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Nuclearity of a class of vector-valued sequence spaces

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Abstract. In this note, we deal with a perfect sequence space λ and a convex bornological space E to introduce and study the space $\lambda(E)$ of all totally λ-summable sequences from E. We prove that $\lambda(E)$ is complete if and only if λ and E are complete, nuclear if and only if λ and E are nuclear, and we make use of a result of Ronald C. Rosier [10] to give a similar characterization of the nuclearity of the space $\lambda\{E\}$ of all absolutely $\lambda-$summable sequences in a locally convex E.

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Introduction

In connection with the nuclearity of a locally convex space E, A. Pietsch in [9] introduced the spaces $\ell_p(E)$ and $\ell_p\{E\}$ respectively of weakly ℓ_p-summable and absolutely ℓ_p-summable sequences in E. In [8], he used these spaces to study the absolutely p-summing operators. Later, he introduced and studied also the space $\lambda\{E\}$ of λ-summable sequences in E, for a perfect sequence space λ in the sense of Köthe endowed with its normal topology. Many other authors were interested in the study of these spaces. Ronald C. Rosier in [10] considered a general polar topology on $\lambda\{E\}$ and got a precise description of the topological dual and its equicontinuous subsets. M. Florencio and P. J. Paúl [3], considering general polar topologies, obtained many interesting results such as barreledness conditions. In [1] and [2], they studied the space $\lambda(E)$ of weakly $\lambda-$summables sequences in E and represented this space as the completion of the injective tensor product $\lambda\widetilde{\otimes}_\epsilon E$. In [6] and [7], L. Oubbi and M. A. Ould Sidaty reconsidered the space $\lambda(E)$ and obtained some of its properties. They mainly described the continuous dual space of $\lambda(E)$. While in [11] and [13], characterizations of the reflexivity of $\lambda(E)$ in terms of that of λ and E and the AK-property are given. A characterization of the nuclearity of of the space of weakly $\lambda-$summable sequences is given in [12].

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In this note, we are concerned with the nuclearity of the convex bornological space $\lambda(E)$ of all totally $\lambda-$summable sequences in E, in the sense of [3], where E is a convex bornological space.
In sections 1 and 2, we endow this space with a structure of b-space, and study some of its properties.
The section 3 is devoted to the nuclearity of $\lambda(E)$. We prove mainly that $\lambda(E)$ possesses this property if and only if both of λ and E have.
In Section 4, we provide an application of the results of Section 3 on the nuclearity of the space $\lambda\{E\}$ of absolutely $\lambda-$summable sequences in a locally convex space E.

1. Preliminaries

For a linear space E, we mean by a convex bornology on E, a collection of subsets of E covering E, hereditary for the inclusion, and closed for the finite unions, the addition, the scalar multiplication and the formation of absolutely convex hulls. We say then that E is a convex bornological space or simply a b-space. The elements of the bornology of E are called bounded sets of E.

A collection $\mathbb{B}$ of bounded sets of E is a basis for its bornology if every bounded set in E is contained in an element of $\mathbb{B}$. In the sequel, we assume that the members of $\mathbb{B}$ are absolutely convex.
A b-space E is said to be Hausdorff if the only bounded linear subspace of E is $\{0\}$.
We say that a sequence $\{x_n\}_{n=1}^{\infty} \subset E$ converges to $x \in E$, or that x is a limit of $\{x_n\}_{n=1}^{\infty}$ in E if there exists an element $B \in \mathbb{B}$ such that $\{x_n - x\}_{n=1}^{\infty}$ is contained and convergent to 0 in the normed space $(E_B, \|\cdot\|_B)$, where E_B is the subspace of E generated by B and $\|\cdot\|_B$ is the gauge of B.

A subset of a b-space E will be said to be closed if it contains the limits of all its sequences.
A Banach disk in a b-space E is an element $B \in \mathbb{B}$ for which the normed space E_B is complete. E is said to be b-complete or simply complete if every bounded set in E is contained in a Banach disk in E.

A linear mapping between two b-spaces E and F is said to be bounded if it transforms bounded sets of E to bounded sets of F. A bounded linear mapping transforms convergent sequences to convergent ones. A bornological isomorphism is a bounded linear bijection whose inverse is also bounded.

