv}]$. Since C is a 2-resolving set in $G \diamond H$, $r_{G\diamond H}(a/C)$ and $r_{G\diamond H}(b/C)$ differ in at least 2 positions. Since $N_{G\diamond H}(a) = N_{H^{uv}}(a) \cup \{u, v\}$ and $N_{G\diamond H}(b) = N_{H^{uv}}(b) \cup \{u, v\}$, $r_{H^{uv}}(a/S_{uv})$ and $r_{H^{uv}}(b/S_{uv})$ must differ in at least 2 positions. By definition of $G \diamond H$, there exist at least two vertices say $p, q \in V(H^{uv}) \cap S_{uv}$ such that either $p, q \in N_{H^{uv}}(a) \backslash N_{H^{uv}}(b)$ or $p, q \in N_{H^{uv}}(b) \backslash N_{H^{uv}}(a)$ or $p \in N_{H^{uv}}(a) \backslash N_{H^{uv}}(b)$ and $q \in N_{H^{uv}}(b) \backslash N_{H^{uv}}(a)$. Similarly, if $a \in S_{uv}$ and $b \in V(H^{uv}) \backslash S_{uv}$, then there exists a vertex $s \in V(H^{uv}) \cap S_{uv}$ such that $s \in N_{H^{uv}}(a) \backslash N_{H^{uv}}(b)$ or $s \in N_{H^{uv}}(b) \backslash N_{H^{uv}}(a)$. Thus, it follows that S_{uv} is a 2-locating set of H^{uv}.

Accordingly, C is a 2-locating set in $G \diamond H$. □

Theorem 8. Let G and H be any nontrivial connected graphs where $G \neq P_2$ and $\Delta(H) \leq |V(H)| - 3$. A set $C \subseteq V(G \diamond H)$ is a 2-locating set of $G \diamond H$ if and only if

$$C = A \cup \left(\bigcup_{uv \in E(G)} S_{uv} \right)$$

where

(i) $A \subseteq V(G)$, $S_{uv} \subseteq V(H^{uv})$ and $V(H^{uv}) \cap C \neq \varnothing$;

(ii) $S_{uv} \subseteq V(H^{uv})$ is a 2-locating set of H^{uv} for all $uv \in E(G)$ or if uv is a pendant edge, then S_{uv} is a $(2,1)$-locating set of H^{uv} whenever $l(\langle\{u, v\}\rangle) \subseteq A$ and S_{uv} is a $(2,2)$-locating set of H^{uv} otherwise.

Proof. Suppose that $C \subseteq V(G \diamond H)$ is a 2-locating set in $G \diamond H$. Let $A = V(G) \cap C$ and $S_{uv} = C \cap V(H^{uv})$ for all $uv \in E(G)$. Then $C = A \cup \left(\bigcup_{uv \in E(G)} S_{uv} \right)$ where $A \subseteq V(G)$ and $S_{uv} \subseteq V(H^{uv})$. Now, suppose that $S_{uv} = \varnothing$ for some $uv \in E(G)$. Let $x, y \in V(H^{uv}) \setminus S_{uv}$. Then $\left| [\left(N_{H^{uv}}(x) \backslash N_{H^{uv}}(y)\right) \cap S_{uv}] \cup [\left(N_{H^{uv}}(y) \backslash N_{H^{uv}}(x)\right) \cap S_{uv}] \right| = 0$, a contradiction to the assumption of C. Thus, $S_{uv} \neq \varnothing$ for all $uv \in E(G)$. Next, we claim that S_{uv} is a 2-locating set in H^{uv} for each $uv \in E(G)$. Let $a, b \in V(H^{uv})$ where $uv \in E(G)$. Then $a, b \in V(G \diamond H)$. Since $N_{H^{uv}}(a) = N_{G\diamond H}(a) \setminus \{u, v\}$ and $N_{H^{uv}}(b) = N_{G\diamond H}(b) \setminus \{u, v\}$ and C is a 2-locating set, this implies that S_{uv} is also a 2-locating set in H^{uv}. Next, suppose that uv is a pendant edge and suppose u is an end-vertex. Then $\langle u \rangle + H^{uv}$ is a subgraph of $G \diamond H$. Since $S_{uv} = C \cap V(H^{uv}) \subseteq C$ and C is a 2-locating set, it follows by Corollary 2, S_{uv} is a (2,1)-locating set of H^{uv} whenever $u \in C$ and S_{uv} is a (2,2)-locating set of H^{uv}, otherwise.

