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Intuitionistic fuzzy ordered subalgebras in ordered BCI-algebras

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Abstract. In this paper, we apply the concept of an intuitionistic fuzzy set to ordered subalgebras in ordered BCI-algebras in the sense of intuitionistic fuzzy point. We introduce the notion of an intuitionistic fuzzy (ordered) subalgebra in ordered BCI-algebras, and investigate some related properties. We provide relations between an intuitionistic fuzzy ordered subalgebra and an intuitionistic fuzzy subalgebra. We give characterizations of an intuitionistic fuzzy (ordered) subalgebra. Finally, we provide relations between a $q_{(t,s)}$-level set of intuitionistic fuzzy set and an intuitionistic fuzzy ordered subalgebra.

2020 Mathematics Subject Classifications: 03G25, 06F35, 08A72

Key Words and Phrases: Intuitionistic fuzzy point, intuitionistic fuzzy (ordered) subalgebra, $q_{(t,s)}$-level set

1. Introduction

The speed of development of mathematics cannot be said to be fast, but it is clear that it is changing and developing through our efforts. There are many examples showing that progress is being made, but so is the appearance of BCI-algebra, which generalizes groups, and ordered BCI-algebra, which generalizes BCI-algebra. However, progress is not fast. BCI-algebra was introduced by Y. Imai and K. Iséki [11] in 1996 (see [10]), and ordered BCI-algebra was introduced by E. Yang, E. H. Roh and Y. B. Jun [8] in 2023. In [8], they introduced the notions of ordered BCI-algebras and (ordered) subalgebras and (ordered) filters of ordered BCI-algebras, and related properties are investigated. Moreover, some specific filters are introduced and their relations are discussed.

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L. A. Zadeh [19] introduced the degrees of membership and truth (t) in 1965 and defined the fuzzy set. As is so well known, a fuzzy set is a mathematical concept in the field of fuzzy logic that represents a set where elements have degrees of membership. Many mathematicians have conducted research to connect the algebraic structure with the fuzzy concept and obtained meaningful results(see [6, 12–18]). The concepts of the fuzzification of ordered subalgebras in ordered BCI-algebras were introduced, and related properties were investigated in [7].

After introduction of fuzzy sets by Zadeh, there have been a number of generalizations of this fundamental concept. The notion of intuitionistic fuzzy sets (IFS) introduced by K. Atanassov [1–4] is one among them. IFS are a mathematical concept in the field of fuzzy set theory. IFS are a generalization of traditional fuzzy sets, allowing for partial membership and uncertainty. In an IFS, an element may have a degree of membership, but also a degree of non-membership, which represents the uncertainty or the lack of information about its membership. These two degrees of membership and non-membership are used to represent the degree of belief and disbelief, respectively, in the membership of an element in the set. Many mathematicians have conducted research to connect the algebraic structure with the concept of intuitionistic fuzzy sets and obtained meaningful results (see [1–5, 9]). In this paper, we apply the concept of an IFS to ordered subalgebras in ordered BCI-algebras. We introduce the notion of an intuitionistic fuzzy (ordered) subalgebra in ordered BCI-algebras, and investigate some related properties. We provide relations between an intuitionistic fuzzy ordered subalgebra and an intuitionistic fuzzy subalgebra. We give characterizations of an intuitionistic fuzzy (ordered) subalgebra.

2. Preliminaries

Definition 1 ([8]). *Let X be a set with a binary operation " $\rightarrow$ ", a constant "e" and a binary relation " $\leq_X$ ". Then $\boldsymbol{X} := (X, \rightarrow, e, \leq_X)$ is called an ordered BCI-algebra (briefly, OBCI-algebra) if it satisfies the following conditions:*

$$(\forall x, y, z \in X)(e \leq_X (x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z))), \tag{1}$$

$$(\forall x, y \in X)(e \leq_X x \rightarrow ((x \rightarrow y) \rightarrow y)), \tag{2}$$

$$(\forall x \in X)(e \leq_X x \rightarrow x), \tag{3}$$

$$(\forall x, y \in X)(e \leq_X x \rightarrow y,\ e \leq_X y \rightarrow x \ \Rightarrow\ x = y), \tag{4}$$

$$(\forall x, y \in X)(x \leq_X y \ \Leftrightarrow\ e \leq_X x \rightarrow y), \tag{5}$$

$$(\forall x, y \in X)(e \leq_X x,\ x \leq_X y \ \Rightarrow\ e \leq_X y). \tag{6}$$

