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Prime Graph Generation through Single Edge Addition: Characterizing a Class of Graphs

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Abstract. A graph G consists of a finite set $V(G)$ of *vertices* with a collection $E(G)$ of unordered pairs of distinct vertices called *edge set* of G.
Let G be a graph. A set M of vertices is a *module* of G if, for vertices x and y in M and each vertex z outside M, $\{z, x\} \in E(G) \Longleftrightarrow \{z, y\} \in E(G)$. Thus, a module of G is a set M of vertices indistinguishable by the vertices outside M. The empty set, the singleton sets and the full set of vertices represent the *trivial modules*. A graph is *indecomposable* if all its modules are trivial, otherwise it is *decomposable*. Indecomposable graphs with at least four vertices are *prime graphs*. The introduction and the study of the construction of prime graphs obtained from a given decomposable graph by adding one edge constitue the central points of this paper.

2020 Mathematics Subject Classifications: 05C60

Key Words and Phrases: Module, Prime, Decomposable, Prime Frame, Isomorphism.

1. Introduction

Our notations and terminology follow [1]. All graphs mentioned in this paper are finite. Without loops and multiple edges, these graphs are called *simple graphs*. A *graph* G consists of a finite set $V(G)$ of vertices called *vertex set* with a collection $E(G)$ of pairs of distinct vertices (*edge set* of G). Such a graph is denoted by $(V(G), E(G))$ (Simply (V, E)). An *empty graph* is a graph without edges while a *complete graph* is a graph with all possible edges.
Two distinct vertices u and v of a graph G are *adjacent* if $\{u, v\} \in E(G)$. An edge $\{u, v\}$ of G is denoted by uv while u and v are called *endpoints* of the edge uv. Two distinct edges e and e' of a graph G are *adjacent edges* if they have a common endpoint. A *neighbor* of a vertex u in a graph G is a vertex adjacent to u, the *neighborhood* of u denoted by $N_G(u)$

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is the set of neighbors of u and the *neighborhood of a subset* X of $V(G)$ represented by $N_G(X)$ is the union of the neighborhoods of every vertex in X. The *degree* of u denoted by $d_G(u)$ is $d_G(u) = |N_G(u)|$. The *complement* of a graph G is the graph $\overline{G}$ such that $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \{\{u, v\} : \ u \neq v \in V(G), \{u, v\} \notin E(G)\}$. For undefined notions and notations in the graph theory, see [8]. In particular, a graph $H = (W, F)$ is a *subgraph* of a graph $G = (V, E)$ if $W \subseteq V$ and $F \subseteq E$. Given a subset X of the vertex set of a graph $G = (V, E)$, the subgraph $G[X] = (X, E \cap \{xy, x \neq y \in X\})$ is called the subgraph *induced* by X. The subgraph $G[V(G) \setminus X]$ is denoted by $G - X$. Considering a vertex x, the subgraph $G - \{x\}$ is also denoted by $G - x$. For a vertex u outside a vertex subset X of a graph G, $u \sim_G X$ denotes when u is either adjacent to all or none of the elements of X.

In a graph G, a vertex subset M is a *module* of G if every vertex outside M is either adjacent to all or none of the elements of M. This concept was introduced in [6]. The empty set, the singleton sets and the full set $V(G)$ of vertices are *trivial modules.* A module of a graph G distinct from $V(G)$ is a *proper module* of G. A graph is *indecomposable* if all its modules are trivial, otherwise it is *decomposable.* Clearly, all graphs with at most two vertices are indecomposable. Given a 3-vertex graph G, if G is complete or empty, then each 2-element vertex subset is a non-trivial module of G, otherwise G has a unique non-trivial module $\{u, v\}$ where uv is the unique edge of G or $\overline{G}$. Thus, all 3-vertex graphs are decomposable. Indecomposable graphs with at least four vertices are called *prime* graphs.

An *isomorphism* f from a graph $G = (V, E)$ onto a graph $G' = (V', E')$ is a bijection from V onto V' such that for all $x, y \in V$, $xy \in E \Leftrightarrow f(x)f(y) \in E'$. We denote $G \simeq G'$ the graphs G and G' which are called *isomorphic* if there is an isomorphism from G onto G'.

In order to state our theorem, we introduce the following new graphs, along with some known graphs.

Recall the known small graphs used in this paper.

First, the graph $P_4 = (\{v_1, v_2, v_3, v_4\}, \{v_1v_2, v_2v_3, v_3v_4\})$ (illustrated in Figure 1).

Second, the graph $\beta = (\{a, a', x, x', y\}, \{ax', ay, aa', a'x', a'y, xy\})$ (shown in Figure 2).