Conversely, let C be the set as described and satisfies the given conditions. Let $x, y \in V(G \diamond H)$ with $x \neq y$. Then it can be easily verified that $r_{G\diamond H}(x/C)$ and $r_{G\diamond H}(y/C)$ differ in at least two positions for all $x, y \in V(G)$ or $x \in V(H^{uv})$ and $y \in V(G)$ for all edges $uv \in E(G)$ or $x \in V(H^{pq})$ and $y \in V(H^{ab})$, for some $pq, ab \in E(G)$.
Hence, consider only the following cases:

Case 1: $x, y \in V(H^{uv}) \setminus S_{uv}$ or $x \in V(H^{uv}) \setminus S_{uv}$ and $y \in S_{uv}$ for some edge $uv \in E(G)$. Now, since S_{uv} is 2-locating set, $r_{H^{uv}}(x/S_{uv})$ and $r_{H^{uv}}(x/S_{uv})$ differ in at least two positions. Then by definition of $G \diamond H$, $r_{G\diamond H}(x/C)$ and $r_{G\diamond H}(y/C)$ differ in at least two positions.

Case 2: $x \in V(H^{uv}) \setminus S_{uv}$ or $x \in S_{uv}$ and $y = u$ for some pendant edge $uv \in E(G)$ and u is an endvertex

Since S_{uv} is a (2,2)-locating set, there exists $a, b \in S_{uv} \setminus N_{H^{uv}}(x)$ but $a, b \in N_{G\diamond H}(y)$. Thus, it follows that $r_{G\diamond H}(x/C)$ and $r_{G\diamond H}(y/C)$ differ at a^{th} and b^{th} positions. Therefore, C is a 2-resolving set in $G \diamond H$. By Theorem 7, C is a 2-locating set in G. □

Corollary 7. Let G and H be any nontrivial connected graphs where $G \neq P_2$ with $|E(G)| = m$ and $\Delta(H) \leq |V(H)| - 3$. Then the following statements hold.

(*i*) If G is a graph with no pendant edges, then $ln_2(G \diamond H) = m \cdot ln_2(H)$.

(*ii*) If G is a graph with $k \geq 1$ pendant edges, then $ln_2(G \diamond H) = \min\{(m-k)ln_2(H) + k \cdot ln_{(2,1)}(H) + k, (m-k)ln_2(H) + k \cdot ln_{(2,2)}(H)\}$ and $ln_2(G \diamond H) = (m-k)ln_2(H) + k \cdot ln_{(2,2)}(H)$ whenever $ln_{(2,2)}(H) = ln_{(2,1)}(H)$.

Proof. (*i*) Suppose G is a graph with no pendant edges. Now, set $A = \varnothing$ and let S_{uv} be an $ln_2 - set$ of H^{uv} for all $uv \in E(G)$. Then $C = A \cup \left(\bigcup_{uv \in E(G)} S_{uv} \right)$ is a 2-locating set in $G \diamond H$ by Theorem 8. Hence,

$$ln_2(G \diamond H) \leq |C| = |A| + |E(G)||S_{uv}| = m(ln_2(H)).$$

Next, let C_0 be an $ln_2 - set$ in $G \diamond H$. Then by Theorem 8, $C_0 = A_0 \cup \left(\bigcup_{uv \in E(G)} S_{uv} \right)$ where $A_0 \subseteq V(G)$ and S_{uv} is a 2-locating set of H^{uv} for all $uv \in E(G)$. Thus,

$$\begin{aligned} ln_2(G \diamond H) &= |C_0| \\ &= |A_0| + |\bigcup_{uv \in E(G)} S_{uv}| \\ &\geq \sum_{uv \in E(G)} |S_{uv}| \\ &\geq m \cdot ln_2(H). \end{aligned}$$

Therefore, $ln_2(G \diamond H) = m \cdot ln_2(H)$.