Proposition 1 ([8]). *Every OBCI-algebra $\boldsymbol{X} := (X, \rightarrow, e, \leq_X)$ satisfies:*

$$(\forall x \in X)(e \rightarrow x = x). \tag{7}$$

$$(\forall x, y, z \in X)(z \rightarrow (y \rightarrow x) = y \rightarrow (z \rightarrow x)). \tag{8}$$

$$(\forall x, y, z \in X)(e \leq_X x \rightarrow y \ \Rightarrow\ e \leq_X (y \rightarrow z) \rightarrow (x \rightarrow z)). \tag{9}$$

$$(\forall x, y, z \in X)(e \leq_X x \rightarrow y,\ e \leq_X y \rightarrow z \ \Rightarrow\ e \leq_X x \rightarrow z). \tag{10}$$

$$(\forall x, y, z \in X)(e \leq_X (z \to (y \to x)) \to (y \to (z \to x))). \tag{11}$$

$$(\forall x, y, z \in X)(e \leq_X z \to (y \to x) \ \Rightarrow \ e \leq_X y \to (z \to x)). \tag{12}$$

$$(\forall x, y \in X)(((x \to y) \to y) \to y = x \to y). \tag{13}$$

$$(\forall x \in X)((x \to x) \to x = x). \tag{14}$$

$$(\forall x, y, z \in X)(e \leq_X (y \to z) \to ((x \to y) \to (x \to z))). \tag{15}$$

$$(\forall x, y, z \in X)(e \leq_X x \to y \ \Rightarrow \ e \leq_X (z \to x) \to (z \to y)). \tag{16}$$

Definition 2 ([8]). *A subset A of X is called*

- *a subalgebra of an OBCI-algebra $\boldsymbol{X} := (X, \to, e, \leq_X)$ if it satisfies:*

$$(\forall x, y \in X)(x, y \in A \ \Rightarrow \ x \to y \in A). \tag{17}$$

- *an ordered subalgebra of an OBCI-algebra $\boldsymbol{X} := (X, \to, e, \leq_X)$ if it satisfies:*

$$(\forall x, y \in X)(x, y \in A, \, e \leq_X x, \, e \leq_X y \ \Rightarrow \ x \to y \in A). \tag{18}$$

A function $f : X \to [0, 1]$ is called a *fuzzy set* in a set X, and the *complement* of f is denoted by $\neg f$, and is given as follows:

$$\neg f : X \to [0, 1], \ x \mapsto 1 - f_{\mathcal{I}}(x).$$

For every fuzzy sets f and g in X, we say $f \leq g$ if $f(x) \leq g(x)$ fo all $x \in X$.

A fuzzy set f in a set X of the form

$$f(b) := \begin{cases} t \in (0, 1] & \text{if } b = a, \\ 0 & \text{if } b \neq a, \end{cases}$$

is said to be a *fuzzy point* with support a and value t and is denoted by a_t.

Definition 3 ([7]). *A fuzzy set f in X is called*

- *a fuzzy subalgebra of an OBCI-algebra $\boldsymbol{X} := (X, \to, e, \leq_X)$ if it satisfies:*

$$(\forall x, y \in X)(\forall t, s \in (0, 1]) \left(\begin{array}{l} x_t \in f, \, y_s \in f \\ \quad \Rightarrow \ \langle (x \to y)_{\min\{t,s\}} \rangle \in f. \end{array} \right). \tag{19}$$

- *a fuzzy ordered subalgebra of an OBCI-algebra $\boldsymbol{X} := (X, \to, e, \leq_X)$ if it satisfies:*

$$(\forall x, y \in X)(e \leq_X x, \, e \leq_X y \ \Rightarrow \ f(x \to y) \geq \min\{f(x), f(y)\}). \tag{20}$$

The concept of intuitionistic fuzzy set was introduced by Atanassov (see [1, 2, 4]) as follows: An *intuitionistic fuzzy set* on a set X is an expression $\mathcal{I}$ given by

$$\mathcal{I} := \{\langle x, f_{\mathcal{I}}, g_{\mathcal{I}} \rangle \mid x \in X\}$$

where $f_{\mathcal{I}}$ and $g_{\mathcal{I}}$ are fuzzy sets in X such that $0 \leq f_{\mathcal{I}}(x) + g_{\mathcal{I}}(x) \leq 1$ for all $x \in X$.

Every fuzzy set f in a set X is obviously an intuitionistic fuzzy set having the form $\{\langle x, f, \neg f\rangle \mid x \in X\}$ (see [2]).

The notion of intuitionistic fuzzy point is considered in the paper [5] as follows: Given elements $b \in X$ and $(t,s) \in (0,1] \times [0,1)$ satisfying $t + s \leq 1$, the intuitionistic fuzzy set

$$b_{(t,s)} := \{\langle x, b_t, \neg b_{1-s}\rangle \mid x \in X\} \tag{21}$$

is called an *intuitionistic fuzzy point* in X.

