EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 17, No. 1, 2024, 310-323
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Fixed Point theorem in symmetric space employing (c)-comparison functions and binary relation

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Abstract. In this paper, we prove the results on existence and uniqueness of fixed points in the setting of symmetric space under ψ-contractions using a binary relation. We also provide some examples to illustrate our newly proved results

2020 Mathematics Subject Classifications: 47H10, 54H25

Key Words and Phrases: Symmetric space binary relation, (c)-comparison functions

1. Introduction

The Banach contraction Principle (BCP), which was developed by the famous Polish mathematician Banach [10], continued to be an inspiration for reseachers in this field. By utilising an amorphous binary relation, Alam and Imdad [5, 6] recently derived an interesting generalisation of the classical Banach contraction principle. The authors did this by introducing relation theoretic analogues of some involved metrical terms, such as completeness,contraction, continuity etc. Indeed, under the universal relation, such newly defined notions reduce to their corresponding usual notion, and subsequently relation-theoretic coincidence point theorem/ metrical fixed point theorem reduced to their corresponding coincidence point theorem/ classical fixed point theorem. Due to its simplicity and wide applicability, this idea has been developed and modified in many different ways in recent years, see [1, 20].

The study of fixed points for contraction mapping in symmetric space was initiated by Cicchese [15] in 1976. Wilson [21] introduced the concept of such spaces by droping the triangle inequality from metric limitation. By now, there exists a considerable literature

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DOI: https://doi.org/10.29020/nybg.ejpam.v17i1.4823

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on fixed point theory in symmetric spaces. In several noted articles written in subsequent years, numerous fixed point results in this setting were established which include Aamri and El Moutawakil [3], Jachymski et al.[17], Aamri et al. [2], Hicks and Rhoades [16], and others. The conclusions of the present work are based on a novel fixed point theorem for regular symmetric spaces that was established by Bessenyei and Pales [12].

The idea of ψ-contraction is primarily investigated by Browder[14] in 1968, wherein the author considered ψ to be increasing and right continuous control function and utilized the same to extend the BCP. Many scholars modified the characteristics of the control function ψ and then generalised the Browder fixed point theorem (e.g.Matkowski contractions [19] and Boyd-Wong contractions [13]). On the other hand, Ahmadullah et al. [4] utilised the idea of (c)-comparison functions to demonstrate a fixed point theorem in a metric space endowed with an amorphous relation that satisfies generalised ψ-contractions.

The aim of this manuscript is to extend the relation-theoretic contraction principle to the class of symmetric spaces involving (c)-comparison functions with the condition $(W3)$. We also deduce the corresponding results for regular symmetric spaces. We provide some examples to demonstrate our results.

2. Preliminaries

Throughout this manuscript $\mathbb{N}_0$, $\mathbb{N}$, $\mathbb{R}^+$, $\mathbb{R}$, and $\mathbb{Q}$ denotes the set of whole numbers, natural numbers, nonnegative real numbers , real numbers and the rational numbers respectively.

Definition 1. *[8, 21] Let $\check{\mathfrak{G}}$ be a nonempty set and p a mapping from $\check{\mathfrak{G}} \times \check{\mathfrak{G}} \to \mathbb{R}^+$ satisfying the following axioms:*

(i) $p(\varpi, \vartheta) = 0$ *if and only if* $\varpi = \vartheta$,

(ii) $p(\varpi, \vartheta) = p(\vartheta, \varpi)$ *for each* $\varpi, \vartheta \in \check{\mathfrak{G}}$.

Then p is a symmetric on $\check{\mathfrak{G}}$ and the pair $(\check{\mathfrak{G}}, p)$ is called a symmetric space.

The concepts of convergent and Cauchy sequences are defined normally in such spaces. A sequence $\{\varpi_n\} \in \check{\mathfrak{G}}$ is said to be convergent to $\varpi \in \check{\mathfrak{G}}$ if $\lim_{\varpi \to \infty} p(\varpi_n, \varpi) = 0$. Also, a sequence is Cauchy if for each $\epsilon > 0$ there exists some $N \in \mathbb{N}$ such that $p(\varpi_n, \vartheta_n) < \epsilon \ \forall n, m \geq N$. The space $\check{\mathfrak{G}}$ is said to be complete if every Cauchy sequence in $\check{\mathfrak{G}}$ converges. The open ball with center $\varpi \in \check{\mathfrak{G}}$ and radius $r > 0$ is defined by

$$B(\varpi, r) = \{\vartheta \in \check{\mathfrak{G}} : p(\varpi, \vartheta) < r\}.$$

If A is a subset of $\check{\mathfrak{G}}$, then

$$diam(A) = \sup\{p(\varpi, \vartheta) : \varpi, \vartheta \in A\}.$$

We require some additional axioms to prove fixed point theorems in such spaces in order to get around the aforementioned difficulties. The following axioms have played a significant role in the literature.

- (W_3): For $\{\varpi_n\}$, ϖ and ϑ in $\check{\mathfrak{G}}$;

$$p(\varpi_n, \varpi) \to 0 \text{ and } p(\varpi_n, \vartheta) \to 0 \implies \varpi = \vartheta.$$

- (W_4): For $\{\varpi_n\}$, $\{\vartheta_n\}$ and ϖ in $\check{\mathfrak{G}}$;

$$p(\varpi_n, \varpi) \to 0 \text{ and } p(\varpi_n, \vartheta_n) \to 0 \implies p(\vartheta_n, \varpi) \to 0.$$

- (HE): For $\{\varpi_n\}$, $\{\vartheta_n\}$ and ϖ in $\check{\mathfrak{G}}$;

$$p(\varpi_n, \varpi) \to 0 \text{ and } p(\varpi_n, \vartheta) \to 0 \implies p(\varpi_n, \vartheta_n) \to 0.$$

- (IC): For $\{\varpi_n\}$, ϖ and ϑ in $\check{\mathfrak{G}}$;

$$p(\varpi_n, \varpi) \to 0 \implies p(\vartheta_n, \varpi) \to p(\vartheta, \varpi).$$

  If $(\check{\mathfrak{G}}, p)$ satisfies the property (IC) then the symmetry p is called 1-continuous.

