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Convex Roman Dominating Functions in a Graph

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Abstract. Let G be a connected graph. A function $f : V(G) \to \{0, 1, 2\}$ is a *convex Roman dominating function* (or CvRDF) if every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$ and $V_1 \cup V_2$ is convex. The weight of a convex Roman dominating function f, denoted by $\omega_G^{CvR}(f)$, is given by $\omega_G^{CvR}(f) = \sum_{v \in V(G)} f(v)$. The minimum weight of a CvRDF on G, denoted by $\gamma_{CvR}(G)$, is called the *convex Roman domination number* of G. In this paper, we determine the convex Roman domination numbers of some graphs and give some realization results involving convex Roman domination, connected Roman domination, and convex domination numbers.

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Key Words and Phrases: Convex set, Roman dominating function, Roman domination number, convex Roman dominating function, convex Roman domination number

1. Introduction

The concept Roman domination was first introduced by Cockayne, Dreyer and Hedetnieme in 2004 [11] as a variant of dominating set problem in graph theory. Roman domination is inspired by the ancient Roman Empire's military strategy, where soldiers would be stationed at strategic points throughout a location to ensure its protection. In the Roman domination strategy, an unsecured location can be secured by sending an army to the location from an adjacent secured location subject to the constraint that one army must be left behind the secured location. Specifically, in this protection strategy, a vertex with label (or image under a function) 1 or 2 may be viewed as one or two armies, respectively, stationed at the given location or vertex. A nearby location (an adjacent vertex) is considered to be unsecured if no armies are stationed there, that is if the label of the vertex is 0. The convex Roman domination strategy in addition ensures that all locations that lie along shortest paths between any two secured locations are also secured.

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Since then, Roman domination function has become a popular topic of study in graph theory, and several variations and related concepts have been studied. Some variations can be found in [1], [2], [3], [4], [5], [10], [11], [13], [14], [18], and [17].

One of the fundamental concepts in mathematics that has been studied extensively in geometry and has been extended to graphs is convexity. Convexity in graphs is discussed in the book of Buckley and Harary [6]. Some studies on convexity in graphs and its related concepts can be found in [7], [8], [9], [12], [15], [16], and [19]. In this paper, we introduce the concept of convex Roman domination on which we combine the notions of convexity and Roman domination.

Let G be a connected graph. For vertices u and v in G, a u-v geodesic is any shortest path in G joining u and v. The length of a u-v geodesic is called the *distance* $d_G(u, v)$ between u and v. For every two vertices u and v of G, the symbol $I_G[u, v]$ is used to denote the set of vertices lying on any of the u-v geodesics.

The set of neighbors of a vertex $u \in G$, denoted by $N_G(u)$, is called the *open neighborhood* of u. The *closed neighborhood* of u is the set $N_G[u] = N_G(u) \cup \{u\}$. The *degree* of a vertex v denoted $deg_G(v)$ in a graph G is the number of vertices in G that are adjacent to v. Hence, $deg_G(v) = |N(v)|$. The largest degree among the vertices of G is called the *maximum degree* of G and is denoted by $\triangle(G)$. The minimum degree of G is denoted by $\delta(G)$. A graph G is *connected* if every pair of its vertices can be joined by a path. A vertex of a connected graph G is an *extreme vertex* or *simplicial* if its open neighborhood induces a complete subgraph of G. The set of extreme vertices of G is denoted by $Ext(G)$.

A set $S \subseteq V(G)$ is said to be a *dominating set* of a graph G if every vertex $v \in V(G)$ is either an element of S or is adjacent to an element of S. Thus, $N[S] = V(G)$. The smallest cardinality of a dominating set S is called the *domination number* of G and is denoted by $\gamma(G)$. That is $\gamma(G) = min\{|S| : S \text{ is a dominating set of } G\}$. Any dominating set S of G with $|S| = \gamma(G)$ is called a γ-*set* of G.

A set $S \subseteq V(G)$ is *convex* if for every two vertices $x, y \in S$, $I_G[x, y] \subseteq S$. The largest cardinality of a proper convex set in G, denoted by $con(G)$, is called the *convexity number* of G. A set $S \subseteq V(G)$ is *convex dominating* if S is both convex and dominating. The minimum cardinality among all convex dominating sets in G, denoted by $\gamma_{con(G)}$ is called the *convex domination number* of G.

A function $f : V(G) \to \{0, 1, 2\}$ is a *Roman dominating function* (or just RDF) if every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The *weight* of an RDF f is given by $\omega_G(f) = \sum_{v\in V(G)} f(v)$. The *Roman domination number* of a graph G, denoted by $\gamma_R(G)$, is the minimum weight of an RDF in G. Any RDF f on G with $\omega_G(f) = \gamma_R(G)$ is called a γ_R-function. If $f = (V_0, V_1, V_2)$ is an RDF in G, then $\omega_G(f) = |V_1| + 2|V_2|$.