Let $\mathcal{I} := \{\langle x, f_{\mathcal{I}}, g_{\mathcal{I}}\rangle \mid x \in X\}$ be an intuitionistic fuzzy set in X. An intuitionistic fuzzy point $b_{(t,s)} := \{\langle x, b_t, \neg b_{1-s}\rangle \mid x \in X\}$ is said to be

- *contained* in $\mathcal{I} := \{\langle x, f_{\mathcal{I}}, g_{\mathcal{I}}\rangle \mid x \in X\}$, denoted by $b_{(t,s)} \in \mathcal{I}$, if $b_t \leq f_{\mathcal{I}}$ and $\neg b_{1-s} \geq g_{\mathcal{I}}$, or equivalently, $f_{\mathcal{I}}(b) \geq t$ and $g_{\mathcal{I}}(b) \leq s$.
- *quasi-coincident* with $\mathcal{I} := \{\langle x, f_{\mathcal{I}}, g_{\mathcal{I}}\rangle \mid x \in X\}$, denoted by $b_{(t,s)}\, q\, \mathcal{I}$, if $f_{\mathcal{I}}(b) + t > 1$ and $g_{\mathcal{I}}(b) + s < 1$.

If $b_{(t,s)}\, \beta\, \mathcal{I}$ is not established for $\beta \in \{\in, q\}$, it is denoted by $b_{(t,s)}\, \overline{\beta}\, \mathcal{I}$. The set

$$\mathcal{I}^{\in}_{(t,s)} := \{b \in X \mid b_{(t,s)} \in \mathcal{I}\}$$

is called the $\in_{(t,s)}$*-level set* of $\mathcal{I}$. It is clear that

$$\mathcal{I}^{\in}_{(t,s)} = U(f_{\mathcal{I}}, t) \cap L(g_{\mathcal{I}}, s)$$

where $U(f_{\mathcal{I}}, t) := \{a \in X \mid f_{\mathcal{I}}(a) \geq t\}$ and $L(g_{\mathcal{I}}, s) := \{a \in X \mid g_{\mathcal{I}}(a) \leq s\}$, which are called the *upper t-level set* and the *lower s-level set* of $\mathcal{I} := \{\langle x, f_{\mathcal{I}}, g_{\mathcal{I}}\rangle \mid x \in X\}$.

3. Intuitionistic fuzzy (ordered) subalgebras

In what follows, let $\mathbf{X} := (X, \to, e, \leq_X)$ denote an OBCI-algebra unless otherwise specified.

Definition 4. *An intuitionistic fuzzy set $\mathcal{I} := \{\langle x, f_{\mathcal{I}}, g_{\mathcal{I}}\rangle \mid x \in X\}$ is called*

- *an intuitionistic fuzzy subalgebra of $\boldsymbol{X} := (X, \to, e, \leq_X)$ if it satisfies:*

$$(\forall x, y \in X)\left(\begin{array}{l} f_{\mathcal{I}}(x \to y) \geq \min\{f_{\mathcal{I}}(x), f_{\mathcal{I}}(y)\} \\ g_{\mathcal{I}}(x \to y) \leq \max\{g_{\mathcal{I}}(x), g_{\mathcal{I}}(y)\} \end{array}\right). \tag{22}$$

- *an intuitionistic fuzzy ordered subalgebra of $\boldsymbol{X} := (X, \to, e, \leq_X)$ if it satisfies:*

$$(\forall x, y \in X)\left(\begin{array}{c} e \leq_X x,\ e \leq_X y,\ x_{(t_1,s_1)} \in \mathcal{I},\ y_{(t_2,s_2)} \in \mathcal{I} \\ \Rightarrow\ (x \to y)_{(\min\{t_1,t_2\},\max\{s_1,s_2\})} \in \mathcal{I} \end{array}\right) \tag{23}$$

for all $(t_1, s_1), (t_2, s_2) \in (0,1] \times [0,1)$.