The Köthe dual of a sequence space λ is defined as

$$\lambda^{\times} = \left\{ (\beta_n) \subset \mathbb{C} : \sum_{n=1}^{\infty} |\alpha_n \beta_n| \text{ converges for all } (\alpha_n) \in \lambda \right\}.$$

We see that $\lambda \subset \lambda^{\times\times} =: (\lambda^{\times})^{\times}$; we say that λ is perfect if the equality holds.
The normal cover of a subset S of λ is the subset of λ formed by the sequences of the

form $(\varepsilon_n\alpha_n)_n$ where $(\alpha_n)_n \in S$ and $(\varepsilon_n)_n \subset \mathbb{C}$ with $|\varepsilon_n| \leq 1$, for all n. We see that S is contained in its normal cover. S is said to be normal or solid if it coincides with its normal cover.
For the general theory of locally convex spaces and Köthe sequence spaces, we refer the reader to [5].
Throughout this paper, λ will be a perfect (and then a normal) sequence space endowed with a normal bornology, that is a convex bornology having a basis $\mathbb{S}$ of solid sets, and for which the standard coordinate projections from λ to $\mathbb{C}$ are bounded.

Following the terminology of [3], a sequence $(x_n)_n \subset E$ is said to be totally $\lambda-$summable in E if there exists an absolutely convex element $B \in \mathbb{B}$ such that $(x_n)_n \subset E_B$ and $(\|x_n\|_B)_n \in \lambda$. In other words, $(x_n)_n = (\alpha_n b_n)_n$, with $(\alpha_n)_n \in \lambda$ and $\{b_n\}_{n=1}^{\infty} \subset B$.

Starting from this definition, we introduce the vector valued sequence space

$$\lambda(E) = \Big\{(x_n)_n \subset E : \exists B \in \mathbb{B}, (x_n)_n \subset E_B \text{ and } (\|x_n\|)_n \in \lambda\Big\}.$$

Due to the properties of $\mathbb{B}$, the triangle inequality of the norms $\|\cdot\|_B$ and the fact that λ is normal, we see that $\lambda(E)$ is a linear space. For $S \in \mathbb{S}$ and $B \in \mathbb{B}$, we define

$$S(B) = \Big\{(x_n)_n \subset E_B, (\|x_n\|_B)_n \in S\Big\}.$$

2. Properties of $\lambda(E)$

In the sequel, the b-spaces E equipped with the convex bornology with basis $\mathbb{B}$ and λ with the normal bornology with basis $\mathbb{S}$, will be supposed to be Hausdorff spaces.
Starting from this setting, one can define, in a natural way, a convex bornology on $\lambda(E)$ with basis $\mathbb{S}(\mathbb{B})$ by setting

$$\mathbb{S}(\mathbb{B}) = \Big\{H \subset \lambda(E) : \exists S \in \mathbb{S}, B \in \mathbb{B} \text{ such that } H = S(B)\Big\}.$$

In view of the hypothesis made on $\mathbb{S}$ and $\mathbb{B}$, $\mathbb{S}(\mathbb{B})$ is indeed a basis for a convex bornology on $\lambda(E)$ for which $\lambda(E)$ is a Hausdorff space.

Lemma 1. *For a fixed $k \in \mathbb{N}$, denote by π_k the projection from $\lambda(E)$ on E defined by*

$$\pi_k(x) = x_k, \text{for all } x = (x_n) \in \lambda(E).$$

Then, π_k is a bounded linear map.

Proof. Let $B \in \mathbb{B}$ and $S \in \mathbb{S}$ and fix $k \in \mathbb{N}$. Since the bornology of λ is normal, the set $\{\alpha_k : (\alpha_n)_n \in S\}$ is bounded in $\mathbb{C}$, and then so is $\{\|x_k\| : (x_n)_n \in S(B)\}$. This means that $\{x_k : (x_n)_n \in S(B)\}$ is bounded in E_B. Thus, π_k is bounded. ■

Proposition 1. *The spaces λ and E can be identified with closed subspaces of $\lambda(E)$.*