- (CC): For $\{\varpi_n\}$, $\{\vartheta_n\}$ and ϖ and ϑ in $\check{\mathfrak{G}}$;

$$p(\varpi_n, \varpi) \to 0 \text{ and } p(\vartheta_n, \vartheta) \to 0 \implies p(\varpi_n, \vartheta_n) \to p(\varpi, \vartheta).$$

  If $(\check{\mathfrak{G}}, p)$ satisfies the property (CC) then the symmetry p is called continuous.

  we observe that

$$(CC) \implies (IC), (W_4) \implies (W_3) \text{ and } (IC) \implies (W_3).$$

  But the converse of the above implications are not true in general. Moreover, (CC) implies all the other four conditions, namely (W_3); (W_4); (HE) and (1C).

Definition 2. *[12] Let $(\check{\mathfrak{G}}, p)$ be a symmetric space. A function $\varphi : \mathbb{R}_+^2 \to \mathbb{R}_+$ is called a triangle function with respect to the symmetry p if*

(a) φ is symmetry,

(b) φ is monotonically increasing in both the arguments,

(c) $\varphi(0,0) = 0$,

(d) $p(\varpi, \vartheta) \leq \varphi(p(\varpi, z), p(\vartheta, z))$ *for all* $\varpi, \vartheta, z \in \check{\mathfrak{G}}$.

Proposition 1. *[12] Every symmetric space* $(\check{\mathfrak{G}}, p)$ *admits a unique triangle function* Φ_p *such that* $\Phi_p \leq \varphi$*, where* φ *is any other triangle function with respect to* p.

Such a unique triangle function Φ_p is called the basic triangle function.

Definition 3. *[12] "A symmetric space* $(\check{\mathfrak{G}}, p)$ *is said to be a regular space if the basic triangle function with respect to the symmetry* p *is continuous at the origin (0,0).*

Lemma 1. *[12] "The topology of a regular symmetric space is always Hausdorff. A convergent sequence in a regular symmetric space possesses a unique limit and it has the Cauchy property. Moreover, a symmetric space* $(\check{\mathfrak{G}}, p)$ *is regular if and only if"*

$$\lim_{\epsilon \to 0} \sup_{p \in 0} B(p, \epsilon) = 0.$$

Proposition 2. *[8] Every regular symmetric space possesses the property* (W_3).

Definition 4. *[18] Let* $\check{\mathfrak{G}}$ *be a nonempty set. A subset* $\check{\mathfrak{R}}$ *of* $\check{\mathfrak{G}}^2$ *is called a binary relation on* $\check{\mathfrak{G}}$*. The subsets,* $\check{\mathfrak{G}}^2$ *and* $\emptyset$ *of* $\check{\mathfrak{G}}^2$ *are called the universal relation and empty relation respectively.*

Definition 5. *[9] Let* $\check{\mathfrak{R}}$ *be a binary relation on a nonempty set* $\check{\mathfrak{G}}$*. For* $\varpi, \vartheta \in \check{\mathfrak{G}}$*, we say that* ϖ *and* ϑ *are* $\check{\mathfrak{R}}$*-comparative if either* $(\varpi, \vartheta) \in \check{\mathfrak{R}}$ *or* $(\vartheta, \varpi) \in \check{\mathfrak{R}}$*. We denote it by* $[\varpi, \vartheta] \in \check{\mathfrak{R}}$.

Proposition 3. *If* $(\check{\mathfrak{G}}, p)$ *is a symmetric space,* $\check{\mathfrak{R}}$ *is a binary relation on* $\check{\mathfrak{G}}$*,* $\check{\mathfrak{T}}$ *a self-mapping on* $\check{\mathfrak{G}}$*. Then these conditions are equivalent:*

(1) $p(\check{\mathfrak{T}}\varpi, \check{\mathfrak{T}}\vartheta) \leq \psi(p(\varpi, \vartheta)) \;\; \forall\, \varpi, \vartheta \in \check{\mathfrak{G}}$ *with* $(\varpi, \vartheta) \in \check{\mathfrak{R}}$,

(2) $p(\check{\mathfrak{T}}\varpi, \check{\mathfrak{T}}\vartheta) \leq \psi(p(\varpi, \vartheta)) \;\; \forall\, \varpi, \vartheta \in \check{\mathfrak{G}}$ *with* $[\varpi, \vartheta] \in \check{\mathfrak{R}}$.

Definition 6. *[6] Let* $\check{\mathfrak{G}}$ *be a non-empty set and* $\check{\mathfrak{R}}$ *a binary relation on* $\check{\mathfrak{G}}$*. A sequence* $\varpi_n \subset \check{\mathfrak{G}}$ *is called* $\check{\mathfrak{R}}$*- preserving if*

$$(\varpi_n, \varpi_{n+1}) \in \check{\mathfrak{R}} \;\forall\; n \in \mathbb{N}_0.$$

Definition 7. *[6] Let* $\check{\mathfrak{G}}$ *be a nonempty set and* $\check{\mathfrak{T}}$ *a self-mapping on* $\check{\mathfrak{G}}$*. A binary relation* $\check{\mathfrak{R}}$ *defined on* $\check{\mathfrak{G}}$ *is called* $\check{\mathfrak{T}}$*-closed if for any* $\varpi, \vartheta \in \check{\mathfrak{G}}$

$$(\varpi, \vartheta) \in \check{\mathfrak{R}} \implies (\check{\mathfrak{T}}\varpi, \check{\mathfrak{T}}\vartheta) \in \check{\mathfrak{R}}.$$

Definition 8. *[6] Let* $\check{\mathfrak{G}}$ *be a nonempty set and* $\check{\mathfrak{T}}$ *a self-mapping on* $\check{\mathfrak{G}}$*. A binary relation* $\check{\mathfrak{R}}$ *defined on* $\check{\mathfrak{G}}$ *is called* $\check{\mathfrak{T}}$*-transitive if for any* $\varpi, \vartheta, z \in \check{\mathfrak{G}}$

$$(\check{\mathfrak{T}}\varpi, \check{\mathfrak{T}}z), (\check{\mathfrak{T}}z, \check{\mathfrak{T}}\vartheta) \in \check{\mathfrak{R}} \implies (\check{\mathfrak{T}}\varpi, \check{\mathfrak{T}}\vartheta) \in \check{\mathfrak{R}}.$$