Finally, the *Taurus* (resp. the *House*) is the graph with the vertex set $\{a, x_1, x'_1, x_2, x'_2\}$ and the edge set $\{ax'_1, ax'_2, x'_1x'_2, x_1x'_1, x_2x'_2\}$, (resp. $\{ax'_1, ax'_2, x'_1x'_2, x_1x'_1, x_2x'_2, x_1x_2\}$) as illustrated in Figure 3.

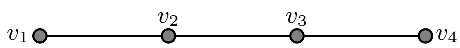

Figure 1: P_4

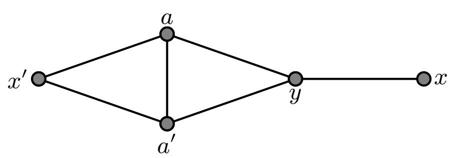

Figure 2: β

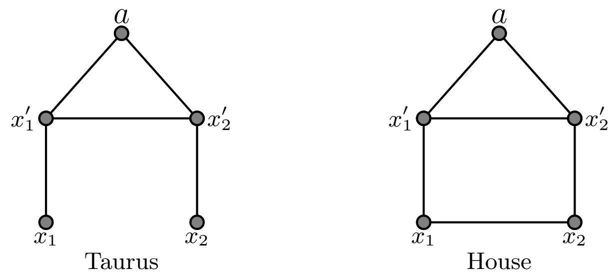

Figure 3: Taurus and House

Let's define the following two graph families:

Definition 1. *Let k and q be two non-negative integers with $k \geq 2$.*
A Palace $P_{k,q}$ is a graph (V, E) satisfying the following:
There is $a \in V$ such that, by denoting $Z = N_{P_{k,q}}(a)$ and $X = V \setminus (Z \cup \{a\})$, $|Z| \geq k$, $|X| = k$ and $|Z| - k = q$. Moreover, by denoting $X = \{x_1, x_2, \ldots, x_k\}$, there is a k-element subset $X' = \{x'_1, x'_2, \ldots, x'_k\}$ of Z such that $P_{k,q}[X']$ is a complete graph, the set of edges between X and X' is $\{x_i x'_i : 1 \leq i \leq k\}$ and the subset $Y = Z \setminus X'$ satisfies the following conditions:

- *(i) For every $y \in Y$, $P_{k,q}[\{a, y\} \cup X']$ is a complete graph.*
- *(ii) For every $y \in Y$, either $1 < |N_{P_{k,q}[X \cup \{y\}]}(y)| < k$ or ($|N_{P_{k,q}[X \cup \{y\}]}(y)| \in \{1, k\}$ and $\exists\, z \in Y \setminus \{y\}$ such that $zy \notin E$).*
- *(iii) For any $y_1 \neq y_2 \in Y$, $N_{P_{k,q}[X \cup y_1]}(y_1) \neq N_{P_{k,q}[X \cup y_2]}(y_2)$.*

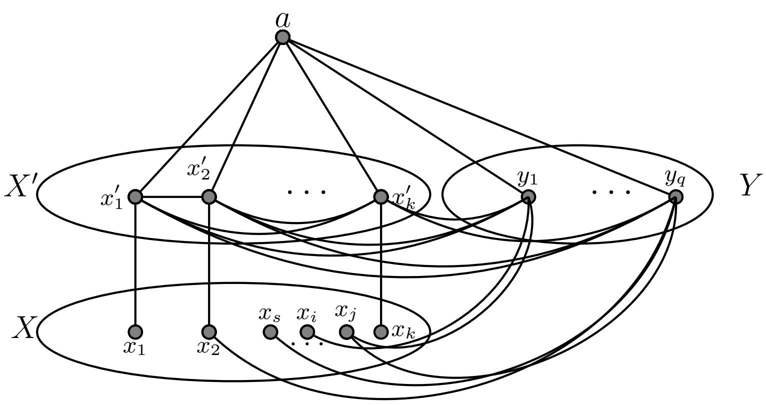

Figure 4: Palace $P_{k,q}$ ($k \geq 2$, $q \geq 0$)

Definition 2. *Let $k \geq 2$ and q be two non-negative integers. A Palace $\beta_{k,q}$ is a graph obtained from a palace $P_{k,q}$ by replacing the module $\{a\}$ with the module $\{a, a'\}$ where $\beta_{k,q}[\{a, a'\}]$ is isomorphic to the graph K_2, as illustrated in Figure 5.*

Notation:

The family of graphs isomorphic to the graph β or a palace $\beta_{k,q}$, for some $k \geq 2$ and $q \geq 0$,