A function $f = (V_0, V_1, V_2)$ is called *connected Roman dominating function* (CRDF) in G if $\langle V_1 \cup V_2\rangle$ is connected. The weight of a connected Roman dominating function $f = (V_0, V_1, V_2)$ in G is given by $\omega^{CR}_G(f) = |V_1| + 2|V_2|$. The *connected Roman domination number* $\gamma_{CR}(G)$ is the minimum weight of a CRDF in G. Any CRDF f in G with $\omega^{CR}_G(f) = \gamma_{CR}(G)$ is called a γ_{CR}-*function.*

A Roman dominating function $f = (V_0, V_1, V_2)$ on G is a *convex Roman dominating function* (or CvRDF) if $V_1 \cup V_2$ is convex. The weight of a convex Roman dominating function $f = (V_0, V_1, V_2)$ in G is given by $\omega_G^{CvR}(f) = |V_1| + 2|V_2|$. The minimum weight of a CvRDF on G, denoted by $\gamma_{CvR}(G)$, is called the *convex Roman domination number* of G. Any CvRDF f in G with $\omega_G^{CvR}(f) = \gamma_{CvR}(G)$ is called a *γ_{CvR}-function.*

A complete k-partite graph G is a k-partite graph with partite sets $S_{n_1}, S_{n_2}, \ldots, S_{n_k}$ having the added property that if $u \in S_{n_i}$ and $v \in S_{n_j}$, $i \neq j$, then $uv \in E(G)$. If $|S_{n_i}| = n_i$, then this graph is denoted by $K_{n_1,n_2,\ldots,n_k}$.

The *join* of two graphs G and H, denoted by $G + H$, is the graph with $V(G+H) = V(G) \cup V(H)$ and $E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G) \text{ and } v \in V(G)\}$, where "$\cup$" refers to a disjoint union of sets.

2. Known Results

Cyman et al. [19] investigated those graphs which have convex domination number close to their orders. They generated the following results which will be used in this study.

Theorem 1. *Let G be a connected graph with $n \geq 5$. If $\gamma_{con}(G) = n$, then $\triangle(G) \leq n - 4$.*

Corollary 1. *If $\gamma_{con}(G) = n$ and $G \neq K_1$, then $2 \leq \delta(G) \leq \triangle(G) \leq n - 4$.*

Corollary 2. *If $\gamma_{con}(G) = n$ and $G \neq K_1$, then $n \geq 6$.*

3. Results

It is well-known that every convex set in a connected graph induces a connected graph.

Remark 1. *Every convex Roman dominating function is a connected Roman dominating function. Hence, $\gamma_{CvR}(G) \geq \gamma_{CR}(G)$.*

The next result shows that every pair of positive integers are realizable as the connected Roman domination number and convex Roman domination number of a connected graph.

Theorem 2. *Let a and b be positive integers such that $4 \leq a \leq b$. Then there exists a connected graph G such that $\gamma_{CR}(G) = a$ and $\gamma_{CvR}(G) = b$.*

Proof. Consider the following cases:

Case 1. $a = b$.
Let $G = C_a$. Then $\gamma_{CR}(G) = \gamma_{CvR}(G) = a$.
Case 2. $a < b$. Let $m = b - a$. Consider the graph G in Figure 1. Let $V_0 = \{x_1, x_2, \ldots, x_m\}$, $V_1 = \{v_1, v_2, \ldots, v_{a-2}\}$, $V_2 = \{v_{a-1}\}$, $V_0' = \varnothing$, $V_1 = \{x_1, x_2, \ldots, x_m, v_1, v_2, \ldots, v_{a-2}\}$, $V_2 = \{v_{a-1}\}$. Then $f = (V_0, V_1, V_2)$ is a γ_{CR}-function and $g = (V_0', V_1', V_2')$ is a γ_{CvR}-function on G. Hence,

$$\gamma_{CR}(G) = \omega_G^{CR}(G) = |V_1| + 2|V_2| = a - 2 + 2(1) = a \text{ and}$$
$$\gamma_{CvR}(G) = \omega_G^{CvR}(G) = |V_1'| + 2|V_2'| = m + (a - 2) + 2(1) = b.$$

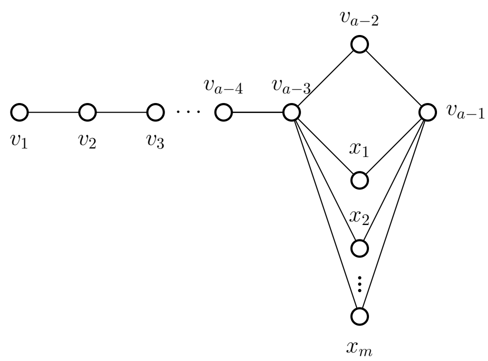

Figure 1: A graph with $\gamma_{CR}(G) = a$ and $\gamma_{CvR}(G) = b$

This proves the assertion. □

Corollary 3. *Let n be a positive integer. Then there exists a connected graph G such that $\gamma_{CvR}(G) - \gamma_{CR}(G) = n$. In other words, the difference $\gamma_{CvR}(G) - \gamma_{CR}(G)$ can be made arbitrarily large.*