(*ii*) Let G be a graph with pendant edges and $A \subseteq V(G)$ consists of pendant edges in a graph G, that is $|A| = k$. By Theorem 8, S_{uv} is a 2-locating set of H^{uv} for all $uv \in E(G)$ and S_{uv} is a (2,1)-locating set of H^{uv} whenever $l(uv) \subseteq A$ and S_{uv} is a (2,2)-locating set of H^{uv}, otherwise. If S_{uv} is a (2,2)-locating sets in H^{uv}, then

$$(m-k)ln_2(H) + k \cdot ln_{(2,2)}(H) \leq |C| = ln_2(G \diamond H).$$

If S_{uv} is a (2,2)-locating sets in H^{uv}, then

$$(m-k)ln_2(H) + k \cdot ln_{(2,1)}(H) + k \leq |C| = ln_2(G \diamond H).$$

Thus,

$$ln_2(G \diamond H) \geq min\{(m-k)ln_2(H) + k \cdot ln_{(2,1)}(H) + k,\\ (m-k)ln_2(H) + k \cdot ln_{(2,2)}(H)\}.$$

Let $(m-k)ln_2(H) + k \cdot ln_{(2,1)}(H) + k \leq (m-k)ln_2(H) + k \cdot ln_{(2,2)}(H)$. Let S_{uv} be the minimum (2,1)-locating set in H^{uv} whenever $l(uv) \subseteq A$ and S_{uv} be the minimum (2,2)-locating set in H^{uv}, otherwise. Then, C is a 2-locating set in $G \diamond H$ by Corollary 7. Hence, $ln_2(G \diamond H) \leq |C| = (m-k)ln_2(H) + k \cdot ln_{(2,1)}(H) + k$. Similarly, if $(m-k)ln_2(H) + k \cdot ln_{(2,2)}(H) \leq (m-k)ln_2(H) + k \cdot ln_{(2,1)}(H) + k$. Then $ln_2(G \diamond H) \leq |C| = (m-k)ln_2(H) + k \cdot ln_{(2,2)}(H)$. Thus,

$$ln_2(G \diamond H) \leq min\{(m-k)ln_2(H) + k \cdot ln_{(2,1)}(H) + k,\\ (m-k)ln_2(H) + k \cdot ln_{(2,2)}(H)\}.$$

Therefore,

$$ln_2(G \diamond H) = min\{(m-k)ln_2(H) + k \cdot ln_{(2,1)}(H) + k,\\ (m-k)ln_2(H) + k \cdot ln_{(2,2)}(H)\}.$$

□

8. Lexicographic Product of Graphs

This section presents characterizations on the 2-locating sets in the lexicographic product of graphs.

Theorem 9. [6] Let G and H be nontrivial connected graphs. Then $W = \bigcup_{x \in S} [\{x\} \times T_x]$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, is a 2-resolving set in $G[H]$ if and only if

(i) $S = V(G)$;

(ii) T_x is a 2-locating set in H for every $x \in V(G)$;

(iii) T_x and T_y are (2,1)-locating sets or one of T_x and T_y is a(2,2)-locating set in H whenever $x, y \in EQ_1(G)$; and (*iv*) T_x and T_y are (2-locating)dominating sets in H or one of T_x and T_y is a 2-dominating set whenever $x, y \in EQ_2(G)$.