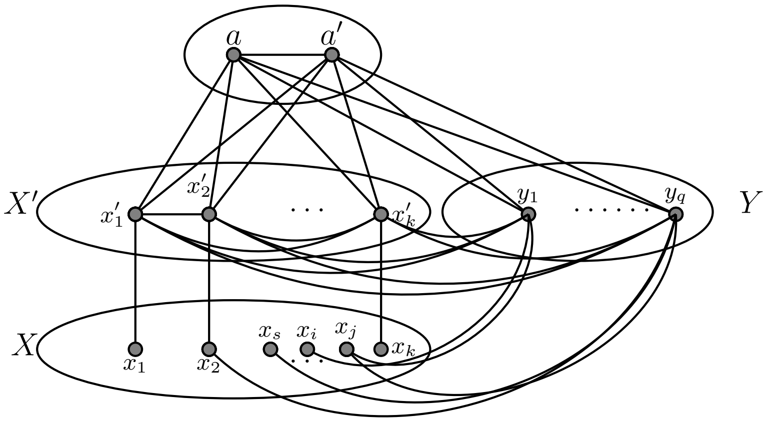

Figure 5: Palace $\beta_{k,q}$ ($k \geq 2$, $q \geq 0$)

is denoted by $\mathcal{B}$.

1.1. Gallai's decomposition

Let $G = (V, E)$ be a graph. An equivalence relation is denoted by $\cong$ between the pairs of vertices of G for x and y as well as u and v. $xy \in E$ if and only if $uv \in E$ defines $\{x, y\} \cong \{u, v\}$.

To recall the basic properties of the modules, we introduce the following notation:

Given a graph G on the vertex set V and two disjoint vertex subsets X and Y of V, X and Y are *equivalent* ($X \sim Y$) if, for any vertices x and x' in X and y and y' in Y, $\{x, y\} \cong \{x', y'\}$.

Proposition 1. *Let G be a graph on the set of vertices V.*

(i) $\emptyset$, V and $\{u\}$ where $u \in V$ are modules of G.

(ii) Considering a non-empty vertex subset W of V, if M is a module of G, then $M \cap W$ is a module of $G[W]$.

(iii) If M and N are modules of G, then $M \cap N$ is a module of G.

(iv) If M and N are modules of G such that $M \cap N \neq \emptyset$, then $M \cup N$ is a module of G.

(v) If M and N are modules of G such that $M \setminus N \neq \emptyset$, then $N \setminus M$ is a module of G.

(vi) If M and N are disjoint modules of G, then $M \sim N$.

A partition $\mathcal{P}$ of the vertex set $V(G)$ of a graph G is a *modular partition* of G if all its elements are modules of G. Based on the last assertion of Proposition 1, it follows that the elements of $\mathcal{P}$ may be considered as the vertices of a new graph. The *quotient* of G by $\mathcal{P}$ is $G/\mathcal{P}$ defined on $\mathcal{P}$ as follows: for the distinct elements X and Y of $\mathcal{P}$, $XY \in E(G/\mathcal{P})$ if $xy \in E(G)$ for every x and y where $x \in X$ and $y \in Y$. A module X of a graph G is a *strong module* of G if, for every module Y of G, $X \cap Y \neq \emptyset$. Hence, either $X \subseteq Y$ or $Y \subseteq X$. If $\mid V(G) \mid \geq 2$, then $\mathcal{P}(G)$ denotes the family of maximal proper strong modules of G, equipped with the inclusion.

The following theorem shows Gallai's decomposition result.

Theorem 1. *[4, 5] Let G be a graph with at least two vertices. The class $\mathcal{P}(G)$ is a modular partition of G and the quotient $G/\mathcal{P}(G)$ is a prime, complete or empty graph.*

Taking in consideration a graph G with more than one vertex, the elements of $\mathcal{P}(G)$ are the *modular components* of G, $\mathcal{P}(G)$ is its *canonical partition* and the quotient $G/\mathcal{P}(G)$ is its *frame*.

2. Preliminary Results

2.1. Prime graphs and their prime subgraphs

Ehrenfeucht and Rozenberg [3] constructed prime subgraphs of a larger size than a given prime subgraph as follows. Let $G = (V, E)$ be a graph. Given a proper subset X of V such that $G[X]$ is prime, consider the following subsets of $V \backslash X$:

- $Ext(X)$ is the set of $x \in V \setminus X$ such that $G[X \cup \{x\}]$ is prime.
- $\langle X \rangle$ is the set of $x \in V \setminus X$ such that X is a module of $G[X \cup \{x\}]$.
- For $u \in X$, $X(u)$ is the set of $x \in V \setminus X$ such that $\{x, u\}$ is a module of $G[X \cup \{x\}]$.

