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British Put Option On Stocks Under Regime-Switching Model

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Abstract. In a plain vanilla option, its holder is given the right, but not the obligation, to buy or sell the underlying stock at a specified price (strike price) at a predetermined date. If the exercise date is at maturity, the option is called a European; if the option is exercised anytime prior to maturity, it is called an American. In a British option, the holder can enjoy the early exercise feature of American option whereupon his payoff is the 'best prediction' of the European payoff given all the information up to exercise date under the hypothesis that the true drift of the stock equals a specified contract drift. In this paper, in contrast to the constant interest rate and constant volatility assumptions, we consider the British option by assuming that the economic state of the world is described by a finite state continuous-time Markov chain. Also, we provide a solution to a free boundary problem by using PDE arguments. However, closed form expression for the arbitrage-free price are not available in our setting.

2020 Mathematics Subject Classifications: 62P05, 97M30

Key Words and Phrases: British put option, american put option, european put option, arbitrage-free price, rational exercise boundary, geometric Brownian motion, optimal stopping time, free boundary problem, regime-switching

1. Introduction

Plain vanilla options such as European options and American options are widely used in the market and their pricing mechanisms are well studied. An option gives the holder the right, but not the obligation, to buy or sell an underlying asset for a specified price, called strike price, on or before a specified future date, called maturity date or expiration date. The option is European if the holder can exercise it only at expiration date; it is American if the option can be exercised anytime even prior to the expiration date.

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One of the pricing mechanisms for European option is provided by the well-known Black-Scholes-Merton formula. This mathematical model assumes, among other things, the absence of arbitrage opportunities and that lending and borrowing are possible at the same risk-free rate. Such method falls within the so-called risk-neutral pricing framework.

In [5], G. Peskir and F. Samee introduced a new type of option, called British option, which is American in nature because it can be exercised prior to maturity but with European payoff. The motivation for this new financial product stems from the disparity between the expected value of the option buyer's investment, in the form of premium paid, and the expected value of his payoff when the actual drift rate of the underlying stock price deviates from the risk-free rate. An added feature is built into this instrument which aim at both providing protection against unfavourable price movements as well as securing higher returns when these movements are favourable [5].

The derivation of the British option price in [5] assumes the usual model as in the Black-Scholes-Merton formula: a geometric Brownian motion for the dynamics of the underlying stock, a constant risk-free interest rate and a constant volatility. In [2], Yao, Zhang and Zhou priced the European options in continuous-time regime-switching via a recursive algorithm. This paper aims to extend the result in [5] by assuming that the economic state of the world is described by a finite state continuous-time Markov chain. The paper is organized as follows. In Section 2 we present the definition of the British put option as given in [5] and the financial setting. In Section 3 we define the stopping set and boundary function and provide results involving these two. In particular, we show that the boundary function satisfies the Volterra type equation, then conclude.

2. Setting of the Problem

In this paper, we assume that the economic state of the world is described by a finite state continuous-time Markov chain $\boldsymbol{\alpha} = (\alpha_t)_{t\in\mathbb{R}_+}$ on $\mathcal{M} = \{1, 2, \dots, m\}$. Suppose that the volatility $\sigma : \mathcal{M} \to (0, \infty)$ depends on the state $\boldsymbol{\alpha}$ of the economy. Under the real world probability measure $\mathbb{P}$, we assume that the dynamics of the stock price process follows a geometric Brownian motion:

$$dX_t \;=\; \mu X_t dt + \sigma(\alpha_t) X_t dW_t, \quad X_0 = x > 0, \tag{1}$$

where $\mu \in \mathbb{R}$ is the true drift, $W = (W_t)_{t\geq 0}$ denotes the standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Here, we assume that W is independent of the Markov-chain $\boldsymbol{\alpha}$ and the filtration $\mathcal{F} = (\mathcal{F}_t)_{t\in\mathbb{R}_+}$ is generated by W and $\boldsymbol{\alpha}$.

We will consider the British put option on stocks in the aforementioned financial market. The British put option with strike price K and time to maturity T (in years) is defined in [5] as follows:

Definition 1. [5] *The British put option is a financial contract between a seller/hedger and a buyer/holder entitling the latter to exercise at any (stopping) time τ prior to maturity T whereupon his payoff (deliverable immediately) is the 'best prediction' of the European* *payoff* $(K - X_T)^+$ *given all the information up to time* τ *under the hypothesis that the true drift* μ *of the stock price equals the contract drift* μ_c.

In [5], the price of the British put option is derived under the hypothesis that the volatility is constant for all $t \in [0, T]$. Hence, this paper presents an extension of the results in [5].

For $0 \leq t \leq T$, let

$$\beta(t) := \frac{\mu_c - \mu}{\sigma(\alpha_t)}, \tag{2}$$

where $\mu_c \neq \mu$. Define an equivalent measure $\mathbb{P}^{\mu_c}$ via the following:

$$\frac{d\mathbb{P}^{\mu_c}}{d\mathbb{P}} = Z_T, \tag{3}$$

where

$$Z_t := \exp\left[\int_0^t \beta(u)dW_u - \frac{1}{2}\int_0^t \beta^2(u)du\right] \tag{4}$$

and $E[Z_t] = 1$ for $0 \leq t \leq T$. Then by Itô′s formula,

$$\frac{dZ_t}{Z_t} = \beta(t)dW_t. \tag{5}$$

This shows that Z_t is a local martingale. From Lemma 1 in [2], we have

$$W_t^{\mu_c} = W_t - \int_0^t \beta(u)du \tag{6}$$

is a $\mathbb{P}^{\mu_c}$-Brownian motion. Under the probability measure $\mathbb{P}^{\mu_c}$, (1) becomes

$$dX_t = \mu_c X_t dt + \sigma(\alpha_t) X_t dW_t^{\mu_c} \tag{7}$$

where $0 \leq t \leq T$ with $X_0 = x \in (0, \infty)$. Thus, making use of (3), we have $E^{\mu_c}(X) = E(Z_T X) = E(Z_T)E(X) = E(X)$ for any random variable X.