Theorem 10. Let G and H be nontrivial connected graphs with $\Delta(H) \leq |V(H)| - 3$. Then $W = \bigcup_{x \in S}[\{x\} \times T_x]$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, is a 2-locating set in $G[H]$ if and only if it is a 2-resolving set and it satisfies the following:

(*i*) $S = V(G)$;

(*ii*) T_x is a 2-locating set in H for every $x \in V(G)$;

(*iii*) T_x and T_y is a $(2, 1)$-locating set or one of T_x and T_y is a $(2, 2)$-locating set in H whenever $x, y \in V(G)$ with $N_G[x] = N_G[y]$; and

(*iv*) T_x and T_y are $(2 - locating)$ dominating sets in H or one of T_x and T_y is a 2-dominating set whenever $x, y \in V(G)$ with $d_G(x, y) = 2$ and $N_G(x) = N_G(y)$.

Proof. Suppose $W = \bigcup\limits_{x \in S} \big[\, \{x\} \times T_x \big]$ is a 2-locating set in $G[H]$. Suppose there exists $x \in V(G) \backslash S$. Pick $a, b \in V(H)$, where $a \neq b$. Then $(x, a), (x, b) \notin W$ and $(x, a) \neq (x, b)$. Since $x \notin S$, $(x, r) \in V(G[H]) \setminus W$. Note that $(z, c) \in N_{G[H]}(x, a) \cup N_{G[H]}(x, b)$ for all $z \in N_G(x)$. Thus, $N_{G[H]}(x, a) \setminus N_{G[H]}(x, b) = \varnothing$ and $N_{G[H]}(x, b) \setminus N_{G[H]}(x, a) = \varnothing$. This implies that W is not a 2-locating set of $G[H]$, a contradiction to the assumption on W. Therefore, $S = V(G)$.

To prove (*ii*), let $x \in V(G)$ and $p, q \in V(H)$ where $p \neq q$. Then $(x, p) \neq (x, q)$. If $p, q \notin T_x$ or $[p \in T_x$ and $q \notin T_x]$, then $(x, p), (x, q) \notin W$ or $[(x, p) \in W$ and $(x, q) \notin W]$. Since W is a 2-locating set in $G[H]$, by definition of $G[H]$ there exist at least two vertices $(x, r), (x, s) \in V(H) \cap T_x$ such that either $(x, r), (x, s) \in N_H((x, p)) \backslash N_H((x, q))$ or $(x, r), (x, s) \in N_H((x, q)) \backslash N_H((x, p))$ or $(x, r) \in N_H((x, p)) \backslash N_H((x, q))$ and $(x, s) \in N_H((x, q)) \backslash N_H((x, p))$. Similarly, if $(x, p) \in W$ and $(x, q) \notin W$, then there exists a vertex $t \in V(H) \cap T_x$ such that $(x, t) \in N_H((x, p)) \backslash N_H((x, q))$ or $(x, t) \in N_H((x, q)) \backslash N_H((x, p))$. Therefore, it follows that T_x is a 2-locating set of H for every $x \in V(G)$. Thus, (*ii*) follows.

To prove (*iii*), let $x, y \in V(G)$ with $N_G[x] = N_G[y]$. Let $a, b \in V(H)$, $a \neq b$. Since W is a 2-locating set, it is not possible that $N_H(a) \cap T_x = T_x$ and $N_H(b) \cap T_y = T_y$. If T_x or T_y is $(2, 2)$-locating, then we are done. Otherwise, T_x and T_y are $(2, 1)$-locating.

To prove (*iv*), let $x, y \in V(G)$ where $d_G(x, y) = 2$ and $N_G(x) = N_G(y)$. Let $a, b \in V(H)$, $a \neq b$. Suppose one of T_x and T_y, say T_x is not a dominating set in H. Pick $a \in V(H) \backslash N_H[T_x]$ and let $b \in V(H) \backslash T_y$. Since $d_{G[H]}((x, a), (y, b)) = 2$, for all (y, b), it follows that $|N_H(b) \cap T_y| \geq 2$, i.e., T_y is a 2-dominating set.

Conversely, let W be the set as described and satisfies the given conditions. Let $(x, a), (y, b) \in V(G[H])$, $(x, a) \neq (y, b)$. Consider the following cases.