The family of the non-empty elements of the union

$$\{Ext(X), \langle X \rangle\} \cup \{X(u) : u \in X\}$$

is denoted by $\mathcal{P}_X$.

Lemma 1. *[3] taking into account a graph $G = (V, E)$, consider a proper subset X of V such that $G[X]$ is prime. The family $\mathcal{P}_X$ realizes a partition of $V \setminus X$. Moreover, the following assertions hold.*

1) Let $u \in X$. For $x \in X(u)$ and $y \in V \setminus (X \cup X(u))$, if $G[X \cup \{x, y\}]$ is not prime, then $\{u, x\}$ is a module of $G[X \cup \{x, y\}]$.

2) For $x \in \langle X \rangle$ and $y \in V \setminus (X \cup \langle X \rangle)$, if $G[X \cup \{x, y\}]$ is not prime, then $X \cup \{y\}$ is a module of $G[X \cup \{x, y\}]$.

3) For two distinct vertices x and y in $Ext(X)$, if $G[X \cup \{x, y\}]$ is not prime, then $\{x, y\}$ is a module of $G[X \cup \{x, y\}]$.

D. P. Sumner obtained the following result:

Lemma 2. *[7] If G is a prime graph, then G contains a path P_4 as an induced subgraph.*

2.2. Prime graphs and their subgraphs with prime frames

The following notations introduced by Y. Boudabbous and P. Ille [2] generalize those mentioned in the previous Section.

Given a proper vertex subset X of a graph G such that $|X| \geq 4$ and the frame of $G[X]$ is prime, consider the following subsets of $V(G) \setminus X$:

- $\langle X \rangle$ is the set of x outside X such that X is a module of $G[X \cup \{x\}]$.
- $Ext(X)$ is the set of x outside X such that the frame of $G[X \cup \{x\}]$ is prime and $\{x\} \in \mathcal{P}(G[X \cup \{x\}])$.
- For each C in $\mathcal{P}(G[X])$, $X(C)$ is the set of x outside X such that the frame of $G[X \cup \{x\}]$ is prime and $C \cup \{x\} \in \mathcal{P}(G[X \cup \{x\}])$.

The family of the non-empty elements of the union

$$\{Ext(X), \langle X \rangle\} \cup \{X(C) : C \in \mathcal{P}(G[X])\}$$

is denoted by $\mathcal{Q}_X$.

The following theorem is essential to prove some results in this paper.

Theorem 2. *[2] Taking into account a graph $G = (V, E)$, consider a proper vertex subset X of G with at least four vertices such that the frame of $G[X]$ is prime.*

1) The family $\mathcal{Q}_X$ forms a partition of $V \setminus X$.

2) If the graph G is prime, then there are two vertices x and y outside X such that the frame of $G[X \cup \{x, y\}]$ is prime and $\{x\}, \{y\} \in \mathcal{P}(G[X \cup \{x, y\}])$. More precisely:

- *(i) If $\langle X \rangle \neq \emptyset$, then there is a vertex x in $\langle X \rangle$ and a vertex y outside $X \cup \langle X \rangle$ such that the frame of $G[X \cup \{x, y\}]$ is prime and $\{x\}, \{y\} \in \mathcal{P}(G[X \cup \{x, y\}])$.*
- *(ii) Given an element C of $\mathcal{P}(G[X])$, if $|C \cup X(C)| \geq 2$ and $Ext(X) = \emptyset$, then there is a vertex x in $X(C)$ and a vertex y outside $X \cup X(C)$ such that the frame of $G[X \cup \{x, y\}]$ is prime and $\{x\}, \{y\} \in \mathcal{P}(G[X \cup \{x, y\}])$.*

3. Main result

Proposition 2. *Let k and q be non-negative integers with $k \geq 2$. Let G be a graph. If G is a $P_{k,q}$ graph, then G is prime.*

Proof. Let k and q be non-negative integers with $k \geq 2$. Let G be a $P_{k,q}$ graph.

First, if $q = 0$, then G is a $P_{k,0}$ graph.
Assume that $|X_k| = |X'_k| = k$ where $X_k = \{x_1, x_2, ..., x_k\}$ and $X'_k = \{x'_1, x'_2, ..., x'_k\}$ are

two disjoint sets. Using the induction on k ($k = |X|$), we prove that G is a prime graph.

On the one hand, if $k = 2$, then G is the Taurus (resp. the House) where x_1 is non-adjacent (resp. adjacent) to x_2. Thus, G is a prime graph.