Definition 9. *[18] Let $\check{\mathfrak{G}}$ be a nonempty set and $\check{\mathfrak{T}}$ a self-mapping on $\check{\mathfrak{G}}$. A binary relation $\check{\mathfrak{R}}$ defined on $\check{\mathfrak{G}}$ and $U \subseteq \check{\mathfrak{G}}$. Then the retriction of $\check{\mathfrak{R}}$ to U is the set $\check{\mathfrak{R}} \cap U^2$ and is denoted by $\check{\mathfrak{R}}|_U$.*

Definition 10. *[7] Let $\check{\mathfrak{G}}$ be a nonempty set and $\check{\mathfrak{T}}$ a self-mapping on $\check{\mathfrak{G}}$. A binary relation $\check{\mathfrak{R}}$ defined on $\check{\mathfrak{G}}$ and $U \subseteq \check{\mathfrak{G}}$. The relation $\check{\mathfrak{R}}$ is said to be locally transitive if for any $\check{\mathfrak{R}}$-preserving sequence $\{\varpi_n\} \subset \check{\mathfrak{G}}$ the binary relation $\check{\mathfrak{R}}|_U$ is transitive, where $U = \{\varpi_n | n \in \mathbb{N}_0\}$.*

Definition 11. *[7] Let $\check{\mathfrak{G}}$ be a nonempty set and $\check{\mathfrak{T}}$ a self-mapping on $\check{\mathfrak{G}}$. A binary relation $\check{\mathfrak{R}}$ defined on $\check{\mathfrak{G}}$ and $U \in \check{\mathfrak{G}}$. The relation $\check{\mathfrak{R}}$ is said to be locally $\check{\mathfrak{T}}$-transitive if for any $\check{\mathfrak{R}}$-preserving sequence $\{\varpi_n\} \subset \check{\mathfrak{T}}(\check{\mathfrak{G}})$ the binary relation $\check{\mathfrak{R}}|_U$ is transitive, where $U = \{\varpi_n | n \in \mathbb{N}_0\}$.*

Definition 12. *[9] Let $\check{\mathfrak{G}}$ be a nonempty set and $\check{\mathfrak{R}}$ a binary relation on $\check{\mathfrak{G}}$. A subset U of $\check{\mathfrak{G}}$ is said to be $\check{\mathfrak{R}}$-connected for $\varpi, \vartheta \in \check{\mathfrak{G}}$, a path of length k (where k is a natural number) in $\check{\mathfrak{R}}$ from ϖ to ϑ is a finite sequence $\{\varpi_0, \varpi_1, \varpi_2, \ldots, \varpi_k\} \subset \check{\mathfrak{G}}$ satisfying the following conditions:*
(i) $\varpi_0 = \varpi$ and $\varpi_k = \vartheta$,
(ii) $(\varpi_i, \varpi_{i+1}) \in \check{\mathfrak{R}}$ for each i $(0 \leq i \leq k-1)$.
Notice that a path of length k involves k+1 elements of $\check{\mathfrak{G}}$, although they are not necessarily distinct.

Definition 13. *[9] Let $(\check{\mathfrak{G}}, p)$ be a symmetric space. A binary relation $\check{\mathfrak{R}}$ defined on $\check{\mathfrak{G}}$ is called p-self closed if, whenever $\{\varpi_n\}$ is an $\check{\mathfrak{R}}$-preserving sequence and $\varpi_n \to_p \varpi$, there exists a subsequence $\{\varpi_{n_k}\}$ of $\{\varpi_n\}$ with $(\varpi_{n_k}, \varpi) \in \check{\mathfrak{R}}$ for all $k \in \mathbb{N}$.*

Definition 14. *[9] Let $(\check{\mathfrak{G}}, p)$ be a symmetric space and a binary relation $\check{\mathfrak{R}}$ defined on $\check{\mathfrak{G}}$. $\check{\mathfrak{T}}$ a self-mapping on $\check{\mathfrak{G}}$ is $\check{\mathfrak{R}}$-continuous at $\varpi \in \check{\mathfrak{G}}$ if for any $\check{\mathfrak{R}}$-preserving sequence $\{\varpi_n\} \in \check{\mathfrak{G}}$ converging to ϖ, we have $\check{\mathfrak{T}}\varpi_n \to \check{\mathfrak{T}}\varpi$. Moreover, $\check{\mathfrak{T}}$ is called $\check{\mathfrak{R}}$-continuous if it is so at each point of $\check{\mathfrak{G}}$".*

Definition 15. *[9] Let $\check{\mathfrak{G}}$ be a nonempty set and a binary relation $\check{\mathfrak{R}}$ defined on $\check{\mathfrak{G}}$. We say that $(\check{\mathfrak{G}}, p)$ is $\check{\mathfrak{R}}$-complete if every $\check{\mathfrak{R}}$-preserving Cauchy sequence in $\check{\mathfrak{G}}$ converges.*

Definition 16. *[11] A mapping $\psi : [0, \infty) \to [0, \infty)$ is termed as comparison function if it enjoys the following ones:*

(i) ψ *is monotonic increasing,*

(ii) $\lim_{n\to\infty} \psi^n(t) = 0, \ \forall \ t > 0.$

Definition 17. *[11] A mapping $\psi : [0, \infty) \to [0, \infty)$ is termed as (c)-comparison function if it enjoys the following ones:*

(i) ψ *is monotonic increasing,*

(ii) $\sum_{n=1}^{\infty} \psi^n(t) < \infty, \quad \forall \quad t > 0.$

Clearly, every (c)-comparison function is a comparison function.

Remark 1. *[11] Let ψ be a (c)-comparison function. Then*

(i) $\psi(0) = 0$,

(ii) $\psi(t) < t$, $\forall t > 0$,

(iii) *ψ is right continuous at* 0.