Case 1. $x = y$

Suppose $(x, a), (y, b) \notin W$. Then $a \neq b$ and $a, b \notin T_x = T_y$. By (*ii*), T_x is a 2-locating set. On the other hand, if $(x, a) \in W$, $(y, b) \notin W$, then $a \in T_x$, $b \notin T_y$. Since T_x is a 2-locating set, there exists $(x, s) \in V(H) \cap T_x$ such that $(x, s) \in N_H((x, a)) \backslash N_H((y, b))$ or $(x, s) \in N_H((y, b)) \backslash N_H((x, a))$. Thus, it follows that W is a 2-locating set of $G[H]$.

Case 2. $x \neq y$.
Subcase 2.1 $xy \in E(G)$.

If $N_G[x] \neq N_G[y]$, then we are done. Suppose $N_G[x] = N_G[y]$, then by (iii), T_x and T_y are $(2,1)$-locating sets in H or one of T_x and T_y is a $(2,2)$-locating set in H.
Subcase 2.2 $xy \notin E(G)$

If $d_G(x,y) > 2$, then we are done. Suppose $d_G(x,y) = 2$ and $N_G(x) = N_G(y)$. Suppose $(x,a),(y,b) \notin W$. Then $a \notin T_x$ and $y \notin T_y$. If T_x and T_y are both dominating, then there exist at least two vertices $(x,r),(x,s) \in V(H) \cap T_x$ such that either $(x,r),(x,s) \in N_H((x,p)) \backslash N_H((x,q))$ or $(x,r),(x,s) \in N_H((x,q)) \backslash N_H((x,p))$ or $(x,r) \in N_H((x,p)) \backslash N_H((x,q))$ and $(x,s) \in N_H((x,q)) \backslash N_H((x,p))$. If one, say T_y, is a 2-dominating set, then there exist at least two vertices $r,s \in V(H) \cap T_x$ such that either $(x,r),(x,s) \in N_H((x,p)) \backslash N_H((x,q))$ or $(x,r),(x,s) \in N_H((x,q)) \backslash N_H((x,p))$ or $(x,r) \in N_H((x,p)) \backslash N_H((x,q))$ and $(x,s) \in N_H((x,q)) \backslash N_H((x,p))$. Similarly, if $(x,a) \in W$, $(y,b) \notin W$, there exists $(x,s) \in V(H) \cap T_x$ such that $(x,s) \in N_H((x,a)) \backslash N_H((y,b))$ or $(x,s) \in N_H((y,b)) \backslash N_H((x,a))$.

Accordingly, W is a 2-locating set of $G[H]$. □

Corollary 8. Let G and H be nontrivial connected graphs with $\Delta(H) \leq |V(H)| - 3$. If G is a totally point determining graph, then

$$ln_2(G[H]) = |V(G)| \cdot ln_2(H).$$

Proof. Supppose that G is totally point determining graph. Let $S = V(G)$ and let T_x be an ln_2-set of H for each $x \in S$. By Theorem 10, $W = \bigcup_{x \in S} [\{x\} \times T_x]$ is a 2-locating set of $G[H]$. It follows that

$$ln_2(G[H]) \leq |W| = |V(G)||T_x| = |V(G)| \cdot ln_2(H).$$

Now, if $W_0 = \bigcup_{x \in S_0} [\{x\} \times T_x]$ is an ln_2-set of $G[H]$, then $S_0 = V(G)$ and T_x is a 2-locating set of H for each $x \in V(G)$ by Theorem 10. Hence,

$$ln_2(G[H]) = |W_0| = |V(G)||T_x| \geq |V(G)| \cdot ln_2(H).$$

Therefore, $ln_2(G[H]) = |V(G)| \cdot ln_2(H)$. □

9. Conclusion

It is shown that the difference of the 2-metric dimension and 2-locating number can be made arbitrarily large. 2-locating sets in the join, corona, edge corona, and lexicographic product of two graphs have been characterized. From these characterizations, 2-locating numbers have been determined. This new invariant can also be studied for graphs under other binary operations.

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