On the other hand, $k \geq 2$. Assume that, for every $P_{k,0}$ graph H, H is prime. We prove that every $P_{k+1,0}$ graph H' is prime.
Let $X_{k+1} = X_k \cup \{x_{k+1}\}$ and $X'_{k+1} = X'_k \cup \{x'_{k+1}\}$ such that $x'_{k+1}x_{k+1} \in E(H')$ and $S = X_k \cup X'_k \cup \{a\}$. Based on Definition 1, there is a $P_{k,0}$ graph H_1 such that $H'[S] \simeq H_1$. Then, by the induction hypothesis, $H'[S]$ is prime. Using the definition of the graph H', $\{a, x'_{k+1}\}$ is adjacent to X'_k but non-adjacent to X_k. Thus, $x'_{k+1} \in S(a)$. According to the definition of the graph H', x_{k+1} is adjacent to x'_{k+1} but non-adjacent to the vertex a. Consequently, $\{a, x'_{k+1}\}$ is not a module in $G[S \cup \{x_{k+1}, x'_{k+1}\}]$. Based on assertion 1 of Lemma 1, H' is prime.

Second, $q \geq 1$. Let $a \in V$. Consider X, X', Y and Z as mentioned in Definition 1. Suppose $W = X \cup X' \cup \{a\}$. $Y \neq \emptyset$. On the contrary, consider a non-trivial module M of G. The fact that $G[W]$ is prime implies that $M \cap W = W$, $M \cap W$ is empty or $M \cap W$ is a singleton.

Firstly, if $M \cap W = W$, then $W \subset M$. Let $y \in V \setminus M$. As a consequence, $y \sim W$. Thus, $|N_{G[X \cup \{y\}]}(y)| = k$ and by definition y is unique in Y. There is $z \in Y \setminus \{y\}$ such that $zy \notin E$ and there is $x \in X$ such that $zx \notin E$ and, for all $x' \in X'$, $zx' \in E$, which is a contradiction.

Secondly, if $M \cap W = \emptyset$, then $M \subset Y$, which contradicts the fact that $N_{G[X \cup \{y_1\}]}(y_1) \neq N_{G[X \cup \{y_2\}]}(y_2)$ for any $y_1 \neq y_2 \in Y$.

Thirdly, there is α in V such that $M \cap W = \{\alpha\}$. Let $t \in M \setminus \{\alpha\}$, $t \in Y$. Then, there is $x \in X$ such that $xt \in E$.
$\alpha \neq a$ because, for all $x \in X$, $ax \notin E$ and $tx \in E$.
$\alpha \notin X$ since, for every $x \in X$, $ax \notin E$ and $at \in E$.
Otherwise, there is $x' \in X'$ such that $\alpha = x'$. Then, there is $x \in X$ such that $xx' \notin E$ and $ax' \in E$, which is a contradiction. Thus, $\alpha \notin X'$.
Therefore, G is prime.

Lemma 3. *Let H be a decomposable graph with a prime frame. For any module $M \in \mathcal{P}(H)$, there are two distinct vertices $y, z \in V(H) \setminus M$ such that $y \in N_H(M)$, $z \notin N_H(M)$ and $yz \notin E(H)$.*

Proof. Let $H = (V, E)$ be a decomposable graph with a prime frame. Let $M \in \mathcal{P}(H)$ be a module of H. As H has a prime frame, $N_H(M) \neq \emptyset$ and $V \setminus (M \cup N_H(M)) \neq \emptyset$.
On the contrary, suppose that, for any $y \in N_H(M)$ and $z \notin (M \cup N_H(M))$, $yz \in E$. It follows that $\forall y \in N_H(M)$ and $\forall t \in V \setminus N_H(M)$, $yt \in E$. Then, $\{N_H(M), (V \setminus N_H(M))\}$

is a modular partition of H with two elements. As $M \neq V \setminus N_H(M)$, $M \subset V \setminus N_H(M)$, which contradict the fact that $M \in \mathcal{P}(H)$.

To prove the main result, we use the following five lemmas.

Lemma 4. *Let G be a decomposable graph with a prime frame such that $G \in \mathcal{B}$. If e is an edge in $\overline{G}$, then $G + e$ is a decomposable graph.*

Proof. Let $G \in \mathcal{B}$ and $e \in \overline{G}$. Consider the graph $H = G + e$.

Firstly, if $|V(G)| = 5$, then $G \simeq \beta$. If e is one of the edges $x'x$ and $x'y$, then $\{a, a'\}$ is still a module in $G + e$. Thus, $G + e$ is a decomposable graph.
Otherwise, $e = ax$ or $e = a'x$. Without loss of generality, we add ax. Then, $\{x, x', a', y\}$ is a non-trivial module. As a result, $G + e$ is a decomposable graph.