Table 1: Cayley table for the binary operation "$\rightarrow$"

$\rightarrow$	1	e	∂	0
1	1	0	0	0
e	1	e	∂	0
∂	1	∂	e	0
0	1	1	1	1

Example 1. *Let $X = \{1, e, \partial, 0\}$ be a set, where 1 and 0 are the greatest element and the least element of X, respectively. Define a binary operation "$\rightarrow$" on X by Table 1*
Let $\leq_e := \{(0,0), (e,e), (\partial,\partial), (1,1), (0,e), (0,\partial), (e,1), (\partial,1)\}$. Then $\boldsymbol{X} := (X, \rightarrow, e, \leq_X)$ is an OBCI-algebra (see [8]). Define an intuitionistic fuzzy set $\mathcal{I} := \{\langle x, f_{\mathcal{I}}, g_{\mathcal{I}}\rangle \mid x \in X\}$ in X as follows:

$$f_{\mathcal{I}} : X \to [0,1],\ x \mapsto \begin{cases} 0.68 & \text{if } x \in \{1, e, 0\}, \\ 0.24 & \text{otherwise,} \end{cases}$$

and

$$g_{\mathcal{I}} : X \to [0,1],\ x \mapsto \begin{cases} 0.31 & \text{if } x \in \{1, e, 0\}, \\ 0.59 & \text{otherwise.} \end{cases}$$

It is routine to verify that $\mathcal{I} := \{\langle x, f_{\mathcal{I}}, g_{\mathcal{I}}\rangle \mid x \in X\}$ is an intuitionistic fuzzy subalgebra of $\boldsymbol{X} := (X, \rightarrow, e, \leq_X)$. Also, if we define an intuitionistic fuzzy set $\mathcal{I} := \{\langle x, f_{\mathcal{I}}, g_{\mathcal{I}}\rangle \mid x \in X\}$ in X by

$$f_{\mathcal{I}} : X \to [0,1],\ x \mapsto \begin{cases} 0.63 & \text{if } x \in \{e, 0\}, \\ 0.27 & \text{otherwise,} \end{cases}$$

and

$$g_{\mathcal{I}} : X \to [0,1],\ x \mapsto \begin{cases} 0.29 & \text{if } x \in \{e, 0\}, \\ 0.62 & \text{otherwise,} \end{cases}$$

then $\mathcal{I} := \{\langle x, f_{\mathcal{I}}, g_{\mathcal{I}}\rangle \mid x \in X\}$ is an intuitionistic fuzzy ordered subalgebra of $\boldsymbol{X} := (X, \rightarrow, e, \leq_X)$.

It is clear that every intuitionistic fuzzy subalgebra is an intuitionistic fuzzy ordered subalgebra, but the converse is not true as seen in the example below.

Example 2. *Let $X = \{0, 1, \frac{3}{4}, \frac{1}{2}, \frac{1}{4}\}$ be a set with a binary operation "$\rightarrow$" given by Table 2 and let $\leq_e$ be the natural order in X.*
Then $\boldsymbol{X} := (X, \rightarrow, e, \leq_X)$, where $e = \frac{3}{4}$, is an OBCI-algebra (see [8]). Define an intuitionistic fuzzy set $\mathcal{I} := \{\langle x, f_{\mathcal{I}}, g_{\mathcal{I}}\rangle \mid x \in X\}$ in X as follows:

$$f_{\mathcal{I}} : X \to [0,1],\ x \mapsto \begin{cases} 0.78 & \text{if } x \in \{\frac{3}{4}, 0\}, \\ 0.23 & \text{otherwise,} \end{cases}$$

Table 2: Cayley table for the binary operation "→"

$\rightarrow$	1	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0
1	1	0	0	0	0
$\frac{3}{4}$	1	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0
$\frac{1}{2}$	1	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{1}{2}$	0
$\frac{1}{4}$	1	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	0
0	1	1	1	1	1

and

$$g_{\mathcal{I}} : X \to [0,1],\ x \mapsto \begin{cases} 0.12 & \text{if } x \in \{\frac{3}{4}, 0\}, \\ 0.61 & \text{otherwise.} \end{cases}$$

It is routine to verify that f is an intuitionistic fuzzy ordered subalgebra of $\boldsymbol{X} := (X, \to, e, \leq_X)$. But it is not an intuitionistic fuzzy subalgebra of $\boldsymbol{X} := (X, \to, e, \leq_X)$ since $f_{\mathcal{I}}(0 \to \frac{3}{4}) = f_{\mathcal{I}}(1) = 0.23 \not\geq 0.78 = \min\{f_{\mathcal{I}}(0), f_{\mathcal{I}}(\frac{3}{4})\}$ and/or $g_{\mathcal{I}}(0 \to \frac{3}{4}) = f_{\mathcal{I}}(1) = 0.61 \not\leq 0.12 = \max\{g_{\mathcal{I}}(0), g_{\mathcal{I}}(\frac{3}{4})\}$.

We provide a condition in which the intuitionistic fuzzy ordered subalgebra becomes the intuitionistic fuzzy subalgebra.