Proof. Let $I : E \longrightarrow \lambda(E), t \longrightarrow te_1$, where t is at the first component. It is clear that I is linear and one to one. Let $B \in \mathbb{B}$, and $S \in \mathbb{S}$ such that $e_1 \in S$, then $I(B) \subset S(B)$ and I is bounded. Inversely, $I^{-1} : I(E) = Ee_1 \to E$ is the restriction of π_1 to the subspace $I(E)$, and then it is bounded by Lemma 1. It remains to show that $I(E)$ is closed in $\lambda(E)$. We have $I(E) = \bigcap_{k\neq 1} \pi_k^{-1}(\{0\})$. Since E is supposed to be a Hausdorff space, then $\{0\}$ is closed and so is $I(E)$.
Now, fix $0 \neq x_0 \in E$ and let $g : \lambda \longrightarrow \lambda(E), \alpha = (\alpha_n)_n \longrightarrow (\alpha_n x_0)_n = \alpha x_0$. It is clear that g is linear and one to one. Let $S \in \mathbb{S}$, and $B \in \mathbb{B}$ with $x_0 \in B$. Then, $g(S) \subset S(B)$; so g is bounded. Inversely, if $S \in \mathbb{S}$ and $B \in \mathbb{B}$, then $g^{-1}(S(B) \cap \lambda x_0) = \dfrac{1}{\|x_0\|_B} S$, and then $g^{-1} : g(E) = \lambda x_0 \to \lambda$ is bounded. It remains to show that $g(\lambda)$ is closed in $\lambda(E)$. Let $\{\alpha^{(k)} x_0 = (\alpha_n^{(k)} x_0)_n\}_{k=1}^{\infty}$ be a sequence in λx_0 which converges to $x = (x_n)_n \in \lambda(E)$. By Lemma 1, $\{\alpha_n^{(k)} x_0\}_{k=1}^{\infty}$ converges to x_n in E, for every n. As, the subspace $\mathbb{C}x_0$ of E is closed in E, x_n must belong to $\mathbb{C}x_0$. Then, there is $\alpha = (\alpha_n)$ such that $x = (x_n)_n = \alpha x_0$. It is easy to see that $\alpha \in \lambda$. We conclude that λx_0 is closed in $\lambda(E)$. ∎

Proposition 2. $\lambda(E)$ *is complete if and only if* λ *and* E *are complete.*

Proof. If $\lambda(E)$ is complete, then so are λ and E by Proposition 1. Inversely, suppose that λ and E are complete. We only show that if B and S are Banach disks in E and λ respectively, then $S(B)$ is a Banach disk in $\lambda(E)$. To simplify the notations, we set $F = \lambda(E)$, $H = S(B)$ and π the gauge of H.
Let $\{(x^i)_i\}_{i=1}^{\infty}$ be a Cauchy sequence in (F_H, π). We have

$$\begin{aligned} \left| \left\|(\|x_n^i\|_B)_n\right\|_S - \left\|(\|x_n^j\|_B)_n\right\|_S \right| \leq \left| \left\|(\|x_n^i\|_B)_n - (\|x_n^j\|_B)_n\right\|_S \right| &\leq \left\|(\|x_n^i\|_B - \|x_n^j\|_B)_n\right\|_S \\ &\leq \left\|(\|x_n^i - x_n^j\|_B)_n\right\|_S \\ &= \pi((x^i - x^j)_n). \end{aligned}$$

This means that $\{(\|x^i\|_B)_i\}_{i=1}^{\infty}$ is a Cauchy sequence in the complete space $(\lambda_S, \|\cdot\|_S)$; let $\alpha = (\alpha_n)_n$ be its limit in λ_S. Fix $n \in \mathbb{N}$. Due to the boundedness of the projections, $\{\|x_n^i\|_B\}_{i=1}^{\infty}$ converges to α_n and $\{x_n^i\}_{i=1}^{\infty}$ is a Cauchy sequence in the complete space E_B; denote by x_n its limit. Thus, $\|x_n\|_B = \alpha_n$, and $x = (x_n)_n \in \lambda(E)$. It remains to prove the convergence of $\{(x^i)_i\}_{i=1}^{\infty}$ to x. This derives from the fact that $\{(\|x^i - x\|_B)_i\}_{i=1}^{\infty}$ is a Cauchy sequence in $(\lambda_S, \|\cdot\|_S)$ and its limit is nothing but the zero sequence in λ. ∎