Remark 2. *If $f = (V_0, V_1, V_2)$ is a γ_{CvR}-function, then $V_1 \cup V_2$ need not be a γ_{con}-set.*

To see this, consider $P_4 = [v_1, v_2, v_3, v_4]$. Let $V_0 = \{v_1\}$, $V_1 = \{v_3, v_4\}$, and $V_2 = \{v_2\}$. Then $f = (V_0, V_1, V_2)$ is a γ_{CvR}-function on P_4. Clearly, $V_1 \cup V_2$ is not a γ_{con}-set in P_4.

Proposition 1. *For any connected graph G of order n,*

$$1 \leq \gamma_{con}(G) \leq \gamma_{CvR}(G) \leq \min\{n, 2\gamma_{con}(G)\}.$$

Proof. Let $f = (V_0, V_1, V_2)$ be a γ_{CvR}-function. Then $V_1 \cup V_2$ is a convex dominating set in G. Hence, $1 \leq \gamma_{con}(G) \leq |V_1| + |V_2| \leq |V_1| + 2|V_2| = \gamma_{CvR}(G)$.

Now, let $V_0' = V_2' = \varnothing$ and $V_1' = V(G)$. Then $g = (V_0', V_1', V_2')$ is a CvRDF and $\gamma_{CvR}(G) \leq |V_1'| = |V(G)| = n$. Next, let S be a γ_{con}-set of G. Define $h = (V_0'', V_1'', V_2'')$ by setting $V_2'' = S$, $V_0'' = V(G) \setminus S$, and $V_1'' = \varnothing$. Then h is a CvRDF on G. Hence, $\gamma_{CvR}(G) \leq \omega_G^{CvR}(G) = 2|S| = 2\gamma_{con}(G)$. Therefore, $\gamma_{CvR}(G) \leq \min\{n, 2\gamma_{con}(G)\}$. □

Theorem 3. *Let G be a connected graph on n vertices. Then each of the the following statements holds.*

(*i*) $\gamma_{CvR}(G) = 1$ *if and only if* $G = K_1$

(*ii*) $\gamma_{CvR}(G) = 2$ *if and only if* $G = K_1 + H$ *for some graph* H

Proof.

(*i*) Assume that $\gamma_{CvR}(G) = 1$ and let $f = (V_0, V_1, V_2)$ be a γ_{CvR}-function on G. Then $|V_1| = 1$ and $|V_2| = 0$. Hence $G = K_1$. The converse is clear.

(*ii*) Suppose that $\gamma_{CvR}(G) = 2$ and let $f = (V_0, V_1, V_2)$ be a γ_{CvR}-function on G. Then $\omega_G^{CvR}(f) = |V_1| + 2|V_2| = 2$. Consider the following cases:

Case 1. $|V_1| \neq 0$

Then $|V_2| = 0$ and $|V_1| = 2$, say $V_1 = \{a, b\}$. Since V_1 is convex, $ab \in E(G)$. Since $|V_2| = 0$, $|V_0| = 0$. Hence, $V(G) = V_1$, that is $G = K_2 = K_1 + K_1$.

Case 2. $|V_1| = 0$

Then $|V_2| = 1$ since $2|V_2| = 2$. If $n = 2$, then $G = K_2$. Suppose $n \geq 3$. Let $V_2 = \{u\}$ and let $w \in V(G) \setminus V_2$. Then $w \in V_0$, that is $V_0 = V(G) \setminus V_2$. This implies that $uw \in E(G)$ for all $w \in V(G) \setminus V_2$. Hence, $G = \langle\{u\}\rangle + \langle V(G) \setminus \{u\}\rangle$. Let $H = \langle V(G) \setminus \{u\}\rangle$. Then $G = K_1 + H$.