The payoff of the British put option at a given stopping time $t = \tau$ is given by

$$E^{\mu_c}[\ (K - X_T)^+ \mid \mathcal{F}_\tau\ ] \tag{8}$$

where the conditional expectation is taken with respect to a new (equivalent) probability measure $\mathbb{P}^{\mu_c}$ under which the stock price X evolves as in (7) with $X_0 = x \in (0, \infty)$. Thus, the effect of exercising the British put option is to substitute the contract drift μ_c to the true (unknown) drift μ of the stock price for the remaining time of the contract. Note that the value of the contract drift μ_c must be equivalent to the buyer's tolerance level for the deviation of the true drift μ from his original belief. Moreover, to avoid arbitrage opportunity, the contract drift naturally satisfies (See [5])

$$\mu_c > r. \tag{9}$$

Note that by Itô's formula, the solution to equation (7) is

$$X_t = X_s Z_{s,t}^{\mu_c} \tag{10}$$

for $0 \leq s \leq t \leq T$ where

$$Z_{s,t}^{\mu_c} = \exp\left[\int_s^t \left(\mu_c - \frac{\sigma^2(\alpha_u)}{2}\right) du + \int_s^t \sigma(\alpha_u) dW_u^{\mu_c}\right] \tag{11}$$

so that the payoff (8) can be written as

$$E^{\mu_c}[\ (K - X_\tau Z_{\tau,T}^{\mu_c})^+ \mid \mathcal{F}_\tau\ ]. \tag{12}$$

Let α_0 be given. Applying the usual hedging scheme, then the arbitrage-free price of the British put option at deal date (time 0) is given by

$$\begin{aligned} V &= V(0, X_0, \alpha_0) \\ &= \sup_{0\leq\tau\leq T} \widetilde{E}\left[e^{-r\tau} E^{\mu_c}\left((K - X_T)^+ \middle| \mathcal{F}_\tau\right) \middle| \mathcal{F}_0\right] \\ &= \sup_{0\leq\tau\leq T} \widetilde{E}\left[e^{-r\tau} E^{\mu_c}\left((K - X_T)^+ \middle| \mathcal{F}_\tau\right)\right] \end{aligned} \tag{13}$$

where the supremum is taken over all stopping time $\tau \in [0, T]$ of X and the $\widetilde{E}$ is taken with respect to the (unique) equivalent martingale measure $\widetilde{\mathbb{P}}$.

Now, fix $t \in [0, T]$. We want a general expression for the price, denoted by $V(t, X_t, \alpha_t)$, of the British put option at any time t at which the stock price $X_t = x > 0$. Denote the payoff in (8) at $\tau = s$ by

$$G^{\mu_c}(s, y, j) = E^{\mu_c}\left[(K - yZ_{s,T}^{\mu_c})^+ \mid \alpha_s = j, X_s = y\right] \tag{14}$$

for $s \in [0, T]$ where $Z_{s,T}^{\mu_c}$ is given in (11) . If the exercise date of the British put option is at time $t + \tau$, where $\tau \in [0, T - t]$, then extending the argument in (13), we have

$$V(t, x, i) = \sup_{0\leq\tau\leq T-t} \widetilde{E}_{t,x}\left[e^{-r\tau} G^{\mu_c}(t + \tau, X_{t+\tau}, j) \mid \alpha_t = i, X_t = x\right] \tag{15}$$

where the supremum is taken overall stopping time $\tau \in [0, T-t]$ of X and $\widetilde{E}_{t,x}$ is taken with respect to the (unique) equivalent martingale measure $\widetilde{\mathbb{P}}_{t,x}$ under which $X_t = x \in \mathbb{R}_+$. Using the same argument as above with μ_c is replaced with r in relations (2) through (7) and that $Z_{t,t+\tau}^r$ (as defined in (11) with μ_c is replaced with r) has stationary and independent increments (i.e., $Z_{t,t+\tau}^r$ is a version of $Z_{0,\tau}^r$), the option price in (15) can be rewritten as

$$V(t, X_t, \alpha_t) = \sup_{0 \leq \tau \leq T-t} E\left[\ e^{-r\tau} G^{\mu_c}(t+\tau, X_t X_\tau, j) \mid \mathcal{F}_t\ \right] \tag{16}$$

where the process $X = X(r)$ under $\mathbb{P}$ solves

$$dX_t = rX_t dt + \sigma(\alpha_t) X_t dW_t^r$$

with $X_0 = 1$. Note that they are equivalent because $\mathcal{F}_t$ knows the values of X_t and α_t.

Proposition 1. For any $t \in [0,T]$ and $j \in \mathcal{M}$ given and fixed, the mapping

$$x \mapsto G^{\mu_c}(t, x, j) \tag{17}$$

is convex on $(0, \infty)$.

Proof. Let $0 \leq \lambda \leq 1$ and $x_2 = \lambda x_1 + (1-\lambda)x_3$ for some $x_1, x_3 \in (0,\infty)$ with $x_1 < x_3$. We have

$$\begin{aligned}
G^{\mu_c}(t, x_2, j) &= G^{\mu_c}(t, \lambda x_1 + (1-\lambda)x_3, j) \\
&= E^{\mu_c}\left[\left(K - \lambda x_1 Z_{t,T}^{\mu_c} - (1-\lambda)x_3 Z_{t,T}^{\mu_c}\right)^+ \mid X_t = x, \alpha_t = j\right] \\
&= E^{\mu_c}\left[\left(\lambda K + (1-\lambda)K - \lambda x_1 Z_{t,T}^{\mu_c} - (1-\lambda)x_3 Z_{t,T}^{\mu_c}\right)^+ \mid X_t = x, \alpha_t = j\right] \\
&\leq \lambda E^{\mu_c}\left[\left(K - x_1 Z_{t,T}^{\mu_c}\right)^+ \mid X_t = x, \alpha_t = j\right] \\
&\quad + (1-\lambda) E^{\mu_c}\left[\left(K - x_3 Z_{t,T}^{\mu_c}\right)^+ \mid X_t = x, \alpha_t = j\right] \\
&= \lambda G^{\mu_c}(t, x_1, j) + (1-\lambda) G^{\mu_c}(t, x_3, j),
\end{aligned}$$

which completes our proof. □

It can also be verified that the mapping in (17) is strictly decreasing on $(0,\infty)$ with $G^{\mu_c}(T, x, j) = (K - X_T)^+$, $G^{\mu_c}(t, 0, j) = K$ and $\lim_{x \to +\infty} G^{\mu_c}(t, x, j) = 0$. By Proposition 1 and equation (15) above, it follows that the mapping

$$x \mapsto V(t, x, i) \tag{18}$$

is convex for any $t \in [0,T]$ and $i \in \mathcal{M}$ given and fixed and strictly decreasing on $(0,\infty)$ with $V(T, x, i) = (K - X_T)^+$, $V(t, 0, i) = K$ and $\lim_{x \to +\infty} V(t, x, i) = 0$. Hence, both mappings (17) and (18) are continuous on $(0,\infty)$ for any $t \in [0,T]$ and $\alpha_t \in \mathcal{M}$ given and fixed.