3. Nuclearity of $\lambda(E)$

A linear mapping $f : E \to F$ between complete normed spaces is said to be nuclear if there exist $(\varepsilon_n)_n \in \ell_1$, a bounded sequence $(a_n)_n$ in the continuous dual E' of E and a bounded sequence $(y_n)_n \subset F$ such that

$$f(x) = \sum_{n=1}^{\infty} \varepsilon_n a_n(x) y_n, \text{ for all } x \in E.$$

A b-space E is said to be nuclear (a Schwartz space) if for every Banach disk A in E there is a Banach disk $B \supset A$ in E such that the inclusion mapping $E_A \to E_B$ is nuclear (compact).

Proposition 3. *The tensor product $\lambda \otimes E$ is identifiable with a subspace of $\lambda(E)$.*

Proof. We see that for all $\alpha = (\alpha_n)_n \in \lambda$ and $x \in E$, $(\alpha_n x)_n \in \lambda(E)$. Define the bilinear mapping $\varphi : \lambda \times E \to \lambda(E)$, such that $\varphi(\alpha, x) = (\alpha_n x)_n$. There exists a linear mapping $\ell : \lambda \otimes E \to \lambda(E)$, with $\ell(\alpha \otimes x) = (\alpha_n x)_n$. Let us show that ℓ is one to one. Suppose that $z \in \lambda \otimes E$ such that $\ell(z) = 0$. We can write $z = \sum_{i=1}^{k} (\alpha_n^i)_n \otimes x_i$, for which $\{(\alpha_n^i)_n\}_{i=1}^{k}$ and $\{x_i\}_{i=1}^{k}$ are linearly independent. But,

$$\ell(z) = \sum_{i=1}^{k} \ell(\alpha^i \otimes x_i) = \sum_{i=1}^{k} (\alpha_n^i x_i)_n = \left(\sum_{i=1}^{k} \alpha_n^i x_i \right)_n .$$

Since $\ell(z) = 0$ then $\left(\sum_{i=1}^{k} \alpha_n^i x_i \right)_n = 0$ and $\sum_{i=1}^{k} \alpha_n^i x_i = 0$, for every n. But, as $\{x_i\}_{i=1}^{k}$ is linearly independent, $\alpha_n^i = 0$, for all $1 \leq i \leq k$ and $n \in \mathbb{N}$. Thus, $z = \sum_{i=1}^{k} (\alpha_n^i)_n \otimes x_i = 0$, and ℓ is one to one. ■

Lemma 2. *Let S and B be Banach disks in λ and E respectively, $N(x) = \left\| (\|x_n\|_B)_n \right\|_S$ for all $x = (x_n)_n \in \lambda_S(E_B)$ and $N_1(z) = N(\ell(z))$ for all $z \in \lambda_S \otimes E_B$. Then,*
1. N_1 is a cross-norm on $\lambda_S \otimes E_B$, that is $N(\alpha \otimes x) = \|\alpha\|_S \|x\|_B$, for every $\alpha \in \lambda_S$ and $x \in E_B$.
2. The mapping $\ell : \lambda_S \otimes E_B \to \lambda_S(E_B)$ is isometric and can be extended to a unique linear mapping $\hat{\ell} : \lambda_S \hat{\otimes}_{N_1} E_B \to \lambda_S(E_B)$, where $\lambda_S \hat{\otimes}_{N_1} E_B$ the completion of the normed space $(\lambda_S \otimes_{N_1} E_B, N_1)$.

Proof. Since N is a solid norm and ℓ is a one to one linear mapping, N_1 is a norm. It is clear that $N_1(\alpha \otimes x) = \|\alpha\|_S \|x\|_B$, and *1.* holds. By the definition of N_1, we see that ℓ is isometric from $\lambda_S \otimes E_B$ to the complete space $\lambda_S(E_B)$, and then it has an extension to the completion $\lambda_S \hat{\otimes}_{N_1} E_B$ of $\lambda_S \otimes_{N_1} E_B$. This gives the second item. ■

We will make use of the following result to represent $\lambda(E)$ as a bornological tensor product.