Conversely, suppose that $G = K_1 + H$ for some graph H. Define $g = (V_0', V_1', V_2')$ by setting $V_0' = V(H)$, $V_1' = \varnothing$ and $V_2' = V(K_1)$. Then g is a γ_{CvR}-function on G and $\omega_G^{CvR}(g) = \gamma_{CvR}(G) = |V_1| + 2|V_2| = 2$. □

Theorem 3(*ii*) can be rephrased as follows:

Corollary 4. *For any connected graph G of order n, $\gamma_{CvR}(G) = 2$ if and only if $G \neq K_1$ and $\gamma(G) = 1$. In particular,*

(*i*) *$\gamma_{CvR}(K_n) = 2$ for $n \geq 2$;*

(*ii*) *$\gamma_{CvR}(F_n) = 2$ for $n \geq 1$;*

(*iii*) *$\gamma_{CvR}(W_n) = 2$ for $n \geq 3$; and*

(*iv*) *$\gamma_{CvR}(S_n) = \gamma_{CvR}(K_{1,n-1}) = 2$ for $n \geq 2$.*

Proposition 2. *There exists no connected graph G with $\gamma_{CvR}(G) = 3$.*

Proof. Suppose G is a connected graph with $\gamma_{CvR}(G) = 3$. Let $f = (V_0, V_1, V_2)$ be a γ_{CvR}-function on G. Then $\gamma_{CvR}(G) = |V_1| + 2|V_2| = 3$. This implies that $|V_2| \leq 1$. Suppose $|V_2| = 0$. Then $|V_0| = 0$ and $|V_1| = |V(G)| = 3$. Hence, $G = K_3$ or $G = P_3$. However, $\gamma_{CvR}(K_3) = \gamma_{CvR}(P_3) = 2$, a contradiction. Next, suppose that $|V_2| = 1$. Then $|V_1| = 1$. Let $V_1 = \{w\}$ and $V_2 = \{v\}$. Since $V_1 \cup V_2$ is convex, $vw \in E(G)$. Also, since $V_0 \subseteq N_G(v)$, V_2 is a dominating set in G. Hence, $\gamma(G) = 1$, implying that $\gamma_{CvR}(G) = 2$, a contradiction. This proves the claim. □

Proposition 3. *Let G be a nontrivial connected graph and let $f = (V_0, V_1, V_2)$ be a γ_{CvR}-function on G. Then the following hold:*

(*i*) *If $|V_0| = 0$, then $|V_2| = 0$.*

(*ii*) *If $|V_0| = 1$, then $|V_2| = 1$.*

(*iii*) *$|V_1| = 0$ if and only if V_2 is a γ_{con}-set in G (hence, $\gamma_{CvR}(G) = 2\gamma_{con}(G)$).*

Proof.

(*i*) Suppose $|V_2| \neq 0$. Let $V_0' = V_0$, $V_1' = V_1 \cup V_2$ and $V_2' = \varnothing$. Then $g = (V_0', V_1', V_2')$ is a CvRDF on G and $\omega_G^{CvR}(g) = |V_1'| = |V_1| + |V_2| < |V_1| + 2|V_2| = \omega_G^{CvR}(f)$, a contradiction.

(*ii*) Suppose $|V_0| = 1$, say $V_0 = \{v_0\}$. Suppose further that $|V_2| \geq 2$. Then $V(G) \setminus \{v_0\}$ is a convex dominating set in G. Let $v \in V_2$ such that $v_0 v \in E(G)$. Let $w \in V_2 \setminus \{v\}$. Let $h = (V_0, V_1', V_2')$, where $V_1' = V_1 \cup \{w\}$ and $V_2' = V_2 \setminus \{w\}$. Since $V_1' \cup V_2' = V_1 \cup V_2$, h is a convex Roman dominating function on G and

$$\omega_G^{CvR}(h) = |V_1'| + 2|V_2'| = |V_1| + 1 + 2(|V_2| - 1) = |V_1| + 2|V_2| - 1 < \omega_G^{CvR}(f),$$

a contradiction. Thus, $|V_2| = 1$.

(*iii*) Suppose $|V_1| = 0$. Suppose further that V_2 is not a γ_{con}-set in G. Then there exists $V_2' \subseteq V(G)$ such that V_2' is a convex dominating set in G with $|V_2'| < |V_2|$. Let $h = (V_0^*, V_1^*, V_2^*)$, where $V_1^* = \varnothing$, $V_2^* = V_2'$, and $V_0^* = V(G) \setminus V_2'$. Then h is a convex Roman dominating function on G and $\omega_G^{CvR}(h) = 2|V_2^*| < 2|V_2|$, a contradiction. Thus, V_2 is a γ_{con}-set in G.

Conversely, suppose that V_2 is a γ_{con}-set in G. Suppose $|V_1| \neq 0$. Then $\gamma_{CvR}(G) = |V_1| + 2|V_2| > 2|V_2|$. Let $V_0'' = V_0 \cup V_1$, $V_1'' = \varnothing$, and $V_2'' = V_2$. Then $V_0'' \subseteq N_G(V_2'')$ and $V_1'' \cup V_2'' = V_2$ is a convex dominating set in G. Thus, $h = (V_0'', V_1'', V_2'')$ is a CvRDF on G and $\omega_G^{CvR}(h) = 2|V_2| < \omega_G^{CvR}(f)$, contrary to our assumption of f. □

We now show that every pair of positive integers under some restrictions are realizable as the convex domination number and convex Roman domination number of a connected graph.