Define the set

$$D := \{(t, x, j) \in [0,T] \times (0,\infty) \times \mathcal{M}\ :\ V(t, x, j) = G^{\mu_c}(t, x, j)\}. \tag{19}$$

Let $(T, x, j) \in \{T\} \times (0, \infty) \times \mathcal{M}$. We note that

$$V(T, X_T, j) = (K - X_T)^+ = G^{\mu_c}(T, X_T, j). \tag{20}$$

Hence, $\{T\} \times (0, \infty) \times \mathcal{M} \subset D$, which is consistent with the fact that the supremum in (15) is taken over $(\mathcal{F}_t)_{t\in[0,T]}$-stopping times $\tau \in [t, T]$. Furthermore, by Corollary 2.9 page 46 in Peskir and Shiryaev [6], the $(\mathcal{F}_t)_{t\in[0,T]}$-stopping time

$$\tau_D(t, X_t, \alpha_t) := \inf \{s \in [0, T - t] \ : \ (t, x, j) \in D\} \tag{21}$$

with $X_t = x \in (0, \infty)$ and $\alpha_t = j \in \mathcal{M}$, is an optimal stopping time for option price in (15) since $x \mapsto V(t, x, j)$ and $x \mapsto G^{\mu_c}(t, x, j)$ are both continuous on $(0, \infty)$ and $G^{\mu_c}(t, x, j) \le K$ for all $t \in [0, T]$ and $j \in \mathcal{M}$. Moreover, by using the equivalent expression for the stopping set D in Relation (46) in Proposition (2), $\tau_D(t, X_t, \alpha_t)$ can be rewritten in terms of the optimal stopping boundary function as

$$\tau_D(t, x, j) := \inf \{s \in [0, T - t] \ : \ x \le b_D(t, j)\}, \tag{22}$$

where $b_D(t, j)$ is defined in (44) below at which $X_t = x$ and $\alpha_t = j$.

We next derive the following continuity results to show that the set D in (24) is closed.

Lemma 1. The mapping $(t, x) \mapsto G^{\mu_c}(t, x, j)$ is jointly continuous on $[0, T] \times (0, \infty)$.

Proof. The continuity of the mapping $x \mapsto G^{\mu_c}(t, x, j)$ follows from the fact that $G^{\mu_c}(t, x, j)$ is convex with respect to $x \in (0, \infty)$ for any time $t \in [0, T]$ given and fixed. It remains to show the uniform continuity of the mapping $t \mapsto G^{\mu_c}(t, x, j)$ at time $t = t_1$. Let $x \in (0, \infty)$ be given and fixed and $0 \le t_1 < t_2 \le T$. Then we have,

$$\begin{aligned} 0 &\le \left| G^{\mu_c}(t_2, x, j) - G^{\mu_c}(t_1, x, j) \right| \\ &\le E^{\mu_c}\left[\left| (K - xZ^{\mu_c}_{t_2,T})^+ - (K - xZ^{\mu_c}_{t_1,T})^+ \right| \,\middle|\, \mathcal{F}_{t_2} \right] \\ &\le xE^{\mu_c}\left[\left| (Z^{\mu_c}_{t_1,T} - Z^{\mu_c}_{t_2,T})^+ \right| \,\middle|\, \mathcal{F}_{t_2} \right] \end{aligned}$$

$$\begin{aligned} &= xE^{\mu_c}\left[\left| Z^{\mu_c}_{t_1,T} \left(1 - \frac{Z^{\mu_c}_{t_2,T}}{Z^{\mu_c}_{t_1,T}} \right)^+ \right| \,\middle|\, \mathcal{F}_{t_2} \right] \\ &= xE^{\mu_c}\left[\left| Z^{\mu_c}_{t_1,T} \left(1 - e^{-\int_{t_1}^{t_2} \left(\mu_c - \frac{\sigma^2(\alpha_u)}{2} \right) du - \int_{t_1}^{t_2} \sigma(\alpha_u) dW^{\mu_c}_u} \right)^+ \right| \,\middle|\, \mathcal{F}_{t_2} \right]. \end{aligned}$$

Therefore, as $t_2 - t_1 \to 0$, we have $G^{\mu_c}(t_2, x, j) - G^{\mu_c}(t_1, x, j) \to 0$ uniformly, which completes our proof. □

Lemma 2. For any $j \in \mathcal{M}$, the mapping $(t,x) \mapsto V(t,x,j)$ is jointly continuous on $[0,T] \times (0,\infty)$.

Proof. The continuity of the mapping $x \mapsto V(t,x,j)$ at a point x_0 follows from the fact that $V(t,x,j)$ is convex with respect to $x \in (0,\infty)$ for any time $t \in [0,T]$ given and fixed. It remains to show that the mapping $t \mapsto V(t,x,j)$ is continuous at t_1 uniformly over $x \in \mathbb{R}$. Let $x \in (0,\infty)$ be given and fixed and suppose $0 \le t_1 < t_2 \le T$. Let $\tau_1 = \tau_D(t,x,i)$ be the optimal stopping time for (15) and $\tau_2 = \tau_1 \wedge (T - t_2)$. Then

$$\begin{aligned}
0 &\le \Big|V(t_1,x,i) - V(t_2,x,i)\Big| \\
&\le \Big|E\left[e^{-r\tau_1} G^{\mu_c}(t_1+\tau_1, X_{t_1+\tau_1}, j) \mid \mathcal{F}_{t_1}\right] \\
&\qquad - E\left[e^{-r\tau_2} G^{\mu_c}(t_2+\tau_2, X_{t_2+\tau_2}, j) \mid \mathcal{F}_{t_2}\right]\Big| \\
&\le \Big|E\left[e^{-r\tau_2} G^{\mu_c}(t_1+\tau_1, X_{t_1+\tau_1}, j) \mid \mathcal{F}_{t_1}\right] \\
&\qquad - E\left[e^{-r\tau_2} G^{\mu_c}(t_2+\tau_2, X_{t_2+\tau_2}, j) \mid \mathcal{F}_{t_2}\right]\Big| \\
&\le \Big|E\left[e^{-r\tau_2}\left\{G^{\mu_c}(t_1+\tau_1, X_{t_1+\tau_1}, j) - G^{\mu_c}(t_2+\tau_2, X_{t_2+\tau_2}, j)\right\} \mid \mathcal{F}_{t_2}\right]\Big| \\
&\le E\left[e^{-r\tau_2}\Big|G^{\mu_c}(t_1+\tau_1, X_{t_1+\tau_1}, j) - G^{\mu_c}(t_2+\tau_2, X_{t_2+\tau_2}, j)\Big| \;\Big|\; \mathcal{F}_{t_2}\right].
\end{aligned}$$

By the continuity of the mapping $t \mapsto G^{\mu_c}(t,x,j)$ from Lemma 1, the mapping $t \mapsto V(t,x,i)$ is continuous on $[0,T]$, uniformly in $x \in (0,\infty)$. □

3. Stopping set and boundary function

Define

$$F(t,x,j) = V(t,x,j) - G(t,x,j) \ge 0, \tag{23}$$

which is nonnegative for $t \in [0,T]$, $x \in (0,\infty)$ and $j \in \mathcal{M}$, so that we have

$$D = \{(t,x,j) \in [0,T] \times (0,\infty) \times \mathcal{M} \;:\; F(t,x,j) = 0\}. \tag{24}$$

By the continuity of both mappings $(t,x) \mapsto V(t,x,i)$ and $(t,x) \mapsto G^{\mu_c}(t,x,j)$ on $[0,T] \times (0,\infty)$, the set D is closed. Thus, the continuation set

$$C = D^c = \{(t,x,j) \in [0,T] \times (0,\infty) \times \mathcal{M} \;:\; F(t,x,j) > 0\} \tag{25}$$

is open.