Secondly, $|V(G)| \geq 6$. If the edge $e = \alpha\beta$ where α and β are two vertices in $X \cup Y$, then $\{a, a'\}$ is still a module in $G + e$. Thus, $G + e$ is a decomposable graph.
Otherwise, $e = ax$ or $e = a'x$ for $x \in X$. Assume, without loss of generality, that $e = ax$ and there is exactly one vertex x' in X' such that $xx' \in E$. Thus, $\{ax'\}$ is a module in $G + e$. Therefore, $G + e$ is a decomposable graph.

Lemma 5. *Let G be a decomposable graph with a prime frame containing only one non-trivial module M. If $G[M] = \overline{K_2}$, then there is an edge e in $\overline{G}$ such that $G + e$ is prime.*

Proof. Let $G = (V, E)$ be a decomposable graph with a prime frame containing only one non-trivial module $M = \{a, a'\}$ such that $G[M] = \overline{K_2}$ and $e \in E(\overline{G})$. Consider $H = G + e$.

Based on Lemma 3, there is $x \notin N_G(M)$ and $x' \in N_G(M)$ such that $xx' \notin E(G)$. If $e = ax$, then $H[\{x, a, x', a'\}]$ is a P_4.
$H - a = G - a$ is prime. As $H[\{x, a, x', a'\}]$ is a path P_4, in H, $a \notin \langle V \setminus \{a\} \rangle$, $a \notin (V \setminus \{a\})(a')$, $a \notin (V \setminus \{a\})(x)$ and $a \notin (V \setminus \{a\})(x')$. If H is not prime, then there is y in $V \setminus \{a, a', x, x'\}$ such that $a \in (V \setminus \{a\})(y)$. Hence, $a'y \notin E(H)$ because $a'a \notin E(H)$. Thus, $ya \notin E(H)$ and $\{xy, x'y\} \subseteq E(H)$. In this case, we choose the graph $H' = G + e'$ where $e' = ay$. Notice that $H'[\{a, a', x, x', y\}]$ is a Taurus.
$H' - a = G - a$ is prime. Since $H'[\{a, a', x, x', y\}]$ is a Taurus, $a \notin \langle V \setminus \{a\} \rangle$ and $a \notin (V \setminus \{a\})(\alpha)$ with $\alpha \in \{a', x, x', y\}$. We prove that H' is prime using contradiction. Assume that H' is not prime. Then, there is z in $V \setminus \{a, a', x, x', y\}$ such that $a \in (V \setminus \{a\})(z)$. Moreover, $za' \notin E(H')$ because $aa' \notin E(H')$. As $\{a, a'\}$ is a module in G, $za \notin E(H')$, $\{x'z, yz\} \subseteq E(H')$ and $xz \notin E(H')$. Knowing that $za \notin E(H')$ and $zy \in E(H')$ contradict the fact that $\{a, y\}$ is a module in H, H' is prime.

Lemma 6. *Let G be a decomposable graph with a prime frame containing only one non-trivial module M. If $G[M] = K_2$ and $G \notin \mathcal{B}$, then an edge e exists in $\overline{G}$ such that $G + e$ is prime.*

Proof. Let $G = (V, E)$ be a decomposable graph with a prime frame containing only one non-trivial module $M = \{a, a'\}$ such that $G[M] = K_2$ and $G \notin \mathcal{B}$. Let $e \in E(\overline{G})$. Consider that $H = G + e$.

Suppose $Z = N_G(M)$, $X = V \setminus (Z \cup \{a, a'\})$. Let $W = V \setminus \{a\}$. Since the frame of G is prime, $X \neq \emptyset$ and $Z \neq \emptyset$.
Let $B = \{b : \ b \in X \text{ and } bz \notin E, \ \forall z \in Z\}$ ($B \neq X$ because the frame is prime).

Firstly, $B \neq \emptyset$.
As $G - a$ is prime, there is $y \in B$ and $x \in X \setminus B$ such that $xy \in E$. Consider $H = G + e$ where $e = ay$.
Notice that $G - a = H - a$ is prime.
Knowing that, $ax \notin E(H)$ and $aa' \in E(H)$, $a \notin \langle W \rangle$ in H.
As $ya \in E(H)$ and $ya' \notin E(H)$, $a \notin W(a')$ in H.
Since, for all $\beta \in Z$, $y\beta \notin E(H)$ and $ya \in E(H)$, $a \notin W(\beta)$ in H.
Knowing that, for all $\alpha \in X \setminus \{y\}$, $a'a \in E(H)$ and $a'\alpha \notin E(H)$, $a \notin W(\alpha)$ in H.
Given that $xa \notin E(H)$ and $xy \in E(H)$, $a \notin W(y)$ in H.
Thus, using Lemma 1 in H, $a \in Ext(W)$. Therefore, $H = G + e$ is prime.