Theorem 4. *Let a and b be positive integers such that $4 \leq a + 2 \leq b \leq 2a$. Then there exists a connected graph G such that $\gamma_{con}(G) = a$ and $\gamma_{CvR}(G) = b$.*

Proof. Suppose $b = 2a$. Consider the graph G' in Figure 2. Let $S = \{v_1, v_2, v_3, \ldots, v_a\}$. Put $V_2 = S$, $V_1 = \varnothing$, and $V_0 = V(G') \setminus V_2$. Clearly, S is γ_{con}-set and $f = (V_0, V_1, V_2)$ is a γ_{CvR}-function on G'. Thus,

$$\gamma_{con}(G') = |S| = a \text{ and } \gamma_{CvR}(G') = 2|V_2| = 2a = b.$$

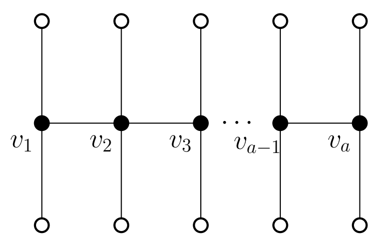

Figure 2: A graph G' with $\gamma_{con}(G') = a$ and $\gamma_{CvR}(G') = 2a$

Next, let $b < 2a$. Then $2 \leq m = b - a < a$. Consider the graph G in Figure 3. Let $S^* = \{v_1, v_2, \dots, v_a\}$. Then S^* is a γ_{con}-set in G. It follows that $\gamma_{con}(G) = a$. Set $V_2 = \{v_1, v_2, \dots, v_{m-1}, v_a\}$, $V_1 = \{v_m, v_{m+1}, \dots, v_{a-1}\}$, and $V_0 = V(G) \setminus (V_1 \cup V_2)$. Clearly, $g = (V_0, V_1, V_2)$ is a γ_{CvR}-function on G. Therefore,

$$\gamma_{CvR}(G) = |V_1| + 2|V_2| = (a - m) + 2m = m + a = b.$$

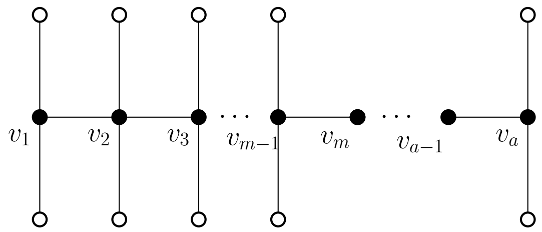

Figure 3: A graph G with $\gamma_{con}(G) = a$ and $\gamma_{CvR}(G) = b < 2a$

This proves the assertion. □

Corollary 5. *Let n be a positive integer with $n \geq 2$. Then there exists a connected graph G such that $\gamma_{CvR}(G) - \gamma_{con}(G) = n$. In other words, the difference $\gamma_{CvR}(G) - \gamma_{con}(G)$ can be made arbitrarily large.*

Proposition 4. *Let n be a positive integer. Then*

$$\gamma_{CvR}(P_n) = \begin{cases} 1 & \text{if } n = 1 \\ 2 & \text{if } n = 2, 3 \\ n & \text{if } n \geq 4. \end{cases}$$

Proof. Clearly, $\gamma_{CvR}(P_1) = 1$ and $\gamma_{CvR}(P_n) = 2$ for $n = 2, 3$. Suppose $n \geq 4$. Let $P_n = [v_1, v_2, \dots, v_n]$ and let $f = (V_0, V_1, V_2)$ be a γ_{CvR}-function on P_n. If $|V_0| = 0$, $|V_2| = 0$ by Proposition 3(i). It follows that $V_1 = V(P_n)$ and $\gamma_{CvR}(P_n) = \omega_{P_n}^{CvR}(f) = |V_1| = n$. Suppose there exists $v_j \in V_0$ such that $j \neq 1, n$. Then $v_{j-1} \in V_2$ or $v_{j+1} \in V_2$. If $v_{j-1} \in V_2$, then $v_k \in V_0$ for all $k > j$ since $V_1 \cup V_2$ is convex. Again, by convexity in $V_1 \cup V_2$, $v_s \in V_0$ for all $s < j$ whenever $v_{j+1} \in V_2$. In either case, f is not an RDF in G, a contradiction. Therefore, $V_0 \subseteq \{v_1, v_n\}$. Suppose $V_0 = \{v_1\}$ (or $\{v_n\}$). Since f is a γ_{CvR}-function, $V_2 = \{v_2\}$ (resp. $\{v_{n-1}\}$) and $V_1 = V(P_n) \setminus \{v_1, v_2\}$ (resp. $V(P_n) \setminus \{v_{n-1}, v_n\}$). Hence, $\gamma_{CvR}(P_n) = \omega_{P_n}^{CvR}(f) = |V_1| + 2|V_2| = n$. Suppose $V_0 = \{v_1, v_n\}$. Then $V_2 = \{v_2, v_{n-1}\}$ and $V_1 = V(P_n) \setminus \{v_1, v_2, v_{n-1}, v_n\}$. Hence $\gamma_{CvR}(P_n) = \omega_{P_n}^{CvR} = |V_1| + 2|V_2| = n$. □