Lemma 3. For any $(t,x,j) \in D$, we have

$$\limsup_{\epsilon \searrow 0} \frac{F(t,x+\epsilon,j) - F(t,x,j)}{\epsilon} \le 0. \tag{26}$$

Proof. For all $x \in (0,\infty)$ and $\epsilon > 0$, consider the $(\mathcal{F}_s)_{s\in[t,T]}$-stopping time

$$\tau_\epsilon^+ = \tau_D(t, x+\epsilon, j) \in [0, T-t] \tag{27}$$

defined in (21), which solves the optimal stopping problem

$$\begin{aligned} V(t, x+\epsilon, \alpha_t) &= \sup_{0\le\tau\le T-t} E\left[e^{-r\tau} G^{\mu_c}(t+\tau, X_{t+\tau}, j) \,\Big|\mathcal{F}_t\right] \\ &= E\left[e^{-r\tau_\epsilon^+} G^{\mu_c}(t+\tau_\epsilon^+, X_{t+\tau_\epsilon^+}, j) \,\Big|\mathcal{F}_t\right]. \end{aligned} \tag{28}$$

We first claim that

$$\tau_\epsilon^+ \to 0 \quad \text{as} \quad \epsilon \to 0. \tag{29}$$

From the definition of $\tau_D(t, x+\epsilon, j)$, we have, on the event $\{\alpha_t = j\}$,

$$\begin{aligned} \tau_D(t, x+\epsilon, j) &= \inf\{s \in [0, T-t] : (t, x+\epsilon, j) \in D\} \\ &= \inf\Bigg\{s \in [0, T-t] : \sup_{0\le s\le T-t} E\left[e^{-rs} E^{\mu_c}\left[(K-(x+\epsilon)X_s Z^{\mu_c}_{t+s,T})^+\Big|\mathcal{F}_{t+s}\right]\Big|\mathcal{F}_t\right] \\ &\qquad = E^{\mu_c}\left[(K-(x+\epsilon)X_s Z^{\mu_c}_{t+s,T})^+\Big|\mathcal{F}_{t+s}\right]\Bigg\} \\ &\le \inf\Bigg\{s \in [0, T-t] : \sup_{0\le s\le T-t} E\left[e^{-rs} E^{\mu_c}\left[(K-xX_s Z^{\mu_c}_{t+s,T})^+\Big|\mathcal{F}_{t+s}\right]\Big|\mathcal{F}_t\right] \\ &\qquad \ge E^{\mu_c}\left[(K-(x+\epsilon)X_s Z^{\mu_c}_{t+s,T})^+\Big|\mathcal{F}_{t+s}\right]\Bigg\} \\ &\le \inf\Bigg\{s \in [0, T-t] : \sup_{0\le s\le T-t} E\left[e^{-rs} E^{\mu_c}\left[(K-xX_s Z^{\mu_c}_{t+s,T})^+\Big|\mathcal{F}_{t+s}\right]\Big|\mathcal{F}_t\right] \\ &\qquad \ge E^{\mu_c}\left[\frac{1}{2}\left(K - xZ^{\mu_c}_{t+s,T} - \epsilon Z^{\mu_c}_{t+s,T} + \left|K - xZ^{\mu_c}_{t+s,T} - \epsilon Z^{\mu_c}_{t+s,T}\right|\right)\Big|\mathcal{F}_{t+s}\right]\Bigg\} \end{aligned}$$

This implies that

$$\begin{aligned} \lim_{\epsilon\to 0} \tau_D(t, x+\epsilon, j) &\le \lim_{\epsilon\to 0} \inf\Bigg\{s \in [0, T-t] : \sup_{0\le s\le T-t} E\left[e^{-rs} E^{\mu_c}\left[(K-xX_s Z^{\mu_c}_{t+s,T})^+\Big|\mathcal{F}_{t+s}\right]\Big|\mathcal{F}_t\right] \\ &\qquad \ge E^{\mu_c}\left[\frac{1}{2}\left(K - xZ^{\mu_c}_{t+s,T} - \epsilon Z^{\mu_c}_{t+s,T} + \left|K - xZ^{\mu_c}_{t+s,T} - \epsilon Z^{\mu_c}_{t+s,T}\right|\right)\Big|\mathcal{F}_{t+s}\right]\Bigg\} \\ &= \inf\Bigg\{s \in [0, T-t] : \sup_{0\le s\le T-t} E\left[e^{-rs} E^{\mu_c}\left[(K-xX_s Z^{\mu_c}_{t+s,T})^+\Big|\mathcal{F}_{t+s}\right]\Big|\mathcal{F}_t\right] \\ &\qquad \ge E^{\mu_c}\left[\frac{1}{2}\left(K - xZ^{\mu_c}_{t+s,T} + \left|K - xZ^{\mu_c}_{t+s,T}\right|\right)\Big|\mathcal{F}_{t+s}\right]\Bigg\} \end{aligned}$$

$$\begin{aligned}
&= \inf\left\{ s\in[0,T-t] : \sup_{0\le s\le T-t} E\left[e^{-rs}E^{\mu_c}\left[(K-xX_sZ^{\mu_c}_{t+s,T})^+\Big|\mathcal{F}_{t+s}\right]\Big|\mathcal{F}_t\right]\right.\\
&\qquad\left.\ge E^{\mu_c}\left[(K-xX_sZ^{\mu_c}_{t+s,T})^+\right]\right\}\\
&= \inf\left\{ s\in[0,T-t] : \sup_{0\le s\le T-t} E\left[e^{-rs}E^{\mu_c}\left[(K-xX_sZ^{\mu_c}_{t+s,T})^+\Big|\mathcal{F}_{t+s}\right]\Big|\mathcal{F}_t\right]\right.\\
&\qquad\left.= E^{\mu_c}\left[(K-xX_sZ^{\mu_c}_{t+s,T})^+\right]\right\}\\
&= \inf\{s\in[0,T-t] : (t,x,j)\in D\}\\
&= 0.
\end{aligned}$$

Now to prove (26), we use (28). Thus, we have

$$\begin{aligned}
&\limsup_{\epsilon\searrow 0}\frac{V(t,x+\epsilon,j)-V(t,x,j)}{\epsilon}\\
&=\limsup_{\epsilon\searrow 0}\frac{1}{\epsilon}\left\{E\left[e^{-r\tau^+_\epsilon}G^{\mu_c}\left(t+\tau^+_\epsilon,x+\epsilon,j\right)\Big|\mathcal{F}_t\right]-\sup_{0\le\tau\le T-t}E\left[e^{-r\tau^+_\epsilon}G^{\mu_c}\left(t+\tau^+_\epsilon,x,j)\right)\Big|\mathcal{F}_t\right]\right\}\\
&\le\limsup_{\epsilon\searrow 0}\frac{1}{\epsilon}\left\{E\left[e^{-r\tau^+_\epsilon}G^{\mu_c}\left(t+\tau^+_\epsilon,x+\epsilon,j\right)\Big|\mathcal{F}_t\right]-E\left[e^{-r\tau^+_\epsilon}G^{\mu_c}\left(t+\tau^+_\epsilon,x,j)\right)\Big|\mathcal{F}_t\right]\right\}\\
&\le\limsup_{\epsilon\searrow 0}\frac{1}{\epsilon}\left\{G^{\mu_c}\left(t+\tau^+_\epsilon,x+\epsilon,j\right)-G^{\mu_c}\left(t+\tau^+_\epsilon,x,j)\right)\right\}\\
&=\frac{\partial G^{\mu_c}}{\partial x}(t,x,j), \qquad (30)
\end{aligned}$$

hence we conclude (26). □

It is well-known that every convex functions on the open interval I are differentiable almost everywhere, e.g. [3]. In the following Lemmas, we use the fact that both $V(t,x,j)$ and $G^{\mu_c}(t,x,j)$ are differentiable $\mathbb{P}$-almost surely for all x on $(0,\infty)$.