3. Main Result

In this manuscript, we utilize the following notations:

(i) $F(\check{\mathfrak{T}})$ = the set of all fixed points of $\check{\mathfrak{T}}$

(ii) $\check{\mathfrak{G}}(\check{\mathfrak{T}}, \check{\mathfrak{R}}) := \{\varpi \in \check{\mathfrak{G}} : (\varpi, \check{\mathfrak{T}}\varpi) \in \check{\mathfrak{R}}\}$

Theorem 1. *Let $(\check{\mathfrak{G}}, p)$ be a symmetric space which enjoys the property (W_3) and $\check{\mathfrak{R}}$ a binary relation on $\check{\mathfrak{G}}$. $\check{\mathfrak{T}} : \check{\mathfrak{G}} \to \check{\mathfrak{G}}$ be mapping satisfying the following conditions.*

(a) $(\check{\mathfrak{G}}, p)$ *is $\check{\mathfrak{R}}$-complete,*

(b) *$\check{\mathfrak{R}}$ is $\check{\mathfrak{T}}$-closed and locally $\check{\mathfrak{T}}$-transitive,*

(c) *either $\check{\mathfrak{T}}$ is $\check{\mathfrak{R}}$-continuous or $\check{\mathfrak{R}}$ is p-self-closed,*

(d) *there is $\varpi_0 \in \check{\mathfrak{G}}(\check{\mathfrak{T}}, \check{\mathfrak{R}})$ such that*

$$\delta(p, \check{\mathfrak{T}}, \varpi_0) = \sup_{i,j\in\mathbb{N}} p(\check{\mathfrak{T}}^i\varpi_0, \check{\mathfrak{T}}^j\varpi_0) < \infty.$$

(e) *There exists (c)-comparison function ψ such that*

$$p(\check{\mathfrak{T}}\varpi, \check{\mathfrak{T}}\vartheta) \leq \psi(p(\varpi, \vartheta)) \qquad \forall\ \varpi, \vartheta\ \in \check{\mathfrak{G}} \text{ with } (\varpi, \vartheta) \in \check{\mathfrak{R}}.$$

Then $\check{\mathfrak{T}}$ posses a fixed point in $\check{\mathfrak{G}}$. In addition if

(f) *$\check{\mathfrak{R}}|_{\check{\mathfrak{T}}(\check{\mathfrak{G}})}$ is complete, then $\check{\mathfrak{T}}$ has a unique fixed point.*

Proof. In the view of (d), there is some $\varpi_0 \in \check{\mathfrak{G}}$, such that

$$\delta(p, \check{\mathfrak{T}}, \varpi_0) = \sup_{i,j\in\mathbb{N}} p(\check{\mathfrak{T}}^i\varpi_0, \check{\mathfrak{T}}^j\varpi_0) < \infty.$$

Take $\varpi_0 \in \check{\mathfrak{G}}(\check{\mathfrak{T}}, \check{\mathfrak{R}})$ and construct the sequence $\{\varpi_n\} \subset \check{\mathfrak{G}}$ such that

$$\varpi_n = \check{\mathfrak{T}}^n(\varpi_0) \qquad \forall\ n \in \mathbb{N}$$

so that

$$\varpi_n = \check{\mathfrak{T}}\varpi_{n-1} \qquad \forall\ n \in \mathbb{N}.$$

As $(\varpi_0, \check{\mathfrak{T}}\varpi_0) \in \check{\mathfrak{R}}$ and $\check{\mathfrak{R}}$ is $\check{\mathfrak{T}}$-closed. we have

$$(\check{\mathfrak{T}}\varpi_0, \check{\mathfrak{T}}^2\varpi_0), (\check{\mathfrak{T}}^2\varpi_0, \check{\mathfrak{T}}^3\varpi_0), \ldots, (\check{\mathfrak{T}}^n\varpi_0, \check{\mathfrak{T}}^{n+1}\varpi_0) \in \check{\mathfrak{R}}$$

so that

$$(\varpi_n, \varpi_{n+1}) \in \check{\mathfrak{R}}.$$

Thus, $\{\varpi_n\}$ is $\check{\mathfrak{R}}$-preserving. Now $\check{\mathfrak{R}}$ is locally $\check{\mathfrak{T}}$-transitive , We have

$$(\check{\mathfrak{T}}^m\varpi_0, \check{\mathfrak{T}}^n\varpi_0) \in \check{\mathfrak{R}} \qquad \forall n > m$$
$$or$$
$$(\varpi_m, \varpi_n) \in \check{\mathfrak{R}} \qquad \forall n > m.$$

Set M $:= \delta(p, \check{\mathfrak{T}}, \varpi_0)$. Then $0 \le M < \infty$. Applying contractivity condition (e), we get

$$p(\varpi_{n+i}, \varpi_{n+j}) \le \psi(p(\varpi_{n+i-1}, \varpi_{n+j-1})).$$

Therefore

$$\begin{aligned} \delta(p, \check{\mathfrak{T}}, \varpi_n) &\le \psi\ \delta(p, \check{\mathfrak{T}}, \varpi_{n-1}) \\ &\le \psi^2\ \delta(p, \check{\mathfrak{T}}, \varpi_{n-2}) \\ &\vdots \\ &\le \psi^n\ \delta(p, \check{\mathfrak{T}}, \varpi_0), \end{aligned}$$

so that

$$\delta(p, \check{\mathfrak{T}}, \varpi_n) \le \psi^n(M) \to 0 \ \ as\ n \to \infty,$$

now

$$p(\varpi_{n+1}, \varpi_{n+m}) \le \delta(p, \check{\mathfrak{T}}, \varpi_n) \to 0 \ \ as\ n \to \infty.$$

Thus, we conclude that the sequence $\{\varpi_n\}$ is a Cauchy sequence and also as the sequence is $\check{\mathfrak{R}}$-preserving, $\check{\mathfrak{R}}$-completeness of $(\check{\mathfrak{G}}, p)$ guarantees the existence of some $\varpi \in \check{\mathfrak{G}}$ such that $\check{\mathfrak{T}}^n\varpi_0 \to \varpi$ or $\varpi_n \to \varpi$.

If $\check{\mathfrak{T}}$ is $\check{\mathfrak{R}}$-continuous, then $\check{\mathfrak{T}}(\varpi_n) \to \check{\mathfrak{T}}(\varpi)$, i.e., $\varpi_{n+1} \to \check{\mathfrak{T}}(\varpi)$. We observed that $\varpi_n \to \varpi$ and $\varpi_n \to \check{\mathfrak{T}}(\varpi)$. As $(\check{\mathfrak{G}}, p)$ posses the property (W_3), we conclude $\check{\mathfrak{T}}(\varpi) = (\varpi)$. Hence $\{\varpi_n\}$ converges to a fixed point of $\check{\mathfrak{T}}$.