Proposition 5. *Let n be a positive integer with $n \geq 3$. Then*

$$\gamma_{CvR}(C_n) = \begin{cases} 2 & \text{if } n = 3 \\ n & \text{if } n \geq 4. \end{cases}$$

Proof. Clearly, $\gamma_{CvR}(C_3) = 2$. Suppose that $n \geq 4$. Let $C_n = [v_1, v_2, \ldots, v_n, v_1]$ and let $g = (V_0, V_1, V_2)$ be a γ_{CvR}-function on C_n. If $V_0 = \varnothing$, then $V_2 = \varnothing$, by Proposition 3(i). Hence, $\gamma_{CvR}(C_n) = n$. Suppose $V_0 \neq \varnothing$. Since $V_1 \cup V_2$ is convex, $\langle V_0 \rangle$ is connected. Hence, $|V_0| \leq 2$. Let $n = 4$. Suppose $|V_0| = 1$. Assume without loss in generality that $V_0 = \{v_1\}$. Then $V_1 \cup V_2 = \{v_2, v_3, v_4\}$ which is not convex, a contradiction. Hence $|V_0| = 2$. Again, we may assume that $V_0 = \{v_1, v_2\}$. Then $V_2 = \{v_3, v_4\}$. It follows that $\gamma_{CvR}(C_4) = 2|V_2| = 4$. Let $n = 5$. By convexity in $V_1 \cup V_2$, it can be verified that $|V_0| = 2$. This implies that $|V_2| = 2$ and $|V_1| = 1$. Thus, $\gamma_{CvR}(C_5) = |V_1| + 2|V_2| = 5$. Suppose $n \geq 6$. Suppose $|V_0| = 2$, say $V_0 = \{v_1, v_2\}$. Then $[v_3, v_2, v_1, v_n]$ is a v_3-v_n geodesic in C_n, implying that $V_1 \cup V_2$ is not convex. This is a contradiction to the assumption that g is a CvRDF. Therefore, $|V_0| = 1$. Since g is a γ_{CvR}-function, $|V_2| = 1$ and $|V_1| = n - 2$. Thus, $\gamma_{CvR}(C_n) = n$. □

Proposition 6. *Let $G = K_{n_1,\ldots,n_k}$ be the complete k-partite graph with $2 \leq n_1 \leq n_2 \ldots \leq n_k$ where $k \geq 2$. Then $\gamma_{CvR}(G) = 4$.*

Proof. Let $S_{n_1}, S_{n_2}, \ldots, S_{n_k}$ be the partite sets in G. Let $i, j \in \{1, 2, \ldots, k\}$ with $i \neq j$. Choose any $u \in S_{n_i}$ and $w \in S_{n_j}$. Put $V_2 = \{u, w\}$, $V_0 = V(G) \setminus V_2$, and $V_1 = \varnothing$. Then $f = (V_0, V_1, V_2)$ is a CvRDF on G. Thus, $\gamma_{CvR}(G) \leq \omega_G^{CvR}(f) = 2|V_2| = 4$. Suppose $\gamma_{CvR}(G) < 4$. Let $g = (V_0', V_1', V_2')$ be a γ_{CvR}-function on G. Then $\gamma_{CvR}(G) = \omega_G^{CvR}(g) = |V_1'| + 2|V_2'| < 4$. This implies that $|V_2'| \leq 1$. If $|V_2'| = 0$, then $|V_0'| = 0$ and $|V_1'| = |V(G)| \geq 4$, a contradiction. Thus, $|V_2'| = 1$. If $\omega_G^{CvR}(g) = 2$, then $|V_1'| = 0$. Let $V_2' = \{p\}$, where $p \in S_{n_j}$ for $j \in \{1, 2, \ldots, k\}$. Pick any $q \in S_{n_j} \setminus \{p\}$. Since $|V_2'| = 1$, $q \in V_0$. This is not possible because $pq \notin E(G)$. This forces $\omega_G^{CvR}(g) = 3$ which is also not possible by Proposition 2. Therefore, $\gamma_{CvR}(G) = 4$. □

Proposition 7. *Let G be a connected non-complete graph and let $f = (V_0, V_1, V_2)$ be a γ_{CvR}-function on G. If $|V_1| = 0$, then the following hold:*

(i) every cut-vertex belongs to V_2,

(ii) no extreme vertex belongs to V_2, and

(iii) for any $u, v \in V_2$ such that $d_G(u, v) \geq 2$, $N_G(u) \cap N_G(v) \subseteq V_2$.