Proposition 4. *[4, Ch VIII, Prop. 4]*
1. There is a convex bornology b on $\lambda \otimes E$ (the finest one) making bounded the inclusion mappings $\lambda_S \otimes_{N_1} E_B \to \lambda(E)$. Moreover, $\lambda \otimes_b E = \varinjlim \lambda_S \otimes_{N_1} E_B$.
2. b is located between the projective bornology π and the injective bornology ε.
3. If λ or E is nuclear, then $\pi = b = \varepsilon$.
4. If λ and E are nuclear, the bornological completion $\lambda \tilde{\otimes}_b E$ of $\lambda \otimes_b E$ is the inductive limit of the Banach spaces $\lambda_S \hat{\otimes}_{N_1} E_B$.

Now, we prove

Theorem 1. *If λ and E are nuclear, the equality $\lambda(E) = \lambda\tilde{\otimes}_b E$ holds algebraically and bornologically.*

Proof. Consider the linear mapping $\ell : \lambda \otimes_b E \to \lambda(E)$ defined in the proof of Proposition 3.
According to the definition of the norms N and N_1, we see that ℓ is bounded, and since $\lambda(E)$ is complete, ℓ can be extended to a bounded linear mapping $\tilde{\ell}$ from the bornological completion $\lambda\tilde{\otimes}_b E$ of $\lambda \otimes_b E$ to $\lambda(E)$.
We will prove that $\tilde{\ell}$ makes $\lambda\tilde{\otimes}_b E$ and $\lambda(E)$ bornologically isomorphic.
Let $z \in \lambda\tilde{\otimes}_b E$ be such that $\tilde{\ell}(z) = 0$. By [4, Ch VIII, Prop. 2], a sequence $\{z_k\}_{k=1}^{\infty}$ of elements of $\lambda \otimes_b E$ converges to z. Then $\{z_k - z\}_{k=1}^{\infty}$ is a null sequence in some subspace $\lambda_S\tilde{\otimes}_b E_B$. Thus,

$$\hat{\ell}(z) = \hat{\ell}(\lim_k \iota(z_k)) = \lim_k(\hat{\ell} \circ \iota)(z_k) = \lim_k \ell(z_k) = \lim_k(\tilde{\ell} \circ \iota)(z_k) = \tilde{\ell}(\lim_k z_k) = \tilde{\ell}(z) = 0.$$

Here ι is the canonical injection from $\lambda \otimes_b E$ to its completion $\lambda\tilde{\otimes}_b E$.
By Lemma 2, $\hat{\ell}$ is isometric and then it is one to one, then $z = 0$, and $\tilde{\ell}$ is one to one.
We will prove that $\tilde{\ell}$ is onto as follows. Let $A \in \mathbb{B}$ be a Banach disk; since E is nuclear we can select a Banach disk $B \in \mathbb{B}$ containing A such that the inclusion $E_A \to E_B$ is nuclear. There are $(\varepsilon_k)_k \in \ell_1$, a bounded sequence $(a_k)_k$ in the continuous dual $(E_A)'$ of E_A and a bounded sequence $(y_k)_k \subset E_B$ such that

$$x = \sum_{k=1}^{\infty} \varepsilon_k a_k(x) y_k, \text{ for all } x \in E_A. \tag{1}$$

Let $x = (x_n)_n \in \lambda_S(E_A)$, and $\alpha^k = (\alpha_n^k)_n =: (a_k(x_n))_n$. We have

$$|\alpha_n^k| = |a_k(x_n)| \leq \|a_k\| \|x_n\|_A \leq \left(\sup_p \|a_p\|\right) \|x_n\|_A, \text{ for all } k, n. \tag{2}$$

The sequence $(a_k)_k$ being bounded in $(E_A)'$, $\sup_p \|a_p\|$ is finite, $\alpha^k = (\alpha_n^k)_n \in \lambda_S(E_A)$, for all k, and, by (2), $\|\alpha^k\|_S \leq (\sup_p \|a_p\|)\|(\|x_n\|_A)_n\|_S$ and then $\sup_k \|\alpha^k\|_S$ is finite. Then,

$$\sum_{k=1}^{r} N_1(\varepsilon_k \alpha^k \otimes y_k) = \sum_{k=1}^{r} |\varepsilon_k| \|\alpha^k\|_S \|y_k\|_B \leq (\sup_p \|a_p\|)(\sup_p \|y_p\|) N(x) \sum_{k=1}^{r} \varepsilon_k. \tag{3}$$