Secondly, $B = \emptyset$. Distinguish two cases:

First, assume that there is $x \in X$ where, for all $x' \in Z$, $\{a, x'\}$ is not a module in $H = G + ax$. Notice that $H - a = G - a$ is prime.
As $X \neq \emptyset$ and $aa' \in E(H)$, $a \notin \langle W \rangle$ in H.
Knowing that, for all $t \in X \setminus \{x\}$, $a't \notin E(H)$ and $a'a \in E$, $a \notin W(t)$ in H.
Since $a'a \in E(H)$ and $a'x \notin E(H)$, $a \notin W(x)$ in H.
Given that, for all $x' \in Z$, $\{a, x'\}$ is not a module in H, $a \notin W(x')$ in H.
As $xa \in E(H)$ and $xa' \notin E(H)$, $a \notin W(a')$ in H.
As a consequence, based on by Lemma 1, $a \in Ext(W)$. Therefore, H is prime.

Second, assume that, for all $x \in X$, there is $x' \in Z$ such that $\{a, x'\}$ is a new module in $H = G + ax$. Since $a'a \in E(H)$ and $xa \in E(H)$, $a'x' \in E(H)$ and $x'x \in E(H)$.

We show that, for all $t \in V \setminus \{a, a', x, x'\}$, $t \sim_H \{a, a', x'\}$.
Indeed, as $\{a, a'\}$ is a module in G, for all $t \in V \setminus \{a, a'\}$, $t \sim_G \{a, a'\}$. Hence, $t \sim_H \{a, a'\}$. Given that $\{a, x'\}$ is a module in H, for all $t \in V \setminus \{a, a', x, x'\}$, $t \sim_H \{a, x'\}$. Thus, for all $t \in V \setminus \{a, a', x, x'\}$, $t \sim_H \{a, a', x'\}$.
Let X' be the greatest clique of Z (i.e. $G[X']$ is a complete graph) such that $X' = \{x' : x' \in Z \text{ and } |N_{G[X \cup \{x'\}]}(x')| = 1\}$ and $X'' = Z \setminus X'$.
As $G - a$ is prime, for all $x \in X$, there is a unique $x' \in X'$ such that $xx' \in E(G)$. As a result, $|X| = |X'|$.
If $X'' = \emptyset$, then G is a $\beta_{k,0}$ graph, which contradicts the fact that $G \notin \mathcal{B}$.
Otherwise, $X'' \neq \emptyset$ and $q \geq 1$. Since G is not a $\beta_{k,q}$ graph and $G - a = H - a$ is prime,

$G - a$ is not a $P_{k,q}$ graph and $k \geq 3$. We distinguish two subcases.

In the first subcase, there is $y \in X''$ such that $|N_{G[X\cup\{y\}]}(y)| = 1$ (resp. $|N_{G[X\cup\{y\}]}(y)| = k$) and $\forall t \in X'' \setminus \{y\},\ ty \in E$. Thus, $y \in X'$, which contradicts the fact that X' is the greatest clique of Z (resp. $y \in \langle V \setminus \{y\}\rangle$, which contradicts the fact that $G - a$ is prime).

In the second subcase, there is $y \neq z \in X''$ such that $N_{G[X\cup\{y\}]}(y) = N_{G[X\cup\{z\}]}(z)$, so $\{y, z\}$ is a non-trivial module in $G - a$, which contradicts the fact that $G - a$ is prime.

Lemma 7. *Let G be a decomposable graph with a prime frame containing only one non-trivial module M. If $G[M]$ is a prime graph, then there is an edge e in $\overline{G}$ such that $G + e$ is prime.*

Proof. Let G be a decomposable graph on V with a prime frame containing only one non-trivial module M such that $G[M]$ is a prime graph. Suppose $a \in M$ and $X = V \setminus \{a\}$. Since the frame of G is prime and the only non-trivial module is M, there is $b \in V \setminus M$ such that $b \notin N_G(M)$. Consider $e = ab$ and $H = G + e$. It's clear that $H - a = G - a$ is a graph with a prime frame having only non-trivial module $M \setminus \{a\}$.
As $b \notin N_H(M \setminus \{a\})$ and $ab \in E(H)$, $a \notin X(M \setminus \{a\})$ in H.
Given that $G[M]$ is prime, there are two vertices $y, z \in M \setminus \{a\}$ such that $ya \in E(G)$ and $za \notin E(G)$. Thus, $a \notin \langle X\rangle$ in H.
For all $t \in (V \setminus M)$, $t \in N_G(M)$ or $t \notin N_G(M)$. If $t \in N_G(M)$, then $zt \in E(G)$. As a result, $a \notin X(t)$ in H. If $t \notin N_G(M)$, then $yt \notin E(G)$. Thus, $a \notin X(t)$ in H.
Therefore, using assertion 1 of Theorem 2, $a \in Ext(X)$ in H.