Proof. Since $|V_1| = 0$, V_2 is a γ_{con}-set in G by Proposition 3(iii). Suppose there exists a cut-vetex v of G such that $v \notin V_2$. Let G_1 and G_2 be distinct components of $G \setminus v$. Pick any $p \in V_2 \cap V(G_1)$ and $q \in V_2 \cap V(G_2)$. Since every p-q geodesic contains v, it follows that V_2 is not convex, a contradiction. Hence, (i) holds. Suppose there exists $w \in Ext(G) \cap V_2$. Since $G \neq K_n$, there exists $z \in N_G(w)$ such that $y \in N_G(z) \setminus N_G(w)$. Suppose $z \in V_0$. Then there exists $x \in N_G(z) \cap V_2$. Since V_2 is γ_{con}-set, $x \notin N_G(w)$. Hence, $[x, z, w]$ is a x-w geodesic, contrary to the fact that V_2 is convex. Hence, $z \in V_2$. Again, this is not possible because V_2 is a γ_{con}-set. Therefore, $w \notin V_2$. Thus, (ii) holds.

Next, let $u, v \in V_2$ such that $d_G(u, v) \geq 2$. If $d_G(u, v) > 2$, then $N_G(u) \cap N_G(v) = \varnothing$, then we are done. Suppose $d_G(u, v) = 2$ and let $a \in N_G(u) \cap N_G(v)$. By convexity of V_2, it follows that $a \in V_2$, showing that $N_G(u) \cap N_G(v) \subseteq V_2$. This shows that (iii) holds. □

Proposition 8. *Let G be a conneted graph of order n. Then $\gamma_{con}(G) = \gamma_{CvR}(G)$ if and only if $\gamma_{con}(G) = n$.*

Proof. If $\gamma_{con}(G) = n$, then $\gamma_{CvR}(G) = n$ by Proposition 1.
For the converse, suppose that $\gamma_{con}(G) = \gamma_{CvR}(G)$. Let $f = (V_0, V_1, V_2)$ be a γ_{CvR}-function on G. Then $\gamma_{con}(G) \leq |V_1| + |V_2| \leq |V_1| + 2|V_2| = \gamma_{CvR}(G)$. By assumption, this implies that $|V_2| = 0$. Hence, $|V_0| = 0$, implying that $\gamma_{con}(G) = |V_1| = n = \gamma_{CvR}(G)$. □

The next two results follow from Proposition 8, Theorem 1, Corollary 1, and Corollary 2.

Corollary 6. *Let G be a connected graph of order n. If $\gamma_{con}(G) = \gamma_{CvR}(G)$ and $G \neq K_1$, then $2 \leq \delta(G) \leq \triangle(G) \leq n - 4$.*

Corollary 7. *Let G be a nontrivial connected graph of order n. If $\gamma_{con}(G) = \gamma_{CvR}(G)$, then $n \geq 6$.*

Theorem 5. *Let G be a nontrivial connected graph on n vertices such that $\gamma_{con}(G) < \gamma_{CvR}(G)$. Then $\gamma_{CvR}(G) = \gamma_{con}(G) + 1$ if and only if there exist a vertex v and a set $S \subseteq V(G)$ such that $S \subseteq N_G(v)$ and $V(G) \setminus S$ is a γ_{con}-set in G.*

Proof. Suppose $\gamma_{CvR}(G) = \gamma_{con}(G) + 1$. Let $f = (V_0, V_1, V_2)$ be a γ_{CvR}-function on G. Since $V_1 \cup V_2$ is a convex dominating set in G, $\gamma_{con}(G) \leq |V_1| + |V_2|$. Consider the following cases:

Case 1. $\gamma_{con}(G) = |V_1| + |V_2|$.
Then $\gamma_{con}(G) + 1 = |V_1| + |V_2| + 1$. By assumption, $|V_1| + |V_2| + 1 = |V_1| + 2|V_2|$. This implies that $|V_2| = 1$ and $|V_1| = \gamma_{con}(G) - 1$. Let $V_2 = \{v\}$ and $S = V_0$. Then $S \subseteq N_G(v)$. Since $V_1 \cup V_2$ is a convex dominating set in G and $|V_1 \cup V_2| = \gamma_{con}(G)$, it follows that $V(G) \setminus S = V_1 \cup V_2$ is a γ_{con}-set in G.