Alternately, if $\check{\mathfrak{R}}$ is p-self closed, then $\exists$ a subsequence $\{\varpi_{n_k}\}$ of $\{\varpi_n\}$ with $[\varpi_{n_k}, \varpi] \in \check{\mathfrak{R}}$, $\forall\ k \in \mathbb{N}$. Hence

$$p(\varpi_{n+1}, \check{\mathfrak{T}}\varpi) = p(\check{\mathfrak{T}}\varpi_{n_k}, \check{\mathfrak{T}}\varpi) \le \psi(p(\varpi_{n_k}, \varpi)).$$

As $p(\varpi_{n_k}, \varpi) \to 0$ we obtain $p(\varpi_{n_k+1}, \check{\mathfrak{T}}\varpi) \to 0$. Owing to property (W_3) of $\check{\mathfrak{G}}$, we obtain $\check{\mathfrak{T}}(\varpi) = \varpi$. Hence $\{\varpi_n\}$ converges to a fixed point of $\check{\mathfrak{T}}$.

For uniqueness part, let ϖ, ϑ be two fixed point of $\check{\mathfrak{T}}$ such that $\varpi \neq \vartheta$. we see that $\varpi, \vartheta \in F(\check{\mathfrak{T}})$ as $\varpi = \check{\mathfrak{T}}(\varpi)$ and $\vartheta = \check{\mathfrak{T}}(\vartheta)$. Now, $\check{\mathfrak{R}}|_{\check{\mathfrak{T}}(\check{\mathfrak{G}})}$ being complete gives rise to $[\varpi, \vartheta] \in \check{\mathfrak{R}}$. Therefore,

$$p(\varpi, \vartheta) = p(\check{\mathfrak{T}}(\varpi), \check{\mathfrak{T}}(\vartheta)) \leq \psi(p(\varpi, \vartheta)) < p(\varpi, \vartheta),$$

which is a contradiction. Hence, the fixed point of $\check{\mathfrak{T}}$ is unique.

Proposition 4. *Let $\check{\mathfrak{R}}$ be a binary relation on a regular symmetric space $(\check{\mathfrak{G}}, p)$ and $\check{\mathfrak{T}}$ a self - mapping on $\check{\mathfrak{G}}$. Let $\check{\mathfrak{R}}$ be $\check{\mathfrak{T}}$-closed and locally $\check{\mathfrak{T}}$-transitive. If there exists (c)-comparison ψ such that*

$$p(\check{\mathfrak{T}}\varpi, \check{\mathfrak{T}}\vartheta) \leq \psi(p(\varpi, \vartheta)) \qquad \forall\ \varpi, \vartheta\ \in \check{\mathfrak{G}}\ \ \textit{with}\ \ (\varpi, \vartheta) \in \check{\mathfrak{R}},$$

then for each $\varpi_0 \in \check{\mathfrak{G}}(\check{\mathfrak{T}}, \check{\mathfrak{R}})$

$$\delta(p, \check{\mathfrak{T}}, \varpi_0) = \sup_{i,j\in\mathbb{N}} p(\check{\mathfrak{T}}^i\varpi_0, \check{\mathfrak{T}}^j\varpi_0) < \infty.$$

Proof. Consider $\varpi_0 \in \check{\mathfrak{G}}(\check{\mathfrak{T}}, \check{\mathfrak{R}})$, then, we have $(\varpi_0, \check{\mathfrak{T}}\varpi_0) \in \check{\mathfrak{R}}$. If $\check{\mathfrak{T}}(\varpi_0) = \varpi_0$, then we are done; as

$$\delta(p, \check{\mathfrak{T}}, \varpi_0) = \sup_{i,j\in\mathbb{N}} p(\check{\mathfrak{T}}^i\varpi_0, \check{\mathfrak{T}}^j\varpi_0) = \sup_{i,j\in\mathbb{N}} p(\varpi_0, \varpi_0) = 0 < \infty.$$

Suppose that $\check{\mathfrak{T}}\varpi_0 \neq \varpi_0$. Since $(\varpi_0, \check{\mathfrak{T}}\varpi_0) \in \check{\mathfrak{R}}$ and $\check{\mathfrak{R}}$ is $\check{\mathfrak{T}}$-closed, we get by induction on n that

$$(\check{\mathfrak{T}}^n\varpi_0, \check{\mathfrak{T}}^{n+1}\varpi_0) \in \check{\mathfrak{R}}\ \forall\ n \in \mathbb{N}.$$

Construct the sequence $\{\varpi_n\} \subset \check{\mathfrak{G}}$ such that

$$\varpi_n = \check{\mathfrak{T}}^n(\varpi_0) \qquad \forall\ n \in \mathbb{N}$$

so that

$$\varpi_n = \check{\mathfrak{T}}(\varpi_{n-1}) \qquad \forall\ n \in \mathbb{N}.$$

As $(\varpi_0, \check{\mathfrak{T}}\varpi_0) \in \check{\mathfrak{R}}$ and $\check{\mathfrak{R}}$ is $\check{\mathfrak{T}}$-closed, we have

$$(\check{\mathfrak{T}}\varpi_0, \check{\mathfrak{T}}^2\varpi_0), (\check{\mathfrak{T}}^2\varpi_0, \check{\mathfrak{T}}^3\varpi_0), \ldots, (\check{\mathfrak{T}}^n\varpi_0, \check{\mathfrak{T}}^{n+1}\varpi_0) \in \check{\mathfrak{R}}$$

so that

$$(\varpi_n, \varpi_{n+1}) \in \check{\mathfrak{R}}.$$

Thus, $\{\varpi_n\}$ is $\check{\mathfrak{R}}$-preserving. Now as $\check{\mathfrak{R}}$ is locally $\check{\mathfrak{T}}$-transitive , we have

$$(\check{\mathfrak{T}}^m\varpi_0, \check{\mathfrak{T}}^n\varpi_0) \in \check{\mathfrak{R}}\ \textit{or}\ (\varpi_m, \varpi_n) \in \check{\mathfrak{R}} \qquad \forall\ m > n$$

we observe that the sequence $p(\varpi_n, \varpi_{n+k}) \to 0 \quad \forall\ k \in \mathbb{N}$,

$$p(\varpi_n, \varpi_{n+k}) = p(\check{\mathfrak{T}}\varpi_{n-1}, \check{\mathfrak{T}}\varpi_{n+k-1})$$