Lemma 4. The functions $\frac{\partial V}{\partial x}(t,x,j)$ and $\frac{\partial G^{\mu_c}}{\partial x}(t,x,j)$ are continuous on $(0,\infty)$ $\mathbb{P}$-almost surely for fixed $t\in[0,T]$ and $j\in\mathcal{M}$.

Proof. Let ϵ and $c\in(0,\infty)$ be arbitrary. Since $V(t,x,j)$ is differentiable for all $x\in(0,\infty)$ for fixed $t\in[0,T]$ and $j\in\mathcal{M}$, we know that there exists $\delta>0$ such that

$$\left|\frac{V(t,x,j)-V(t,c,j)}{c-x}-\frac{\partial V}{\partial x}(t,c,j)\right|<\frac{\epsilon}{2} \qquad (31)$$

whenever $0<|c-x|<\delta/2$. Moreover, by Mean-Value Theorem, there is an element

$y \in (x, c)$ such that

$$\frac{V(t,x,j) - V(t,c,j)}{c-x} = \frac{\partial V}{\partial x}(t,y,j)$$

and inequality (31) becomes

$$\left| \frac{\partial V}{\partial x}(t,y,j) - \frac{\partial V}{\partial x}(t,c,j) \right| < \frac{\epsilon}{2}. \tag{32}$$

Note that we have $0 < |y-c| < |x-c| < \delta/2$. For $t \in [0,T]$ and $j \in \mathcal{M}$ given and fixed, we know from Proposition 1 that $V(t,x,j)$ is convex for all $x \in (0,\infty)$, then $\dfrac{\partial V}{\partial x}(t,x,j)$ is monotonically increasing. Thus, if $0 < |x-y| < \delta/2$ we have

$$\left| \frac{\partial V}{\partial x}(t,x,j) - \frac{\partial V}{\partial x}(t,y,j) \right| < \frac{\epsilon}{2}. \tag{33}$$

Therefore, combining inequalities (32) and (33) we have

$$\left| \frac{\partial V}{\partial x}(t,x,j) - \frac{\partial V}{\partial x}(t,c,j) \right| < \epsilon,$$

whenever $0 < |x-c| \leq |x-y| + |y-c| < \delta$.

Furthermore, by Lemma 6 and the fact that $\dfrac{\partial G^{\mu_c}}{\partial x}$ is monotonically increasing and that $V = G^{\mu_c}$ in the stopping set D which is defined in (24) above, we have

$$\begin{aligned} \frac{V(t,x_2,j) - V(t,x_1,j)}{x_2 - x_1} &= \frac{\partial V}{\partial x}(t,x,j) \\ &\geq \frac{\partial G^{\mu_c}}{\partial x}(t,x,j) \\ &= \frac{G^{\mu_c}(t,x_2,j) - G^{\mu_c}(t,x_1,j)}{x_2 - x_1} \geq 0 \end{aligned}$$

for $x_1, x_2 \in (0,\infty)$. Therefore, continuity of $\frac{\partial G^{\mu_c}}{\partial x}(t,x,j)$ follows from the continuity of $\frac{\partial V}{\partial x}(t,x,j)$ on $(0,\infty)$. This completes our proof. □

Define the infinitesimal generator

$$\begin{aligned} \mathbb{L}f(s,x,\alpha_s) &= \left(\frac{\partial}{\partial t} + rx\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2(\alpha_s)x^2\frac{\partial^2}{\partial x^2} - r \right) f(s,x,j) \\ &\quad + \sum_{i=1}^{m} q_{ji} f(s,x,i) \end{aligned} \tag{34}$$

of the Markov process $(X_s)_{s\in[0,T]}$, where $Q = (q_{ij})_{i,j=1,2,\ldots,m}$ is the infinitesimal matrix generator of the Markov process $(\alpha_s)_{s\in[0,T]}$, for any sufficiently differentiable function f of $(s, x, j) \in [0, T] \times (0, \infty) \times \mathcal{M}$.

Lemma 5. For all $(t, x, j) \in [0, T] \times (0, \infty) \times \mathcal{M}$, we have

$$\mathbb{L}G^{\mu_c}(t, x, j) < 0 \tag{35}$$

when the contract drift μ_c satisfies $\mu_c < r$.

Proof. The payoff function in (14) can be rewritten as

$$G^{\mu_c}(t, x, j) = E^{\mu_c}[(K - X_T)^+ \big| X_t = x, \alpha_t = j]$$

for all $j \in \mathcal{M}$ and hence a martingale by tower property. By (7) and the Itô's formula we have

$$\begin{aligned} dG^{\mu_c}(t, x, j) &= \frac{\partial G^{\mu_c}}{\partial t}(t, x, j) + \mu_c x \frac{\partial G^{\mu_c}}{\partial x}(t, x, j) \\ &+ \frac{1}{2}\sigma^2(j)x^2 \frac{\partial^2 G^{\mu_c}}{\partial x^2}(t, x, j) + \sum_{i=1}^{m} q_{ji} G^{\mu_c}(t, x, i) \\ &+ \sigma(j)x \frac{\partial G^{\mu_c}}{\partial x} dW_t^{\mu_c}. \end{aligned}$$

Since $G^{\mu_c}(t, x, j)$ is a martingale, we find

$$\begin{aligned} &\frac{\partial G^{\mu_c}}{\partial t}(t, x, j) + \mu_c x \frac{\partial G^{\mu_c}}{\partial x}(t, x, j) + \frac{1}{2}\sigma^2(j)x^2 \frac{\partial^2 G^{\mu_c}}{\partial x^2}(t, x, j) \\ &+ \sum_{i=1}^{m} q_{ji} G^{\mu_c}(t, x, i) = 0. \end{aligned} \tag{36}$$

Substituting (36) to (34) we have

$$\mathbb{L}G^{\mu_c}(t, x, j) = (r - \mu_c)x \frac{\partial G^{\mu_c}}{\partial x}(t, x, j) - rG^{\mu_c}(t, x, j). \tag{37}$$

Since $G^{\mu_c}(t, x, j)$ is convex and decreasing with respect to $x \in (0, \infty)$, then we have $\frac{\partial G^{\mu_c}}{\partial x}(t, x, j) < 0$. This completes our proof. □