Based on assertion 2 of Theorem 2, if $N \in \mathcal{P}(H)$ where N is a non-trivial module, then N is a non-singleton module of $H[M \setminus \{a\}]$. Hence, N is a module of $H - a$. Moreover, $a \sim_H N$. Then, $a \sim_G N$ contradicts the fact that $G[M]$ is prime. Consequently, H does not contain any non-trivial module. Therefore, H is a prime graph.

Lemma 8. *Let G be a decomposable graph with an empty frame containing only one non-trivial module M. If $|M| = |V(G)| - 1$ and $G[M]$ is a prime graph, then an edge e exists in $\overline{G}$ such that $G + e$ is prime.*

Proof. Let G be a decomposable graph on V with an empty frame containing only one non-trivial module M such that $V(G) = M \cup \{b\}$ and $G[M]$ is a prime graph.
Let $P = (x_1, x_2, \ldots, x_k)$ be the longest prime path in $G[M]$ with length k. Using Lemma 2, $k \geq 4$. Consider $e = bx_1$ and $H = G + e$. $H - b = G - b = G[M]$ is a prime graph. As $H[\{b, x_1, x_2, \ldots, x_k\}]$ is a path of a length greater than 4, $H[\{b, x_1, x_2, \ldots, x_k\}]$ is prime. Hence, $b \notin \langle M\rangle$ in H and $b \notin M(x_i)$ in H for all $1 \leq i \leq k$. If H is not prime, then, using Lemma 1, there is $t \in M \setminus \{x_1, x_2, \ldots, x_k\}$ such that $b \in M(t)$. Thus, $H[\{t, x_1, x_2, \ldots, x_k\}] = G[\{t, x_1, x_2, \ldots, x_k\}]$ is a path of length $k + 1$, which is a contradiction. As a consequence, based on Lemma 1, $b \in Ext(M)$. Therefore, H is a prime graph.

Our main result is the following theorem:

Theorem 3. *Let G be a decomposable graph with at least 4 vertices having exactly one non-trivial module M. There is an edge e in $\overline{G}$ such that $G+e$ is a prime graph if and only if one of the following assertions holds.*

(i) G has a prime frame and $G[M]$ is a prime graph or $\overline{K_2}$.

(ii) G has a prime frame, $G[M]$ is K_2 and $G \notin \mathcal{B}$.

(iii) G has an empty frame and $G[M]$ is a prime graph with $|M| = |V(G)| - 1$.

Proof. Let G be a decomposable graph with at least 4 vertices having exactly one non-trivial module M.

On the one hand, if G has a prime frame, then $G[M]$ is $\overline{K_2}$, K_2 or prime.

First, if $G[M] = \overline{K_2}$, then, according to Lemma 5, there is an edge e in $\overline{G}$ such that $G+e$ is prime.

Second, if $G[M] = K_2$ and $G \notin \mathcal{B}$, then, based on Lemma 6, there is an edge e in $\overline{G}$ such that $G+e$ is prime.

Third, if $G[M]$ is a prime graph, then, using Lemma 7, there is an edge e in $\overline{G}$ such that $G+e$ is prime.

On the other hand, if G has an empty frame and $V(G) = M \cup \{b\}$ where $G[M]$ is a prime graph, then, based on Lemma 8, there is an edge e in $\overline{G}$ such that $G+e$ is prime.

Inversely, assume that there is e in $\overline{G}$ where $G+e$ is prime. It's clear that the frame of G is not complete.

On the one hand, if the frame of G is prime since M is the only non-trivial module in G, then M does not contain any non-trivial module. Thus, $G[M]$ is a prime graph or $G[M] \in \{K_2, \overline{K_2}\}$. Assume that $G[M] = K_2$. As $G+e$ is prime, the Lemma 4 implies $G \notin \mathcal{B}$.

On the other hand, if the frame is empty knowing that each element of $\mathcal{P}(G)$ is a module of G and G contains only one non-trivial module M, then $V \setminus M$ is a module. Given that M is the unique non-trivial module, $V \setminus M$ is a trivial module. Thus, $V \setminus M$ is a singleton. Therefore, the frame of G is isomorphic to $\overline{K_2}$, $G[M]$ is a prime graph and $|M| = |V(G)| - 1$.

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