Theorem 1. *Let $\mathcal{I} := \{\langle x, f_{\mathcal{I}}, g_{\mathcal{I}}\rangle \mid x \in X\}$ be an intuitionistic fuzzy ordered subalgebra of $\boldsymbol{X} := (X, \to, e, \leq_X)$. If its $\in_{(t,s)}$-level set $\mathcal{I}^{\in}_{(t,s)}$ satisfies $e \leq_X x$ for all $x \in \mathcal{I}^{\in}_{(t,s)}$, then $\mathcal{I} := \{\langle x, f_{\mathcal{I}}, g_{\mathcal{I}}\rangle \mid x \in X\}$ is an intuitionistic fuzzy subalgebra of $\boldsymbol{X} := (X, \to, e, \leq_X)$.*

Proof. Straightforward.

Theorem 2. *An intuitionistic fuzzy set $\mathcal{I} := \{\langle x, f_{\mathcal{I}}, g_{\mathcal{I}}\rangle \mid x \in X\}$ in X is an intuitionistic fuzzy subalgebra of $\boldsymbol{X} := (X, \to, e, \leq_X)$ if and only if it satisfies:*

$$x_{(t_1,s_1)} \in \mathcal{I},\ y_{(t_2,s_2)} \in \mathcal{I} \ \Rightarrow\ (x \to y)_{(\min\{t_1,t_2\},\max\{s_1,s_2\})} \in \mathcal{I} \tag{24}$$

for all $x, y \in X$ and $(t_i, s_i) \in (0,1] \times [0,1)$ for $i = 1, 2$.

Proof. Assume that $\mathcal{I} := \{\langle x, f_{\mathcal{I}}, g_{\mathcal{I}}\rangle \mid x \in X\}$ is an intuitionistic fuzzy subalgebra of $\mathbf{X} := (X, \to, e, \leq_X)$. Let $x, y \in X$ be such that $x_{(t_1,s_1)} \in \mathcal{I}$ and $y_{(t_2,s_2)} \in \mathcal{I}$ for all $(t_i, s_i) \in (0,1] \times [0,1)$ for $i = 1, 2$. Then $f_{\mathcal{I}}(x) \geq t_1$, $f_{\mathcal{I}}(y) \geq t_2$, $g_{\mathcal{I}}(x) \leq s_1$, and $g_{\mathcal{I}}(y) \leq s_2$. It follows from (22) that

$$f_{\mathcal{I}}(x \to y) \geq \min\{f_{\mathcal{I}}(x), f_{\mathcal{I}}(y)\} \geq \min\{t_1, t_2\}$$

and $g_{\mathcal{I}}(x \to y) \leq \max\{g_{\mathcal{I}}(x), g_{\mathcal{I}}(y)\} \leq \max\{s_1, s_2\}$. Hence

$$(x \to y)_{(\min\{t_1,t_2\},\max\{s_1,s_2\})} \in \mathcal{I}.$$

Conversely, suppose that $\mathcal{I} := \{\langle x, f_{\mathcal{I}}, g_{\mathcal{I}}\rangle \mid x \in X\}$ satisfies (24) for all $x, y \in X$ and $(t_i, s_i) \in (0,1] \times [0,1)$ for $i = 1, 2$. then $f_{\mathcal{I}}(a \to b) < \min\{f_{\mathcal{I}}(a), f_{\mathcal{I}}(b)\}$ or $g_{\mathcal{I}}(a \to b) > \max\{g_{\mathcal{I}}(a), g_{\mathcal{I}}(b)\}$ for some $a, b \in X$. Taking $t := \min\{f_{\mathcal{I}}(a), f_{\mathcal{I}}(b)\}$ and $s := \max\{g_{\mathcal{I}}(a), g_{\mathcal{I}}(b)\}$ induces $a_{(t,s)} \in \mathcal{I}$, and $b_{(t,s)} \in \mathcal{I}$. It follows from (24) that $(a \to b)_{(t,s)} = (a \to b)_{(\min\{t,t\}, \max\{s,s\})} \in \mathcal{I}$. But $f_{\mathcal{I}}(a \to b) < t$ or $g_{\mathcal{I}}(a \to b) > s$ imply that $(a \to b)_{(t,s)} \overline{\in} \mathcal{I}$, a contradiction. Therefore $\mathcal{I} := \{\langle x, f_{\mathcal{I}}, g_{\mathcal{I}}\rangle \mid x \in X\}$ is an intuitionistic fuzzy subalgebra of $\mathbf{X} := (X, \to, e, \leq_X)$.