Lemma 6. For all $(t, y, j) \in C$, we have

$$\frac{\partial V}{\partial y}(t, y, j) > \frac{\partial G^{\mu_c}}{\partial y}(t, y, j). \tag{38}$$

Proof. Let (t, x_b, j) be a fixed point on the boundary function $b_D(t, j)$ so that $x_b = b_D(t, j)$. Let $x_b < y \le K$ so that $(t, y, j) \in C$. Since $x \mapsto V(t, x, j)$ is continuous on $(0, \infty)$

by Lemma 2 and differentiable $\mathbb{P}$-almost surely, by Mean Value Theorem, there exists at least one $c \in (x_b, y)$ such that

$$\frac{V(t,y,j) - V(t,x_b,j)}{y - x_b} = \frac{\partial V}{\partial x}(t,c,j).$$

Similarly, we have

$$\frac{G^{\mu_c}(t,y,j) - G^{\mu_c}(t,x_b,j)}{y - x_b} = \frac{\partial G^{\mu_c}}{\partial x}(t,c,j).$$

Since $V(t,y,j) > G^{\mu_c}(t,y,j)$ for all $(t,y,j) \in C$, we have

$$\frac{\partial V}{\partial x}(t,c,j) = \frac{V(t,y,j) - V(t,x_b,j)}{y - x_b} > \frac{G^{\mu_c}(t,y,j) - G^{\mu_c}(t,x_b,j)}{y - x_b} = \frac{\partial G^{\mu_c}}{\partial x}(t,c,j).$$

Since V and G^{μ_c} are continuous and convex, the above inequality holds for all $(t,c,j) \in C$. □

Lemma 7. For any (t,x,j) in the optimal stopping boundary $\partial C \subset D$, we have

$$\frac{\partial V}{\partial x}(t,x+,j) = \frac{\partial V}{\partial x}(t,x-,j). \tag{39}$$

Proof. For nay $\epsilon > 0$, consider the stopping time $\tau_\epsilon^+ = \tau_D(t,x+\epsilon,j)$ as in (27). Noting that $\tau_\epsilon^+ \to 0$ as $\epsilon \to 0$ as claimed in (29), by (30) we have

$$\frac{\partial G^{\mu_c}}{\partial x}(t,x,j) \geq \limsup_{\epsilon \searrow 0} \frac{V(t,x+\epsilon,j) - V(t,x,j)}{\epsilon}.$$

On the other hand, since $(t,x,j) \in \partial C \subset D$, we have

$$\begin{aligned}\liminf_{\epsilon \searrow 0} \frac{V(t,x+\epsilon,j) - V(t,x,j)}{\epsilon} &\geq \liminf_{\epsilon \searrow 0} \frac{G^{\mu_c}(t,x+\epsilon,j) - G^{\mu_c}(t,x,j)}{\epsilon} \\ &= \frac{\partial G^{\mu_c}}{\partial x}(t,x,j).\end{aligned}$$

Since $V = G^{\mu_c}$ on a closed set D, we have

$$\frac{\partial V}{\partial x}(t,x-,j) = \frac{\partial G^{\mu_c}}{\partial x}(t,x,j) = \frac{\partial V}{\partial x}(t,x-,j). \tag{40}$$

□

Lemma 8. We have

$$\{(t,x,i) \in [0,T] \times (0,\infty) \times \mathcal{M} \ : \ \mathbb{L}G^{\mu_c}(t,x,i) > 0\} \subset C$$

where $C = D^c$ is the continuation set.

Proof. Let $(t,x,i) \in [0,T] \times (0,\infty) \times \mathcal{M}$ be such that $\mathbb{L}G^{\mu_c}(t,x,i) > 0$. By Lemma 1 in [2] we have

$$\begin{aligned} e^{-rs}G^{\mu_c}(t+s, X_{t+s}, \alpha_{t+s}) &= G^{\mu_c}(t,x,i) \\ &+ \int_t^{t+s} e^{-ru}\mathbb{L}G^{\mu_c}(u, X_u, \alpha_u)du + M_s, \end{aligned} \tag{41}$$

where $M_s = \int_t^{t+s} e^{-ru}\sigma(\alpha_u)X_u \frac{\partial G^{\mu_c}}{\partial x}(u, X_u, \alpha_u)dW_u^{\mu_c}$ defines a continuous martingale for $s \in [0, T-t]$ with $t \in [0,T)$. By Lemma (1), Lemma (4) and equation (37), the infinitesimal generator $\mathbb{L}G^{\mu_c}(t,x,j)$ is continuous with respect to $(t,x) \in [0,T] \times (0,\infty)$. Thus there exists an open neighborhood $U \times V \subset [0,T) \times (0,\infty)$ of (t,x) such that $\mathbb{L}G^{\mu_c}(s,y,j) > 0$ for all $(s,y) \in U \times V$. Let

$$\tau_U = \inf\{\tau \ : \ (t+\tau, X_{t+\tau}) \in U \times V, (X_t, \alpha_t) = (x,i) \in V \times \mathcal{M}\}.$$

By Optional Sampling Theorem, the Relation (41) with $s = \tau_U$ shows that

$$\begin{aligned} E\left[e^{-r\tau_U}G^{\mu_c}(t+\tau_U, X_{t+\tau_U}, \alpha_{t+\tau_U})\Big|\mathcal{F}_t\right] &= G^{\mu_c}(t,x,i) \\ &+ E\left[\int_t^{t+\tau_U} e^{-ru}\mathbb{L}G^{\mu_c}(u, X_u, \alpha_u)du\Big|\mathcal{F}_t\right]. \end{aligned} \tag{42}$$

Since $\mathbb{L}G^{\mu_c}(u, X_u, \alpha_u) > 0$ for $u \in (t, t+\tau_U)$, the right hand side of equation (42) is strictly greater than $G^{\mu_c}(t,x,i)$, while from equation (15) we have

$$V(t,x,i) \geq E\left[e^{-r\tau_U}G^{\mu_c}(t+\tau_U, X_{t+\tau_U}, \alpha_{t+\tau_U}) \mid \mathcal{F}_t\right]$$

showing that $V(t,x,i) > G^{\mu_c}(t,x,i)$, which implies that $(t,x,i) \in C$. This completes our proof. □

Next, we define the boundary function $b_D(t,j)$ via the following:

For any stopping time $\tau \in [0, T-t]$, it can be verified from equations (37) and (42) that there is a continuous function $h : [0,T] \times \mathcal{M} \to \mathbb{R}$ such that the infinitesimal generator (37) satisfies

$$\mathbb{L}G^{\mu_c}(t, h(t,j), j) = 0. \tag{43}$$

Since $\mu_c > r$, we see that $\mathbb{L}G^{\mu_c}(t, h(t,j), j) > 0$ for $x > h(t,j)$ and $\mathbb{L}G^{\mu_c}(t, h(t,j), j) < 0$ for $x < h(t,j)$ when $t \in [0,T]$ and $j \in \mathcal{M}$ are given and fixed. In view of equation (42), this implies that for any stopping time $\tau \in [0, T-t]$, there is no point $(t,x) \in [0,T] \times (0,\infty)$ with $x > h(t,j)$ is a stopping point. From here, we define the optimal stopping boundary as follows:

$$b_D(t,j) := \sup\left\{x \in (0,\infty) : (t,x,j) \in D\right\}. \tag{44}$$

Now, we characterize the stopping set defined in (24) in terms of the boundary function $b_D(t,j)$.