Therefore

$$\begin{aligned} p(\varpi_n, \varpi_{n+k}) &\leq \psi\ (p(\varpi_{n-1}, \varpi_{n+k-1})) \\ &\leq \psi^2\ (p(\varpi_{n-2}, \varpi_{n+k-2})) \\ &\vdots \\ &\leq \psi^n\ (p(\varpi_0, \varpi_k)), \to 0 \quad as\ n \to \infty. \end{aligned}$$

Now we show that $\{\varpi_n\}$ *is Cauchy. Let* $\epsilon > 0$ *be any positive number. As* $(\check{\mathfrak{G}}, p)$ *is regular, the basic triangle function* Φ_p *is continuous at the origin* $(0,0)$*. Therefore, there exists a neighbourhood* U *of the origin such that* $\Phi_p(u,v) \in U$*. In other words, there exists* $\delta > 0$ *such that,*

$\Phi_p(u,v) < \epsilon$ *for all* $u, v : 0 \leq u, v \leq \delta$*. Take* $\delta < \epsilon$*. We can find* $N \in \mathbb{N}$ *such that* $\psi^N \epsilon < \delta$*. Set* $F = \check{\mathfrak{T}}^N$*, then we have*

$$p(F\varpi, F\vartheta) = p(\check{\mathfrak{T}}^n\varpi, \check{\mathfrak{T}}^n\vartheta) \leq \psi^N p(\varpi, \vartheta) \quad when \quad (\varpi, \vartheta) \in \check{\mathfrak{R}}.$$

Define $m_k : p(\varpi_n, \check{\mathfrak{T}}^k F\varpi_n) < \delta\ \forall\ n \leq m_k$ *and set* $m = max\{m_0, m_1, m_2, \ldots, m_N\}$*. If* $V = \{\varpi_m, \varpi_{m+1}, \varpi_{m+2}, \ldots, \varpi_{m+k}, \ldots, \}$ *then for any* $\vartheta \in B(\varpi_m, \epsilon) \cap V$, $\vartheta \neq \varpi_m$

$$\begin{aligned} p(\check{\mathfrak{T}}^k F\varpi_m, \check{\mathfrak{T}}^k F\vartheta) &= p(F\check{\mathfrak{T}}^k\varpi_m, F\check{\mathfrak{T}}^k\vartheta) \leq \psi^N p(\check{\mathfrak{T}}^k r_m, \check{\mathfrak{T}}^K\vartheta) \quad as \quad (\check{\mathfrak{T}}^k r_m, \check{\mathfrak{T}}^K\vartheta) \in \check{\mathfrak{R}} \\ &\leq \psi^N\psi^k p(\varpi_m, \vartheta) < \psi^N p(\varpi_m, \vartheta) < \psi^N(\epsilon) < \delta, \end{aligned}$$

yielding thereby

$$\begin{aligned} p(\check{\mathfrak{T}}^k F\vartheta, \varpi_m) &\leq \Phi_p(p(\check{\mathfrak{T}}^k F\vartheta, \check{\mathfrak{T}}^k F\varpi_m), p(\check{\mathfrak{T}}^k F\varpi_m, \varpi_m)) \\ &\leq \Phi_p(\delta, \delta) \quad \forall\ k = 0, 1, 2, \ldots, N \end{aligned}$$

which implies that

$$p(\check{\mathfrak{T}}^k F\vartheta, \varpi_m) < \epsilon, \quad \forall\ k = 0, 1, 2, \ldots, N.$$

Also, for $\vartheta = \varpi_m, p(\check{\mathfrak{T}}^k F\varpi_m, \varpi_m) < \delta < \epsilon, \forall k = 0, 1, 2, \ldots, N$*. Thus, we see that* $\check{\mathfrak{T}}^k F$ *maps* $V \cap B(\varpi_m, \epsilon)$ *into itself for all* k = *0,1,2,. . . ,* N*. In particular, each iterate of* $\check{\mathfrak{T}}$ *maps* $V \cap B(\varpi_m, \epsilon)$ *into itself (as* $F = \check{\mathfrak{T}}^N$*). Now, if* $n > m$ *be an arbitrarily given natural number, i.e.,* $n = Nk + M$ *where* $k \in \mathbb{N}_0$ *and* $0 \leq M < N$*, then*

$$\check{\mathfrak{T}}^n F = \check{\mathfrak{T}}^{Nk+M} F = \check{\mathfrak{T}}^M F^{k+1}.$$

Henceforth,

$$\check{\mathfrak{T}}^n F(V \cap B(\varpi_m, \epsilon)) \quad = \quad \check{\mathfrak{T}}^M F^{k+1}(V \cap B(\varpi_m, \epsilon))$$

$$
\begin{aligned}
&= \check{\mathfrak{T}}^{M} F(F^{k+1}(V \cap B(\varpi_m, \epsilon))) \\
&\subset \check{\mathfrak{T}}^{M} F(V \cap B(\varpi_m, \epsilon)) \\
&\subset V \cap B(\varpi_m, \epsilon) \text{ as } 0 \leq M < N.
\end{aligned}
$$

Therefore, $\check{\mathfrak{T}}^n F(\varpi_m) \in B(\varpi_m, \epsilon)\ \forall\ n > m$, *i.e.,*$\varpi_{m+N+k} \in B(\varpi_m, \epsilon)\ \forall\ k \in \mathbb{N}$. *As* $(\check{\mathfrak{G}}, p)$ *is regular,* $diam(\varpi_m, \epsilon) \to 0$ *when* $\epsilon \to 0$, *which means the sequence* $\{\varpi_n\}$ *is a Cauchy sequence. Therefore, for each* $\varpi_0 \in \check{\mathfrak{G}}(\check{\mathfrak{T}}, \check{\mathfrak{R}})$

$$
\delta(p, \check{\mathfrak{T}}, \varpi_0) = \sup_{i,j \in \mathbb{N}} p(\check{\mathfrak{T}}^i \varpi_0, \check{\mathfrak{T}}^j \varpi_0) = \sup_{i,j \in \mathbb{N}} p(\varpi_i, \varpi_j) < \infty,
$$

as $p(\varpi_i, \varpi_j) \to 0$ *when* $i, j \to \infty$. *This accomplish the proof.*

By the use of Propositions 2 and 4, Theorem 1 yields the following consequence.