Theorem 3. *An intuitionistic fuzzy set $\mathcal{I} := \{\langle x, f_{\mathcal{I}}, g_{\mathcal{I}}\rangle \mid x \in X\}$ in X is an intuitionistic fuzzy ordered subalgebra of $\boldsymbol{X} := (X, \to, e, \leq_X)$ if and only if it satisfies:*

$$(\forall x, y \in X) \left(\begin{array}{l} e \leq_X x,\ e \leq_X y \\ \quad \Rightarrow \left\{ \begin{array}{l} f_{\mathcal{I}}(x \to y) \geq \min\{f_{\mathcal{I}}(x), f_{\mathcal{I}}(y)\} \\ g_{\mathcal{I}}(x \to y) \leq \max\{g_{\mathcal{I}}(x), g_{\mathcal{I}}(y)\} \end{array} \right. \end{array} \right). \tag{25}$$

Proof. Assume that $\mathcal{I} := \{\langle x, f_{\mathcal{I}}, g_{\mathcal{I}}\rangle \mid x \in X\}$ is an intuitionistic fuzzy ordered subalgebra of $\mathbf{X} := (X, \to, e, \leq_X)$. If the assertion (25) is not valid, then $f_{\mathcal{I}}(a \to b) < t < \min\{f_{\mathcal{I}}(a), f_{\mathcal{I}}(b)\}$ or $g_{\mathcal{I}}(a \to b) > s > \max\{g_{\mathcal{I}}(a), g_{\mathcal{I}}(b)\}$ for some $(t, s) \in (0,1) \times (0,1)$ and $a, b \in X$ with $e \leq_X a$, $e \leq_X b$. Then $t + s \leq 1$, $a_t \leq f_{\mathcal{I}}$, $b_t \leq f_{\mathcal{I}}$, $\neg a_{1-s} \geq g_{\mathcal{I}}$, and $\neg b_{1-s} \geq g_{\mathcal{I}}$. Hence $a_{(t,s)} \in \mathcal{I}$ and $b_{(t,s)} \in \mathcal{I}$. It follows from (23) that $(a \to b)_{(t,s)} = (a \to b)_{(\min\{t,t\}, \max\{s,s\})} \in \mathcal{I}$. Thus $(a \to b)_t \leq f_{\mathcal{I}}$ and $\neg(a \to b)_s \geq g_{\mathcal{I}}$, that is, $f_{\mathcal{I}}(a \to b) \geq t$ and $g_{\mathcal{I}}(a \to b) \leq s$. This is a contradiction, and so (25) is valid.

Conversely, suppose that $\mathcal{I} := \{\langle x, f_{\mathcal{I}}, g_{\mathcal{I}}\rangle \mid x \in X\}$ satisfies (25). Let $x, y \in X$ be such that $e \leq_X x$, $e \leq_X y$, $x_{(t_1,s_1)} \in \mathcal{I}$ and $y_{(t_2,s_2)} \in \mathcal{I}$ for every $(t_1, s_1), (t_2, s_2) \in (0,1] \times [0,1)$. Then $f_{\mathcal{I}}(x) \geq t_1$, $g_{\mathcal{I}}(x) \leq s_1$, $f_{\mathcal{I}}(y) \geq t_2$, and $g_{\mathcal{I}}(y) \leq s_2$. It follows from (25) that

$$f_{\mathcal{I}}(x \to y) \geq \min\{f_{\mathcal{I}}(x), f_{\mathcal{I}}(y)\} \geq \min\{t_1, t_2\}$$

and $g_{\mathcal{I}}(x \to y) \leq \max\{g_{\mathcal{I}}(x), g_{\mathcal{I}}(y)\} \leq \max\{s_1, s_2\}$. Hence

$$(x \to y)_{(\min\{t_1,t_2\}, \max\{s_1,s_2\})} \in \mathcal{I}.$$

Therefore $\mathcal{I} := \{\langle x, f_{\mathcal{I}}, g_{\mathcal{I}}\rangle \mid x \in X\}$ is an intuitionistic fuzzy ordered subalgebra of $\mathbf{X} := (X, \to, e, \leq_X)$.

Lemma 1. *An intuitionistic fuzzy set $\mathcal{I} := \{\langle x, f_{\mathcal{I}}, g_{\mathcal{I}}\rangle \mid x \in X\}$ in X is an intuitionistic fuzzy ordered subalgebra of $\boldsymbol{X} := (X, \to, e, \leq_X)$ if and only if $f_{\mathcal{I}}$ and $g_{\mathcal{I}}^c$ are fuzzy ordered subalgebras of $\boldsymbol{X} := (X, \to, e, \leq_X)$, where $g_{\mathcal{I}}^c$ is defined by $g_{\mathcal{I}}^c(x) = 1 - g_{\mathcal{I}}(x)$ for all $x \in X$.*