Proposition 2. For any $(t,x,j) \in [0,T] \times (0,\infty) \times \mathcal{M}$ such that $(t,x,j) \in D$ we have

$$\{t\} \times (0,x] \times \{j\} \subset D \tag{45}$$

and

$$D = \{(t,x,j) \in [0,T] \times (0,\infty) \times \mathcal{M} : x \leq b_D(t,j)\}. \tag{46}$$

Proof. Let $(t,y,j) \in \{t\} \times (0,x] \times \{j\}$. Since $(t,x,j) \in D$ and $V(t,x,j) \geq G^{\mu_c}(t,x,j)$ for all $x \in (0,\infty)$, we have

$$\begin{aligned}\frac{V(t,x,j) - V(t,y,j)}{x-y} &= \frac{G^{\mu_c}(t,x,j) - V(t,y,j)}{x-y} \\ &\leq \frac{G^{\mu_c}(t,x,j) - G^{\mu_c}(t,y,j)}{x-y}.\end{aligned}$$

Taking the limit on both sides as $x - y \to 0$ and by Lemma (3), we have $(t,y,j) \in D$ and conclude (45). From the definition of the boundary function in (44), we have the following equivalence

$$(t,x,j) \in D \Longleftrightarrow \{t\} \times (0,x] \times \{j\} \subset D \Longleftrightarrow x \leq b_D(t,j). \quad \square$$

Lemma 9. For any $(x,j) \in (0,\infty) \times \mathcal{M}$, the mapping

$$t \mapsto F(t,x,j) = V(t,x,j) - G^{\mu_c}(t,x,j) \tag{47}$$

is nonincreasing in $t \in [0,T]$.

Proof. Let $s_1, s_2 \in [0, T-t]$ with $s_1 < s_2$ and consider the stopping time $\tau_{s_2} = \tau_D(s_2,x,j) \in [0, T-s_2]$. From definition of the function F in (23) and replacing τ_U with τ_{s_2} in (42), we have

$$\begin{aligned}F(s_2,x,j) &= V(s_2,x,j) - G^{\mu_c}(s_2,x,j) \\ &= E\left[e^{-r\tau_{s_2}} G^{\mu_c}(s_2+\tau_{s_2}, X_{s+\tau_{s_2}}, \alpha_{s+\tau_{s_2}}) \middle| \alpha_{s_2} = j\right] - G^{\mu_c}(s_2,x,j) \\ &= E\left[\int_{s_2}^{s_2+\tau_{s_2}} e^{-ru} \mathbb{L} G^{\mu_c}(u, X_u, \alpha_u) du \middle| \alpha_{s_2} = j\right] \\ &= E\left[\int_0^{\tau_{s_2}} e^{-ru} \mathbb{L} G^{\mu_c}(s_2+u, X_{s_2+u}, \alpha_{s_2+u}) du \middle| \alpha_0 = j\right]. \end{aligned} \tag{48}$$

Combining (48) with (49) below

$$F(s_1,x,j) = V(s_1,x,j) - G^{\mu_c}(s_1,x,j)$$

$$
\begin{aligned}
&\geq E\left[e^{-r\tau_{s_2}}G^{\mu_c}(s_1+\tau_{s_2},X_{s_1+\tau_{s_2}},\alpha_{s_1+\tau_{s_2}})\Big|\alpha_{s_1}=j\right]-G^{\mu_c}(s_1,x,j)\\
&= E\left[\int_{s_1}^{s_1+\tau_{s_2}} e^{-ru}\mathbb{L}G^{\mu_c}(u,X_u,\alpha_u)du\Big|\alpha_{s_1}=j\right]\\
&= E\left[\int_{0}^{\tau_{s_2}} e^{-ru}\mathbb{L}G^{\mu_c}(s_1+u,X_{s_1+u},\alpha_{s_1+u})du\Big|\alpha_{0}=j\right], \qquad (49)
\end{aligned}
$$

we have

$$
\begin{aligned}
&F(s_2,x,j)-F(s_1,x,j)\\
&\quad\leq E\left[\int_{0}^{\tau_{s_2}} e^{-ru}\mathbb{L}G^{\mu_c}(s_2+u,X_{s_2+u},\alpha_{s_2+u})du\Big|\alpha_{0}=j\right]\\
&\qquad\qquad -E\left[\int_{0}^{\tau_{s_2}} e^{-ru}\mathbb{L}G^{\mu_c}(s_1+u,X_{s_1+u},\alpha_{s_1+u})du\Big|\alpha_{0}=j\right]\\
&\quad\leq E\left[\int_{0}^{\tau_{s_2}} \{\mathbb{L}G^{\mu_c}(s_2+u,X_{s_2+u},\alpha_{s_2+u})du-\mathbb{L}G^{\mu_c}(s_1+u,X_{s_1+u},\alpha_{s_1+u})du\}\Big|\alpha_{0}=j\right].
\end{aligned}
$$

From Relation (37) with $r<\mu_c$, since $t\mapsto G^{\mu_c}(t,x,j)$ is nondecreasing on $[0,T]$, we say that $\mathbb{L}G^{\mu_c}(t,x,j)$ is nonincreasing in t, we find that the right hand side is nonpositive, thereby conclude that $F(t,x,j)$ is nonincreasing in $t\in[0,T]$. □

Proposition 3. The boundary function $b_D(t,j)$ is continuous in $t\in[0,T]$ for all $j\in\mathcal{M}$.

Proof.
Let $\alpha_t=j\in\mathcal{M}$ be fixed. We first show that the boundary function $b_D(t,j)$ is left-continuous. Suppose to the contrary that it is not left-continuous at time $t=t_0$. Consider the following cases:

Case 1. $b_D(t_0-,j)<b_D(t_0,j)$

Let $(t',x',j)\in(0,t_0)\times(b_D(t_0-,j),b_D(t_0,j))\times\mathcal{M}$ be a point in the continuation set C with t' close to t_0 and $t'\uparrow t_0$. We know that, by Lemma 4, $x\mapsto\dfrac{\partial V}{\partial x}$ and $x\mapsto\dfrac{\partial G^{\mu_c}}{\partial x}$ are both continuous. Since both $\dfrac{\partial V}{\partial x}$ and $\dfrac{\partial G^{\mu_c}}{\partial x}$ are bounded by $-\mathbb{P}(y\leq K)$ for $(t,y,j)\in D$, by Newton-Leibniz formula and Lemma 6 we have

$$
\begin{aligned}
0 &< \int_{x'}^{b_D(t_0,j)}\left[V_x(t',u,j)-G_x^{\mu_c}(t',u,j)\right]du\\
&= G^{\mu_c}(t',x',j)-V(t',x',j)
\end{aligned}
$$

as $t'\to t_0$. This implies that $V(t_0,x',j)<G^{\mu_c}(t_0,x',j)$ which contradicts the fact that $(t_0,x',j)\in D$ since $x'<b_D(t_0,j)$, i.e., $V(t_0,x',j)=G^{\mu_c}(t_0,x',j)$.