As, $\lambda_S(E_B)$ is a complete normed spaces, the series $\sum_{k=1}^{\infty} \varepsilon_k \alpha^k \otimes y_k$ converges in $\lambda_S(E_B)$ to a limit $g(x)$. Moreover,

$$\tilde{\ell}(g(x)) = x. \tag{4}$$

Indeed, if $z = (z_n)_n \in \lambda_S(E_B)$ is such that $z = \tilde{\ell}(g(x))$, then

$$z = (z_n)_n = \tilde{\ell}\Big(\sum_{k=1}^{\infty} \varepsilon_k (a_k(x_n))_n \otimes y_k\Big) \quad = \quad \sum_{k=1}^{\infty} \varepsilon_k \tilde{\ell}((a_k(x_n))_n \otimes y_k)$$

$$= \sum_{k=1}^{\infty} \varepsilon_k \ell((a_k(x_n))_n \otimes y_k)$$
$$= \sum_{k=1}^{\infty} \varepsilon_k (a_k(x_n) y_k)_n.$$

But the projections are bounded by Lemma 1, then

$$z_n = \sum_{k=1}^{\infty} \varepsilon_k a_k(x_n) y_k, \text{ for all } n.$$

By (1), $z_n = x_n$, for all n, and $\tilde{\ell}(g(x)) = x$. This means that $\tilde{\ell}$ is onto. In the other hand, if K is bounded in $\lambda(E)$, then K is contained and bounded in some $\lambda_S(E_B)$, and $\tilde{\ell}(g(K)) = K$, from what, we conclude that the inverse of $\tilde{\ell}$ is bounded. ■

We are now ready to prove the main result of this section.

Theorem 2. *Let E be a complete b-space and λ be a normal sequence space. Then $\lambda(E)$ is nuclear if and only if λ and E are nuclear.*

Proof. If $\lambda(E)$ is nuclear then, by Proposition 1, E and λ are closed subspaces of $\lambda(E)$ and then they are nuclear also.
Inversely, suppose that E and λ are nuclear. By Proposition 4, $\lambda\tilde{\otimes}_b E$ is nuclear. So by Theorem 1, $\lambda(E)$ is nuclear. ■

Theorem 3. *Let E be a complete b-space and λ be a normal sequence space.*

(i) If λ is nuclear then, $\lambda(E)$ is a Schwartz space if and only if E is a Schwartz space.

(ii) If E is nuclear then, $\lambda(E)$ is a Schwartz space if and only if λ is a Schwartz space.

Proof. Suppose that E is nuclear. If $\lambda(E)$ is a Schwartz space, then λ, being a closed subspace of $\lambda(E)$ by Proposition 1, is a Schwartz space. Inversely, suppose that E is nuclear and λ is a Schwartz space. Let $A \in \mathbb{B}$ and $S \in \mathbb{S}$ be a Banach disks in E and λ respectively. Since E is nuclear we can select a Banach disk $B \in \mathbb{B}$ containing A such that the inclusion $E_A \to E_B$ is nuclear. So, there are $(\varepsilon_k)_k \in \ell_1$, a bounded sequence $(a_k)_k$ in the continuous dual $(E_A)'$ of E_A and a bounded sequence $(y_k)_k \subset E_B$ such that

$$x = \sum_{k=1}^{\infty} \varepsilon_k a_k(x) y_k, \text{ for all } x \in E_A. \tag{5}$$

Since λ is a Schwartz space, there is a Banach disk T in λ such that the injection $\lambda_S \to \lambda_T$ is compact. We will show that the injection $\lambda_S(E_A) \to \lambda_T(E_B)$ is compact. Let

$$\{x^i = (x_n^i)_n\}_{i=1}^{\infty} \tag{6}$$

be a sequence in $S(A)$. By (5), we have

$$x_n^i = \sum_{k=1}^{\infty} \varepsilon_k a_k(x_n^i) y_k, \text{ for all } n, i. \tag{7}$$