Proof. Let $\mathcal{I} := \{\langle x, f_{\mathcal{I}}, g_{\mathcal{I}}\rangle \mid x \in X\}$ be an intuitionistic fuzzy ordered subalgebra of $\mathbf{X} := (X, \to, e, \leq_X)$. Obviously, $f_{\mathcal{I}}$ is a fuzzy ordered subalgebra of $\mathbf{X} := (X, \to, e, \leq_X)$ by Theorem 3. Let $x, y \in X$ be such that $e \leq_X x$ and $e \leq_X y$. Using Theorem 3 induces

$$\begin{aligned} g_{\mathcal{I}}^c(x \to y) &= 1 - g_{\mathcal{I}}(x \to y) \geq 1 - \max\{g_{\mathcal{I}}(x), g_{\mathcal{I}}(y)\} \\ &= \min\{1 - g_{\mathcal{I}}(x), 1 - g_{\mathcal{I}}(y)\} = \min\{g_{\mathcal{I}}^c(x), g_{\mathcal{I}}^c(y)\}. \end{aligned}$$

Hence $g_{\mathcal{I}}^c$ is a fuzzy ordered subalgebra of $\mathbf{X} := (X, \to, e, \leq_X)$.

Conversely, suppose that $f_{\mathcal{I}}$ and $g_{\mathcal{I}}^c$ are fuzzy ordered subalgebras of $\mathbf{X} := (X, \to, e, \leq_X)$. For every $x, y \in X$ with $e \leq_X x$ and $e \leq_X y$, we have

$$f_{\mathcal{I}}(x \to y) \geq \min\{f_{\mathcal{I}}(x), f_{\mathcal{I}}(y)\}$$

and

$$\begin{aligned} 1 - g_{\mathcal{I}}(x \to y) &= g_{\mathcal{I}}^c(x \to y) \geq \min\{g_{\mathcal{I}}^c(x), g_{\mathcal{I}}^c(y)\} \\ &= \min\{1 - g_{\mathcal{I}}(x), 1 - g_{\mathcal{I}}(y)\} \\ &= 1 - \max\{g_{\mathcal{I}}(x), g_{\mathcal{I}}(y)\}, \end{aligned}$$

that is, $g_{\mathcal{I}}(x \to y) \leq \max\{g_{\mathcal{I}}(x), g_{\mathcal{I}}(y)\}$. Therefore $\mathcal{I} := \{\langle x, f_{\mathcal{I}}, g_{\mathcal{I}}\rangle \mid x \in X\}$ is an intuitionistic fuzzy ordered subalgebra of $\mathbf{X} := (X, \to, e, \leq_X)$ by Theorem 3.

Theorem 4. *An intuitionistic fuzzy set $\mathcal{I} := \{\langle x, f_{\mathcal{I}}, g_{\mathcal{I}}\rangle \mid x \in X\}$ in X is an intuitionistic fuzzy ordered subalgebra of $\boldsymbol{X} := (X, \to, e, \leq_X)$ if and only if $\Box\mathcal{I} := \{\langle x, f_{\mathcal{I}}, f_{\mathcal{I}}^c\rangle \mid x \in X\}$ and $\Diamond\mathcal{I} := \{\langle x, g_{\mathcal{I}}^c, g_{\mathcal{I}}\rangle \mid x \in X\}$ are intuitionistic fuzzy ordered subalgebras of $\boldsymbol{X} := (X, \to, e, \leq_X)$*

Proof. It is straightforward by Lemma 1.

Theorem 5. *If $\mathcal{I} := \{\langle x, f_{\mathcal{I}}, g_{\mathcal{I}}\rangle \mid x \in X\}$ is an intuitionistic fuzzy ordered subalgebra of $\boldsymbol{X} := (X, \to, e, \leq_X)$, then the set*

$$\mathcal{I}_{(0,1)} := \{x \in X \mid f_{\mathcal{I}}(x) > 0,\ g_{\mathcal{I}}(x) < 1\},$$

which is called the intuitionistic support of $\mathcal{I}$, is an ordered subalgebra of $\boldsymbol{X} := (X, \to, e, \leq_X)$.

Proof. Let $x, y \in X$ be such that $e \leq_X x$, $e \leq_X y$ and $x, y \in \mathcal{I}_{(0,1)}$. Then $f_{\mathcal{I}}(x) > 0$, $g_{\mathcal{I}}(x) < 1$, $f_{\mathcal{I}}(y) > 0$, and $g_{\mathcal{I}}(y) < 1$. Using Theorem 3, we have $f_{\mathcal{I}}(x \to y) \geq \min\{f_{\mathcal{I}}(x), f_{\mathcal{I}}(y)\} > 0$ and $g_{\mathcal{I}}(x \to y) \leq \max\{g_{\mathcal{I}}(x), g_{\mathcal{I}}(y)\} < 1$. Hence $x \to y \in \mathcal{I}_{(0,1)}$, and so $\mathcal{I}_{(0,1)}$ is an ordered subalgebra of $\mathbf{X} := (X, \to, e, \leq_X)$.

