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On the Diophantine Equation $(p+n)^x + p^y = z^2$ where p and $p+n$ are prime numbers

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Abstract. In this paper, we study the Diophantine equation $(p+n)^x + p^y = z^2$, where $p, p+n$ are prime numbers and n is a positive integer such that $n \equiv 0 \pmod 4$. In case $p = 3$ and $n = 4$, Rao [7] showed that the non-negative integer solutions are $(x, y, z) = (0, 1, 2)$ and $(1, 2, 4)$. In case $p > 3$ and $p \equiv 3 \pmod 4$, if $n - 1$ is a prime number and $2n - 1$ is not prime number, then the non-negative integer solution (x, y, z) is $(0, 1, \sqrt{p+1})$ or $(1, 0, \sqrt{p+n+1})$. In case $p \equiv 1 \pmod 4$, the non-negative integer solution (x, y, z) is also $(0, 1, \sqrt{p+1})$ or $(1, 0, \sqrt{p+n+1})$.

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1. Introduction

Many mathematicians have been studying the Diophantine equations of the type $(p+n)^x+p^y = z^2$ with a constant n and a specific condition of p, for example, in case that p is a prime number. In 2015, Tatong and Suvarnamani [10] found that $(p, x, y, z) = (3, 1, 0, 2)$ is a unique non-negative integer solution of the Diophantine equation $p^x+(p+1)^y = z^2$ where p is an odd prime number. In 2018, Burshtein [1] showed that the Diophantine equation $p^x + (p+4)^y = z^2$ when $p > 3, p+4$ are primes has no positive integer solution (x, y, z). In the same year, Fernando [3] showed that $p^x + (p+8)^y = z^2$ has no positive integer solution, when $p > 3$ and $p + 8$ are primes. In addition, Kumar, Gupta and Kishan [5] proved that the solution of $p^x + (p+12)^y = z^2$ has no non-negative integer solution where p and $p+12$ are prime numbers and $p = 6n + 1$ for some natural number n. In 2021, Dokchan and Pakapongpun [2] studied a Diophantine equation $p^x + (p + 20)^y = z^2$, when p and $p + 20$ are primes and showed that the equation has no positive integer solution (x, y, z). In the same year, Gayo Jr and Bacani [4] solved the Diophantine equation $M_p^x + (M_q + 1)^y = z^2$ where M_p and M_q are Mersenne primes. In 2022, Tadee [8] gave the solutions of equations $p^x+(p+14)^y = z^2$, where p and $p+14$ are primes. In 2023, Viriyapong and Viriyapong [11] studied the Diophantine equation $a^x + (a + 2)^y = z^2$, where $a \equiv 5 \pmod{21}$ and showed

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that the equation has no non-negative integer solution (x, y, z). In the same year, Tadee and Siraworakun [9] studied the Diophantine equation $p^x+(p+2q)^y = z^2$, where $p, q, p+2q$ are prime numbers and showed that the equation has no positive integer solution.

In this paper, we give a solution of the Diophantine equation $(p+n)^x+p^y = z^2$, where $p, p+n$ are odd prime numbers such that $n \equiv 0 \pmod 4$. To obtain our result, we consider two cases of p in modulo 4, i.e., the case that $p \equiv 1 \pmod 4$ and the case that $p \equiv 3 \pmod 4$. In case $p \equiv 3 \pmod 4$, we first consider case $p = 3$ and $n = 4$. The considered equation in this case is $3^x + 7^y = z^2$ in which the solutions were given by Rao [7] that $(x, y, z) = (0, 1, 2)$ or $(x, y, z) = (1, 2, 4)$. Next, we consider $p \equiv 3 \pmod 4$ such that $p > 3$ with a specific condition that $n - 1$ is a prime number and $2n - 1$ is not a prime number. Then, our final case is when $p \equiv 1 \pmod 4$ and n is a positive integer.

2. Preliminaries

Proposition 1. *(Catalan's conjecture) The Diophantine equation* $a^x - b^y = 1$ *where* $\min\{a, b, x, y\} > 1$ *has a unique solution* $(a, b, x, y) = (3, 2, 2, 3)$.

This proposition was proved in 2004 by Mihailescu [6].

Lemma 1. *Let p be odd prime number. The non-negative integer solution to the Diophantine equation* $1 + p^y = z^2$ *is* $(y, z) = (1, \sqrt{p + 1})$ *if* $\sqrt{p + 1}$ *is a positive integer.*

Proof. Let (y, z) be a non-negative integer solution of $1 + p^y = z^2$. If $y = 0$, then $z^2 = 2$, which is impossible. If $y > 1$, then $z > 1$. By Catalan's conjecture, there is no non-negative integer solution. If $y = 1$, then $z^2 = p + 1$. So $z = \sqrt{p + 1}$. This means that $\sqrt{p + 1}$ is a non-negative integer as z is a non-negative integer.