Case 2. $\gamma_{con}(G) < |V_1| + |V_2|$.
Then $\gamma_{con}(G) + 1 \leq |V_1| + |V_2|$. The assumption $\gamma_{CvR}(G) = \gamma_{con}(G) + 1$ forces the equality $|V_1| + |V_2| = |V_1| + 2|V_2|$. Hence, $|V_2| = 0$, $|V_0| = 0$, and $|V_1| = n$. Consequently, $\gamma_{con}(G) = n - 1$. Let $D = V(G) \setminus \{w\}$ be a γ_{con}-set in G and set $S = \{w\}$. Since D is a dominating set and G is non-trivial, there exists $v \in D$ such that $S \subseteq N_G(v)$.

In either case, the desired properties hold.

For the converse, suppose that there exist a vertex v and a set $S \subseteq V(G)$ such that $S \subseteq N_G(v)$ and $V(G) \setminus S$ is a γ_{con}-set in G. Let $V_2 = \{v\}$, $V_1 = V(G) \setminus (S \cup V_2)$, and $V_0 = V(G) \setminus (V_1 \cup V_2) = S$. Then, by assumption, $V(G) \setminus S = V_1 \cup V_2$ is convex and $V_0 \subseteq N_G(v)$. Therefore, $g = (V_0, V_1, V_2)$ is CvRDF on G and

$$\begin{aligned} \gamma_{CvR}(G) &\leq w_G^{CvR}(g) \\ &= |V_1| + 2|V_2| \\ &= [n - (n - \gamma_{con}(G) + 1)] + 2 \end{aligned}$$

$$= \gamma_{con}(G) + 1.$$

Since $\gamma_{con}(G) < \gamma_{CvR}(G)$, $\gamma_{con}(G)+1 \leq \gamma_{CvR}(G)$. Therefore, $\gamma_{CvR}(G) = \gamma_{con}(G)+1$. □

Graphs G such that $\gamma_{CvR}(G) = 2\gamma_{con}(G)$ are called convex Roman graphs.

Theorem 6. *Let G be a nontrivial connected graph. The following statements are equivalent.*

(*i*) *G is a convex Roman graph.*

(*ii*) *G has a γ_{CvR}-function $f = (V_0, V_1, V_2)$ such that $|V_1| = 0$.*

(*iii*) *G has a γ_{CvR}-function $f = (V_0, V_1, V_2)$ such that V_2 is a γ_{con}-set in G.*

Proof. Let G be a convex Roman graph and let $f = (V_0, V_1, V_2)$ be a γ_{CvR}-function on G. Then $2\gamma_{con}(G) = 2|V_1| + 2|V_2| = |V_1| + 2|V_2| = \gamma_{CvR}(G)$. This implies that $|V_1| = 0$. Hence (*i*) implies (*ii*).

Next, let $f = (V_0, V_1, V_2)$ be a γ_{CvR}-function on G with $|V_1| = 0$. By Proposition 3(*iii*), V_2 is a γ_{con}-set in G and $\gamma_{CvR}(G) = 2|V_2| = 2\gamma_{con}(G)$. Thus (*ii*) implies (*i*).

The equivalence of statements (*ii*) and (*iii*) follows from Proposition 3(*iii*). □

Corollary 8. *Let G be a nontrivial connected graph. If $\gamma(G) = 1$, then G is a convex Roman graph.*

Theorem 7. *Let G and H be any connected graphs. Then*

$$\gamma_{CvR}(G+H) = \begin{cases} 2 & \text{if } \gamma(G) = 1 \text{ or } \gamma(H) = 1 \\ 4 & \text{otherwise.} \end{cases}$$

Proof. Since $G + H \neq K_1$, $\gamma_{CvR}(G+H) \geq 2$, by Theorem 3(*i*). Suppose $\gamma(G) = 1$ or $\gamma(H) = 1$. Then $\gamma(G+H) = 1$. By Corollary 4, $\gamma_{CvR}(G+H) = 2$. Suppose $\gamma(G) \neq 1$ and $\gamma(H) \neq 1$. By Proposition 2, $\gamma_{CvR}(G+H) \geq 4$. Pick any $x \in V(G)$ and $y \in V(H)$. Let $V_2 = \{x, y\}$, $V_0 = V(G) \setminus V_2$, and $V_1 = \varnothing$. Then $f = (V_0, V_1, V_2)$ is a CvRDF on $G + H$ and $\omega_{G+H}^{CvR}(f) = 4$. Therefore, $\gamma_{CvR}(G+H) = 4$. This proves the assertion. □

4. Conclusion

The concept of convex Roman domination was introduced and initially investigated in this study. The convex Roman domination numbers of some graphs and the join of two graphs were determined. It was shown that every pair of positive integers (with some restrictions) are realizable as the connected Roman domination number and convex Roman domination number of some connected graph. A realization result involving convex domination number and convex Roman domination number was also obtained. The newly defined variant of Roman domination in this study can be studied for other graphs including those ones under some binary operations.

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