The sequence $(a_k)_k$ being bounded in $(E_A)'$, there is a constant $c > 0$ such that

$$|a_k(x_n^i)| \leq c\|x_n^i\|_A \text{ for all } i, k, n.$$

This means that $\{(a_k(x_n^i))_n\}_{i=1}^{\infty} \subset \lambda_S$ and that

$$\{(a_k(x_n^i))_n\}_{i=1}^{\infty} \subset cS. \tag{8}$$

A subsequence $\{(a_k(x_n^j))_n\}_{j=1}^{\infty}$ of $\{(a_k(x_n^i))_n\}_{i=1}^{\infty}$ should converge in λ_T to $\alpha^k = (\alpha_n^k)_n$. In the other hand, the equation (8) shows that the sequence $\{(a_k(x_n^j))_n\}_{k,j=1}^{\infty}$ is bounded in λ_S. For every $n \in \mathbb{N}$, there $c_n > 0$ such that for all j, k

$$|a_k(x_n^j)| \leq c_n \text{ and then } |\alpha_n^k| \leq c_n. \tag{9}$$

For every $n \in \mathbb{N}$, since $\{\alpha_n^k y_k\}_{k=1}^{\infty}$ is bounded in the complete normed space E_B, the series $\sum_k \varepsilon_k \alpha_n^k y_k$ converges to a limit $x_n \in E_B$. Let $x = (x_n)_n$. Since $\{(\alpha_n^k)_n\}_{k=1}^{\infty}$ is bounded in λ_S and $\{y_k\}_{k=1}^{\infty}$ is bounded in E_B, the sequence $\{(\alpha_n^k y_k)_n\}_{k=1}^{\infty}$ is bounded in $\lambda_S(E_B)$ and then in $\lambda_T(E_B)$. Thus, the series $\sum_k \varepsilon_k (\alpha_n^k y_k)_n$ converges in $\lambda_T(E_B)$ to $z = (z_n)_n$. Since the projections are bounded by Lemma 1, one has $z_n = \sum_k \varepsilon_k \alpha_n^k y_k$ for all n, and then $x = z \in \lambda_T(E_B)$.

It remains to prove that $\{x^j\}_{i=1}^{\infty}$ converges in $(\lambda_T(E_B), N)$ to x. We have,

$$x^j - x = \sum_k \varepsilon_k (a_n(x_n^j) - \alpha_n^j)_n y_k$$

and

$$N(x^j - x) \leq \sum_k |\varepsilon_k| \|(a_n(x_n^j) - \alpha_n^j)_n\|_S \|y_k\|_B \tag{10}$$

For j, k, let

$$\beta_k^j = \|a_k(x_n^j) - \alpha_n^k\|_T \text{ and } \gamma_k = \|y_k\|_B. \tag{11}$$

Then, $(\gamma_k)_k \in c_0$ and $\{(\varepsilon_k \beta_k^j)_k\}_{j=1}^{\infty}$ is a sequence in ℓ_1 which is $\sigma(\ell_1, c_0)-$bounded, then it has a convergent subsequence say,

$$\{(\varepsilon_k \beta_k^r)_k\}_{r=1}^{\infty}. \tag{12}$$

As, $\lim_{r\to\infty} \varepsilon_k \beta_k^r = 0$, for all k, then the sequence in (12) converges to 0 in $(\ell_1, \sigma(\ell_1, c_0))$. By (11) and (10), we have

$$N(x^r - x) \leq \sum_k |\varepsilon_k \beta_k^r| \gamma_k, \text{ for all } r \in \mathbb{N}.$$

Thus, $\{x^r - x\}_{r=1}^{\infty}$ converges to 0 in $\lambda_T(E_B)$, and (6) has a convergent subsequence. This finishes the proof of *(i)*. The proof of *(ii)* is similar by interchanging the roles of E and λ in the proof. ■

4. Nuclearity of $\lambda\{E\}$

Notice that a locally convex space is said to be nuclear (resp. a Schwartz space) if the convex bornology of equicontinuous subsets of its topological dual is nuclear (resp. of Schwartz).
Let λ be a perfect sequence space and E a locally convex space whose topology is defined by a family $\mathcal{M}$ of absolutely convex equicontinuous subsets of its topological dual E'. Define

$$\lambda\{E\} = \{(x_n)_n \subset E : (P_M(x_n))_n \in \lambda\}, \text{ where } P_M(x_n) = \sup_{a\in M} |a(x_n)|.$$