Corollary 1. *Let* $(\check{\mathfrak{G}}, p)$ *be a regular symmetric space endowed with a binary relation* $\check{\mathfrak{R}}$. *Let* $\check{\mathfrak{T}}$ *be a self - mapping on* $\check{\mathfrak{G}}$ *and the following conditions hold:*
(a) $\check{\mathfrak{G}}(\check{\mathfrak{T}}, \check{\mathfrak{R}})$ *is nonempty,*
(b) $(\check{\mathfrak{G}}, p)$ *is* $\check{\mathfrak{R}}$*-complete,*
(c) $\check{\mathfrak{R}}$ *is locally* $\check{\mathfrak{T}}$*-transitive and* $\check{\mathfrak{T}}$*-closed,*
(d) either $\check{\mathfrak{T}}$ *is* $\check{\mathfrak{R}}$ *is p-self closed or* $\check{\mathfrak{R}}$*-continuous,*
(e) there exists (c)-comparison function ψ *such that.*

$$
p(\check{\mathfrak{T}}\varpi, \check{\mathfrak{T}}\vartheta) \leq \psi(p(\varpi, \vartheta)) \qquad \forall\ \varpi, \vartheta\ \in \check{\mathfrak{G}} \text{ with } (\varpi, \vartheta) \in \check{\mathfrak{R}}.
$$

Then $\check{\mathfrak{T}}$ *has a fixed point, moreover, if*
(f) $\check{\mathfrak{R}}|_{\check{\mathfrak{T}}(\check{\mathfrak{G}})}$ *is complete, then the fixed point of* $\check{\mathfrak{T}}$ *is unique.*

Proof. As $(\check{\mathfrak{G}}, p)$ *is regular space, using Proposition 2, we infer that it has the property* (W_3). *Also, in view of assumption (a),* $\exists\ \varpi_0 \in \check{\mathfrak{G}}(\check{\mathfrak{T}}, \check{\mathfrak{R}})$. *From Proposition 4 , we have* $\delta(p, \check{\mathfrak{T}}, \varpi_o) < \infty$. *Hence we observe that all the hypotheses of Theorem 1 holds. Therefore,* $\check{\mathfrak{T}}$ *has a unique fixed point in* $\check{\mathfrak{G}}$.

Theorem 2. *In the hypotheses of Corollory 1, if we replace assumption (f) by the following weaker condition:*

(f') $\check{\mathfrak{T}}(\check{\mathfrak{G}})$ *is* $\check{\mathfrak{R}}^s -$ *connected;.*
Then the fixed point of $\check{\mathfrak{T}}$ *is unique.*

Proof. The existence of fixed point is guaranteed from the assumption (a)-(e) of Corollary 1. To prove the uniqueness let ϖ, ϑ *be two fixed points of* $\check{\mathfrak{T}}$ *such that* $\varpi \neq \vartheta$. *we see that* $\varpi, \vartheta \in F(\check{\mathfrak{T}})$ *as* $\varpi = \check{\mathfrak{T}}(\varpi)$ *and* $\vartheta = \check{\mathfrak{T}}(\vartheta)$. *As* $\check{\mathfrak{T}}(\check{\mathfrak{G}})$ *being* $\check{\mathfrak{R}}^s$*- connected, there exists* $\varpi_0, \varpi_1, \varpi_2, \ldots, \varpi_k \in \check{\mathfrak{G}}$ *satisfying the following conditions:*
(i) $\varpi_0 = \varpi$, $\varpi_k = \vartheta$
(ii) $[\varpi_i, \varpi_{i+1}] \in \check{\mathfrak{R}}$ *for each* i $(0 \leq i \leq k-1)$.

Due to condition (ii), we have $p(\check{\mathfrak{T}}\varpi_i, \check{\mathfrak{T}}\varpi_{i+1}) \leq \psi(p(\varpi_i, \varpi_{i+1}))$*. By using induction, we get* $p(\check{\mathfrak{T}}^n\varpi_i, \check{\mathfrak{T}}^n\varpi_{i+1}) \leq \psi^n(p(\varpi_i, \varpi_{i+1}))$*. For* $\epsilon > 0$, $\exists\delta > 0$ *such that*

$$\Phi_p(\varpi, \vartheta) < \epsilon \ \ \forall \ \ \varpi, \vartheta : 0 \quad \leq \varpi, \vartheta < \delta.$$

Let $\delta_1 = \delta$ *and define* $\delta_i (2 \leq i \leq k-1)$*:* $\Phi_p(\varpi, \vartheta) < \delta_{i-1} \quad \forall \quad \varpi, \vartheta : 0 \quad \leq \varpi, \vartheta < \delta_i$ *and set* $\gamma = \min\{\delta_1, \delta_2, \ldots, \ \delta_{k-1}\}$ *also, set* $M' = \max\{N_1, N_2, \ldots, \ N_{K-1}\}$ *where,*

$$N_i : p(\check{\mathfrak{T}}^n\varpi_i, \check{\mathfrak{T}}^n\varpi_{i+1}) \leq \psi^n p(\varpi_i, \varpi_{i+1}) < \gamma \quad \forall n \leq N_i$$