Lemma 2. *Let n be a positive integer such that* $n \equiv 0 \pmod 4$ *and let p, $p + n$ be prime numbers. The non-negative integer solutions of the Diophantine equation* $1+(p+n)^x = z^2$ *is* $(x, z) = (1, \sqrt{p + n + 1})$ *if* $\sqrt{p + n + 1}$ *is a positive integer.*

Proof. Let (x, z) be a non-negative integer solution of $1 + (p+n)^x = z^2$. If $x = 0$, then $z^2 = 2$, which is impossible. If $x > 1$, then $z > 1$. By Catalan's conjecture, there is no of a non-negative integer solution. If $x = 1$, then $z^2 = p + n + 1$. So $z = \sqrt{p + n + 1}$. This means that $\sqrt{p + n + 1}$ is a non-negative integer as z is a non-negative integer.

Theorem 1. *([7]) Let* $n = 4$ *and* $p = 3$. *The non-negative integer solutions of the Diophantine equation* $(p + n)^x + p^y = z^2$ *are* $(x, y, z) = (0, 1, 2)$ *and* $(1, 2, 4)$.

This theorem was proved in 2017 by Rao [7].

3. Main results

In Theorem 2, we use the same method of proof as appeared in [1, 2] to derive the result.

Theorem 2. *Let n and p be positive integers where $n \equiv 0 \pmod 4$, $p \equiv 3 \pmod 4$ and $p > 3$ such that $p, p+n, n-1$ are prime numbers and $2n-1$ is not a prime number. If $\sqrt{p+1}$ and $\sqrt{p+n+1}$ are integers, then all of the non-negative integer solutions of the Diophantine equation $(p+n)^x + p^y = z^2$ are given by $(x, y, z) \in \{(0, 1, \sqrt{p+1}), (1, 0, \sqrt{p+n+1})\}$, where x, y and z are non-negative integer.*

Proof. Let (x, y, z) be a non-negative integer solution of $(p+n)^x + p^y = z^2$. If $x = 0$ or $y = 0$, then $(x, y, z) = (0, 1, \sqrt{p+1})$ or $(x, y, z) = \left(1, 0, \sqrt{p+n+1}\right)$ by Lemma 1 and Lemma 2. Now, we suppose $x > 0$ and $y > 0$. We consider the following cases. If x and y are even, then $(p+n)^x \equiv 1 \pmod 4$ and $p^y \equiv 1 \pmod 4$. Thus $(p+n)^x + p^y \equiv 2 \pmod 4$ which is impossible since $z^2 \equiv 0, 1 \pmod 4$. If x and y are odd, then $(p+n)^x \equiv 3 \pmod 4$ and $p^y \equiv 3 \pmod 4$. Thus $(p+n)^x + p^y \equiv 2 \pmod 4$ which is impossible since $z^2 \equiv 0, 1 \pmod 4$. Now, there are two remaining cases to be considered.

Case 1. x is even and y is odd.

There exist $k \geq 1$ and $s \geq 0$ such that $x = 2k, y = 2s+1$. We have $(p+n)^{2k} + p^{2s+1} = z^2$, which can be rewritten as

$$p^{2s+1} = z^2 - (p+n)^{2k} = \left[z - (p+n)^k\right]\left[z + (p+n)^k\right].$$

Thus, there exist non-negative integers α, β that $p^\alpha = z - (p+n)^k$ and $p^\beta = z + (p+n)^k$, where $\alpha < \beta$ and $\alpha + \beta = 2s+1$. Hence

$$2(p+n)^k = p^\alpha \left[p^{\beta-\alpha} - 1\right].$$

If $\alpha \geq 1$, then $p \mid (p+n)$ which is impossible since p and $p+n$ are different primes. In Case $\alpha = 0$, we have $2(p+n)^k = p^{2s+1} - 1$. If $s = 0$, then $2(p+n)^k + 1 = p$ which is impossible. If $s \geq 1$, then we have

$$2(p+n)^k = (p-1)\left[p^{2s} + p^{2s-1} + \cdots + p + 1\right].$$

Since $p-1$ is even and $p^{2s} + p^{2s-1} + \cdots + p + 1$ is odd, it follows that $p-1$ is an even positive divisor of $2(p+n)^k$ that is $p - 1 = 2(p+n)^j$, for some integer j such that $0 \leq j < k$. If $j = 0$, then $p = 3$ which contradicts $p > 3$. If $1 \leq j < k$, then $2(p+n)^j + 1 = p$ which is also impossible.