If a topology on λ is defined by family $\mathcal{S}$ of normal, absolutely convex and $\sigma(\lambda^\times, \lambda)-$bounded subsets of $\lambda^\times$, then a locally convex topology can be defined on $\lambda\{E\}$ by the family of semi-norms $(\pi_{S,M})_{S\in\mathcal{S},M\in\mathcal{M}}$, such that, if $x = (x_n)_n \in \lambda\{E\}$ then

$$\pi_{S,M}((x_n)_n) = P_S((P_M(x_n))) = \sup\{\sum_{n=1}^{\infty} |\alpha_n P_M(x_n)| : (\alpha_n)_n \in S\}.$$

For the topology so defined, Ronald C. Rosier in [10] proved that the dual space $(\lambda\{E\})^*$ of $\lambda\{E\}$ is $\lambda^\times(E')$ and that a subset of $(\lambda\{E\})^*$ is equicontinuous if and only if it is contained in some $S(M)$ for $S \in \mathcal{S}$ and $M \in \mathcal{M}$.
Starting from this setting, Theorem 2 gives

Theorem 4. *$\lambda\{E\}$ is nuclear if and only if λ and E are nuclear.*

Also, Theorem 3 gives

Theorem 5. *If E (resp. λ) is nuclear, then $\lambda\{E\}$ is a Schwartz space if and only if λ (resp. E) is a Schwartz space.*

5. Conclusion

In this paper we have characterized the bornological structure, the completeness and the nuclearity of $\lambda(E)$ in terms of that of λ and E. An application to the nuclearity of the locally convex space $\lambda\{E\}$ is given.

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References

[1] M. Florencio and Pedro J. Paúl. Una representación de cietros ϵ-productos tensoriales. In *Actas de las Jornadas Matematicas Hispano Lusas, Murcia.*, pages 191–203, Murcia, Spain, 1985. Universidad de Murcia.

[2] M. Florencio and Pedro J. Paúl. La propiedad ak en ciertos espacios de suecsiones vectoriales. In Dep. Mat. Univ. Extremadura, editor, *Proc. Eleventh Spanish-Portuguese Conference on Mathematics*, pages 197–203, 1986.

[3] M. Florencio and Pedro J. Paúl. Barrelledness conditions on vector valued sequence spaces. *Arch. Math.*, 48:153–164, 1987.

[4] H. Henri-Hogbé-N'Lend. *Théorie de Bornologies et Applications.* Springer-Verlag, Lecture Notes, 213, Berlin and Heidelberg, 1971.

[5] G. Köthe. *Topological Vector Spaces I and II.* Springer-Verlag, Berlin, Heidelberg, New York, 1979.

[6] L. Oubbi and M. A. Ould Sidaty. Dual space of certain locally convex sequence spaces. *Revista de la Real Academia de Ciencias de Zargoza*, 59:79–88, 2004.

[7] L. Oubbi and M. A. Ould Sidaty. Reflexivity of spaces of weakly summable sequences. *Rev. R. Acad. Cien. Serie A. Mat.*, 101(1):51–62, 2007.

[8] E. Pietsch. *Verallgemeinerte Vollkommene Folgenräume.* Akademie-Verlag, Berlin, Heidelberg, New York, 1962.

[9] E. Pietsch. *Nuclear locally convex spaces.* Springer-Verlag, Berlin, Heidelberg, New York, 1972.

[10] R. C. Rosier. Dual space of certain vector sequence spaces. *Pacific J. Math.*, 46(2):487–501, 1973.

[11] M. A. Ould Sidaty. Reflexivity and AK-property of certain vector sequence spaces. *Bull. Belg. Math. Soc.*, 10(4):579–583, 2003.

[12] M. A. Ould Sidaty. Nuclearity of certain vector-valued sequence spaces. *Rev. Real Academia de Ciencias. Zaragoza.*, 62:81–89, 2007.

[13] M. A. Ould Sidaty. Reflexivity of vector-valued Köthe-Orlicz sequence spaces. *Turk J Math.*, 42(3):911–923, 2018.