Case 2. $b_D(t_0-,j)>b_D(t_0,j)$

Let $(t^*, x^*, j) \in (0, t_0) \times (b_D(t_0, j), b_D(t_0-, j) \times \mathcal{M})$ be a point on the stopping set D with t^* close to t_0 and $t^* \uparrow t_0$. By (40), we have $V_x(t^*, x^*, j) = G^{\mu_c}(t^*, x^*, j)$ on D. Similarly, by Newton-Leibniz formula, we have

$$\begin{aligned} 0 &= \int_{b_D(t_0,j)}^{x} \left[V_x(t^*, v, j) - G_x^{\mu_c}(t^*, v, j)\right] dv \\ &= V(t^*, x^*, j) - G^{\mu_c}(t^*, x^*, j) \end{aligned}$$

as $t^* \to t_0$. This shows that $V(t_0, x^*, j) = G^{\mu_c}(t_0, x^*, j)$ which contradicts the fact that $(t_0, x^*, j) \in C$ since $x^* > b_D(t_0, j)$, i.e., $V(t^*, x^*, j) > G^{\mu_c}(t^*, x^*, j)$.

Therefore, in either case, b_D is left-continuous. To prove the right-continuity can be done similarly. □

Proposition 4. The boundary function $b_D(t, j)$ satisfies the Volterra type equation

$$G^{\mu_c}(t, b_D(t, j), j) = F(t, b_D(t, j), j) - \int_t^T J(t, b_D(t, j), u, b_D(u, \alpha_u), \alpha_u) du, \tag{50}$$

for $0 \le t \le T$, where

$$F(t, x, j) = E\left[(K - X_T)^+ \middle| \alpha_t = j, X_t = x\right] \tag{51}$$

and

$$J(t, x, u, b_D(u, \alpha_u), \alpha_u) = E\left[\mathbb{L}_X V(u, X_u, j) I(X_u < b_D(u, j)) \middle| \alpha_t = j, X_t = x\right], \tag{52}$$

for $0 \le t \le T$ and $x \in (0, \infty)$.

Proof. From Relation (16), we see that $V(t, x, j) \ge G^{\mu_c}(t, x, j)$ for all $(t, x, j) \in [0, T] \times (0, \infty) \times \mathcal{M}$ and recall the continuation set

$$C = D^c = \{(t, x, j) \in [0, T] \times (0, \infty) \times \mathcal{M} \,\big|\, V(t, x, j) > G^{\mu_c}(t, x, j)\}.$$

Noting that the stopping time $\tau_D = \tau_D(t, x, j)$ defined in (21) is optimal for (16), we have

$$V(t, x, i) = E\left[e^{-r\tau} G^{\mu_c}(t + \tau_D, X_{t+\tau_D}, j) \;\middle|\; \alpha_t = i, X_t = x\right].$$

It is well known from the theory of Markov processes that $V(t, x, i)$ is $C^{1,2}$ in the continuation set and it solves the Cauchy-Dirichlet free-boundary problem

$$\begin{cases} \mathbb{L}_X V(t, x, j) = 0, & (t, x, j) \in C \\ V(t, x, j) = G^{\mu_c}(t, x, j), & (t, x, j) \in \partial C, \end{cases} \tag{53}$$

where ∂C is the boundary of the open set C. By the local time space formula of [4], we have

$$\begin{aligned} V(T, X_T, j) &= E\left[(K - X_T)^+ \Big| \alpha_t = j, X_t = x\right] \\ &= V(t, x, j) + E\left[M_t^b \;\Big|\; \alpha_t = j, X_t = x\right] \\ &+ E\left[\int_t^T \mathbb{L}_X V(u, X_u, \alpha_u) I(X_u \neq b_D(u, \alpha_u)) du \Big| \alpha_t = j, X_t = x\right] \\ + \frac{1}{2} E\Bigg[\int_t^T &\left(\frac{\partial V}{\partial y}(u, X_u+, \alpha_u) - \frac{\partial V}{\partial y}(u, X_u-, \alpha_u)\right) I(X_u = b_D(u, \alpha_u) d\ell_u^b(X^x) \Big| \alpha_t = j, X_t = x\Bigg] \end{aligned} \tag{54}$$

where $M_t^b = \displaystyle\int_t^T \sigma(\alpha_u) X_u \frac{\partial V}{\partial x} dW_u$ is a continuous local martingale and $\ell^b = (\ell_u^b(X^x))_{t \leq u \leq T}$ is the local time of $X^x = (X_u)_{t \leq u \leq T}$ at the curve $u \mapsto b_D(u, j)$. Using that $\dfrac{\partial G^{\mu_c}}{\partial x}(t, x, j) = -\mathbb{P}(x \leq K) \leq \dfrac{\partial V}{\partial x}(t, y, j) \leq 0$ for all $t \in [0, T)$, it can easily be verified from Proposition 4.4, page 45 in [1] that $E\left[M_t^b\right] = 0$. By the smooth-fit property shown in Lemma 7, the last two terms in (54) above vanishes. Furthermore, by (53) above and the fact that $V = G^{\mu_c}$ in the closed set D, equation (54) becomes

$$\begin{aligned} E\left[(K - X_T)^+ \Big| \alpha_t = j, X_t = x\right] &= G^{\mu_c}(t, x, j) \\ &+ \int_t^T E\left[\mathbb{L}_X V(u, X_u, \alpha_u) I(X_u < b_D(u, \alpha_u)) \Big| \alpha_t = j, X_t = x\right] du. \end{aligned} \tag{55}$$

Substituting x with $b_D(t, j)$, we have

$$\begin{aligned} G^{\mu_c}(t, b_D(t, j), j) &= E\left[(K - X_T)^+ \Big| \alpha_t = j, X_t = b_D(t, j)\right] \\ - \int_t^T & E\left[\mathbb{L}_X V(u, X_u, \alpha_u) I(X_u < b_D(u, \alpha_u)) \Big| \alpha_t = j, X_t = b_D(t, j)\right] du \\ &= F(t, b_D(t, j), j) - \int_t^T J(t, b_D(t, j), u, b_D(u, \alpha_u), \alpha_u). \end{aligned}$$

□

4. Conclusion and Recommendations

This paper extends the results for British put option that was introduced by G. Peskir and F. Samee (2011) by considering stochastic volatility, particularly in a regime-switching. We have shown that the boundary function satisfies the Volterra equation, instead of deriving the closed form expression for the arbitrage-free price for the British put option. For further studies, a similar extension may be done for the British call option. In addition, one may provide a practical implication of this study.

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