Case 2. x is odd and y is even.

There exist $k \geq 0$ and $s \geq 1$ such that $x = 2k+1$ and $y = 2s$. We now have $(p+n)^{2k+1} + (p)^{2s} = z^2$, which can be rewritten as

$$(p+n)^{2k+1} = z^2 - p^{2s} = (z + p^s)(z - p^s).$$

Thus, there exist non-negative integers α, β such that $(p+n)^\alpha = z - p^s$ and $(p+n)^\beta = z + p^s$, where $\alpha < \beta$ and $\alpha + \beta = 2k+1$. Then

$$2p^s = (p+n)^\alpha \left[(p+n)^{\beta-\alpha} - 1\right].$$

If $\alpha \geq 1$, then $(p+n) \mid p$ which is impossible. In Case $\alpha = 0$, we have $2p^s = (p+n)^{2k+1} - 1$. If $k = 0$, then $2p^s - p = n - 1$. Hence $p\left[2p^{s-1} - 1\right] = n - 1$. Since $n - 1$ is prime, it follows that $p = n - 1$ this contradicts the fact that $p + n = 2n - 1$ is not prime. If $k \geq 1$, then

$$2p^s = (p+n-1)\left[(p+n)^{2k} + (p+n)^{2k-1} + \cdots + (p+n) + 1\right].$$

Since $p + n - 1$ is even and $(p+n)^{2k} + (p+n)^{2k-1} + \cdots + (p+n) + 1$ is odd, it follows that $p + n - 1$ is an even positive divisor of $2p^s$ that is $p + n - 1 = 2p^l$, for some integer l such that $0 \leq l < s$. If $l = 0$, then $p + n = 3$, which is impossible since $p > 3$ and n are positive integer. If $1 \leq l < s$, then $p\left[2(p)^{l-1} - 1\right] = n - 1$ which is also impossible.

Example 1. *There are infinitely many n, p of the form $n \equiv 0 \pmod 4$,$p \equiv 3 \pmod 4$ where $p > 3$ such that $p, p+n, n-1$ are prime numbers and $2n-1$ is not a prime number. Some Diophantine equations of particular values of n where n is between 1 to 70 with positive integers $\sqrt{p+1}$ and $\sqrt{p+n+1}$ are given in the table below.*

n	$(p+n)^x + p^y = z^2$	(x, y, z)
8	$(p+8)^x + p^y = z^2$	$\{(0, 1, \sqrt{p+1}\} \cup \{(1, 0, \sqrt{p+9})\}$
20	$(p+20)^x + p^y = z^2$ [2]	$\{(0, 1, \sqrt{p+1}\} \cup \{(1, 0, \sqrt{p+21})\}$
32	$(p+32)^x + p^y = z^2$	$\{(0, 1, \sqrt{p+1}\} \cup \{(1, 0, \sqrt{p+33})\}$
44	$(p+44)^x + p^y = z^2$	$\{(0, 1, \sqrt{p+1}\} \cup \{(1, 0, \sqrt{p+45})\}$
48	$(p+48)^x + p^y = z^2$	$\{(0, 1, \sqrt{p+1}\} \cup \{(1, 0, \sqrt{p+49})\}$
60	$(p+60)^x + p^y = z^2$	$\{(0, 1, \sqrt{p+1}\} \cup \{(1, 0, \sqrt{p+61})\}$
68	$(p+68)^x + p^y = z^2$	$\{(0, 1, \sqrt{p+1}\} \cup \{(1, 0, \sqrt{p+69})\}$

Table 1: Diophantine equations satisfying the condition in Theorem 2.

Theorem 3. *Let p and n be a positive integer where $n \equiv 0 \pmod 4, p \equiv 1 \pmod 4$ such that p and $p+n$ are prime numbers. If $\sqrt{p+1}$ and $\sqrt{p+n+1}$ are integers, then the all of the non-negative integer solutions of $(p+n)^x + p^y = z^2$ are given by $(x, y, z) \in \{(0, 1, \sqrt{p+1}), (1, 0, \sqrt{p+n+1})\}$.*

Proof. Let (x, y, z) be a non-negative integer solution of $(p+n)^x + p^y = z^2$. If $x = 0$ or $y = 0$, then $(x, y, z) = (0, 1, \sqrt{p+1})$ or $(x, y, z) = \left(1, 0, 2\sqrt{p+n+1}\right)$ by Lemma 1 and Lemma 2. If $x > 0$ and $y > 0$, then $(p+n)^x \equiv 1 \pmod 4$ and $p^y \equiv 1 \pmod 4$. Thus $(p+n)^x + p^y \equiv 2 \pmod 4$ which is impossible since $z^2 \equiv 0, 1 \pmod 4$.

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