hence, for $n \leq M'$*, we have,*

$$\begin{array}{rcl}
p(\check{\mathfrak{T}}^n\varpi_{k-1}, \check{\mathfrak{T}}^n\vartheta) & = & p(\check{\mathfrak{T}}^n\varpi_{k-i}, \check{\mathfrak{T}}^n\varpi_k) < \gamma \quad \leq \delta_{k-1} \\
p(\check{\mathfrak{T}}^n\varpi_{k-2}, \check{\mathfrak{T}}^n\vartheta) & \leq & \Phi_p(p(\check{\mathfrak{T}}^n\varpi_{k-2}, \check{\mathfrak{T}}^n\varpi_{k-1}), p(\check{\mathfrak{T}}^n\varpi_{k-1}, \check{\mathfrak{T}}^n\vartheta)) \\
& \leq & \Phi_p(\gamma, \delta_{k-1}) \leq \Phi_p(\delta_{k-1}, \delta_{k-1}) < \delta_{k-2} \\
p(\check{\mathfrak{T}}^n\varpi_{k-3}, \check{\mathfrak{T}}^n\vartheta) & \leq & \Phi_p(p(\check{\mathfrak{T}}^n\varpi_{k-3}, \check{\mathfrak{T}}^n\varpi_{k-2}), p(\check{\mathfrak{T}}^n\varpi_{k-2}, \check{\mathfrak{T}}^n\vartheta)) \\
& \leq & \Phi_p(\gamma, \delta_{k-2}) \leq \Phi_p(\delta_{k-2}, \delta_{k-2}) < \delta_{k-3} \\
& \vdots & \\
p(\check{\mathfrak{T}}^n\varpi_1, \check{\mathfrak{T}}^n\vartheta) & \leq & \Phi_p(p(\check{\mathfrak{T}}^n\varpi_1, \check{\mathfrak{T}}^n\varpi_2), p(\check{\mathfrak{T}}^n\varpi_2, \check{\mathfrak{T}}^n\vartheta)) \\
& \leq & \Phi_p(\gamma, \delta_2) \leq \Phi_p(\delta_2, \delta_2) < \delta_1 \\
p(\check{\mathfrak{T}}^n\varpi, \check{\mathfrak{T}}^n\vartheta) & \leq & \Phi_p(p(\check{\mathfrak{T}}^n\varpi, \check{\mathfrak{T}}^n\varpi_1), p(\check{\mathfrak{T}}^n\varpi_1, \check{\mathfrak{T}}^n\vartheta)) \\
& \leq & \Phi_p(\gamma, \delta_1) \leq \Phi_p(\delta_1, \delta_1) < \epsilon.
\end{array}$$

It is true for any $\epsilon > 0$*. Therefore,* $p(\check{\mathfrak{T}}^n\varpi, \check{\mathfrak{T}}^n\vartheta) = p(\varpi, \vartheta) = 0$ *i.e.,* $\varpi = \vartheta$*. Hence, fixed point of* $\check{\mathfrak{T}}$ *is unique.*

Now, we present two examples to demonstrate our main results.

Example 1. *Let* $\check{\mathfrak{G}} = [0, 1)$*. Define* $p : \check{\mathfrak{G}} \times \check{\mathfrak{G}} \to \mathbb{R}^+$ *by*

$$p(\varpi, \vartheta) = \begin{cases} 0 & if \ \varpi = \vartheta = \vartheta, \\ 1 & if \ \varpi = \vartheta \neq 0, \\ \varpi + \vartheta & if \ \varpi \neq \vartheta. \end{cases}$$

Here, it easy to check that $(\check{\mathfrak{G}}, p)$ *is a symmetric space having the property* (W_3)*. Consider the binary relation* $\check{\mathfrak{R}}$ *on* $\check{\mathfrak{G}}$ *as given below:*

$$\check{\mathfrak{R}} = \{[\frac{1}{m}, \frac{1}{n}] | m, n \in \mathbb{N}, 5 \leq m < n\}.$$

Also define $\check{\mathfrak{T}} : \check{\mathfrak{G}} \to \check{\mathfrak{G}}$ *by*

$$\check{\mathfrak{T}}(\varpi) = \begin{cases} \frac{\varpi}{3} & if\ 0 \leq \varpi \leq \frac{1}{5}, \\ \frac{1}{6}(7\varpi - 1) & if\ \frac{1}{5} < \varpi < 1. \end{cases}$$

Define $\psi : [0, \infty) \to [0, \infty)$ *by*

$$\psi(t) = \frac{t}{3}.$$

Then, for all $(\varpi, \vartheta) \in \check{\mathfrak{R}}$, *we have*

$$p(\check{\mathfrak{T}}\varpi, \check{\mathfrak{T}}\vartheta) = p(\frac{\varpi}{3}, \frac{\vartheta}{3}) = \frac{\varpi}{3} + \frac{\vartheta}{3} \leq \frac{1}{3}p(\varpi, \vartheta) = \psi(p(\varpi, \vartheta)).$$

It follows that $\check{\mathfrak{T}}$ *is a contraction for the elements related by* $\check{\mathfrak{R}}$. *Thus, all the conditions of Theorem 1 are also satisfied and hence* $\check{\mathfrak{T}}$ *has a fixed point (namely,* $\varpi = 0$*).*

Example 2. *let* $\check{\mathfrak{G}} = \mathbb{R}$ *and define a symmetric* p *on* $\check{\mathfrak{G}}$ *by* $p(\varpi, \vartheta) = (\varpi - \vartheta)$, *then* $(\check{\mathfrak{G}}, p)$ *is a regular symmetric space then* $(\check{\mathfrak{G}}, p)$ *is* $\check{\mathfrak{R}}$*-complete. Take a binary relation* $\check{\mathfrak{R}}$ *on* $\check{\mathfrak{G}}$ *as follows:*

$$\check{\mathfrak{R}} = \{(\varpi, \vartheta) \in \mathbb{R}^2 : \varpi \geq \vartheta \geq 0, \varpi \in \mathbb{R}\}.$$

Define a mapping $\check{\mathfrak{T}} : \check{\mathfrak{G}} \to \check{\mathfrak{G}}$ *such that*

$$\check{\mathfrak{T}}(\varpi) = \begin{cases} \frac{\varpi}{2} & if\ \varpi \geq 0, \\ (3\varpi + 1) & if \varpi < 0. \end{cases}$$

Define $\psi : [0, \infty) \to [0, \infty)$ *by*

$$\psi(t) = \frac{t}{2}.$$

Consider $(\varpi, \vartheta) \in \check{\mathfrak{R}}$, *then*

$$p(\check{\mathfrak{T}}\varpi, \check{\mathfrak{T}}\vartheta) = p(\frac{\varpi}{2}, \frac{\vartheta}{2}) = \frac{\varpi}{2} - \frac{\vartheta}{2} \leq \frac{1}{2}p(\varpi, \vartheta) = \psi(p(\varpi, \vartheta)).$$

It follows that $\check{\mathfrak{T}}$ *is a contraction for the elements related by* $\check{\mathfrak{R}}$. *Thus, all the hypotheses of Corollary 1 are also satisfied and hence* $\check{\mathfrak{T}}$ *has a fixed point.*

4. Conclusions

We have proved some fixed point theorems for relation-theoretic ψ-contraction in symmetric space. Analogously, we can prove the variants of similar results in the settings of quasi-metric space, dislocated space, b-metric space, cone metric space etc.

Acknowledgements

Research Supporting Project number (RSP2023167), King Saud University, Riyadh, Saudi Arabia.

Funding

This project is funded by King Saud University, Riyadh, Saudi Arabia

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