Theorem 6. *If an intuitionistic fuzzy set $\mathcal{I} := \{\langle x, f_{\mathcal{I}}, g_{\mathcal{I}}\rangle \mid x \in X\}$ in X satisfies:*

$$(\forall x, y \in X)\left(\begin{array}{c} e \leq_X x,\ e \leq_X y,\ x_{(t_1,s_1)} \in \mathcal{I},\ y_{(t_2,s_2)} \in \mathcal{I} \\ \Rightarrow\ (x \to y)_{(\min\{t_1,t_2\},\max\{s_1,s_2\})}\, q\, \mathcal{I} \end{array}\right) \tag{26}$$

where $(t_i, s_i) \in (0,1] \times [0,1)$ for $i = 1, 2$, then its intuitionistic support $\mathcal{I}_{(0,1)}$ is an ordered subalgebra of $\boldsymbol{X} := (X, \to, e, \leq_X)$.

Proof. Let $x, y \in X$ be such that $e \leq_X x$, $e \leq_X y$ and $x, y \in \mathcal{I}_{(0,1)}$. Then $(f_{\mathcal{I}}(x), g_{\mathcal{I}}(x)), (f_{\mathcal{I}}(y), g_{\mathcal{I}}(y)) \in (0,1] \times [0,1)$. Note that $x_{(f_{\mathcal{I}}(x), g_{\mathcal{I}}(x))} \in \mathcal{I}$ and $y_{(f_{\mathcal{I}}(y), g_{\mathcal{I}}(y))} \in \mathcal{I}$. Using (26), we get

$$(x \to y)_{(\min\{f_{\mathcal{I}}(x), f_{\mathcal{I}}(y)\}, \max\{g_{\mathcal{I}}(x), g_{\mathcal{I}}(y)\})}\, q\, \mathcal{I}.$$

Hence $f_{\mathcal{I}}(x \to y) > 1 - \min\{f_{\mathcal{I}}(x), f_{\mathcal{I}}(y)\} \geq 0$ and

$$g_{\mathcal{I}}(x \to y) < 1 - \max\{g_{\mathcal{I}}(x), g_{\mathcal{I}}(y)\} \leq 1.$$

This shows that $x \to y \in \mathcal{I}_{(0,1)}$, and therefore $\mathcal{I}_{(0,1)}$ is an ordered subalgebra of $\mathbf{X} := (X, \to, e, \leq_X)$.

Theorem 7. *If an intuitionistic fuzzy set $\mathcal{I} := \{\langle x, f_{\mathcal{I}}, g_{\mathcal{I}}\rangle \mid x \in X\}$ in X satisfies:*

$$(\forall x, y \in X)\left(\begin{array}{c} e \leq_X x,\ e \leq_X y,\ x_{(t_1,s_1)}\, q\, \mathcal{I},\ y_{(t_2,s_2)}\, q\, \mathcal{I} \\ \Rightarrow\ (x \to y)_{(\min\{t_1,t_2\}, \max\{s_1,s_2\})} \in \mathcal{I} \end{array}\right) \tag{27}$$

for all $(t_i, s_i) \in (0,1] \times [0,1)$ for $i = 1, 2$, then its intuitionistic support $\mathcal{I}_{(0,1)}$ is an ordered subalgebra of $\boldsymbol{X} := (X, \to, e, \leq_X)$.

Proof. Let $x, y \in X$ be such that $e \leq_X x$, $e \leq_X y$ and $x, y \in \mathcal{I}_{(0,1)}$. Then $(f_{\mathcal{I}}(x), g_{\mathcal{I}}(x)), (f_{\mathcal{I}}(y), g_{\mathcal{I}}(y)) \in (0,1] \times [0,1)$, and so $f_{\mathcal{I}}(x) + 1 > 1$, $g_{\mathcal{I}}(x) + 0 < 1$, $f_{\mathcal{I}}(y) + 1 > 1$, and $g_{\mathcal{I}}(y) + 0 < 1$. This shows that $x_{(1,0)}\, q\, \mathcal{I}$ and $y_{(1,0)}\, q\, \mathcal{I}$. It follows from (27) that $(x \to y)_{(1,0)} \in \mathcal{I}$. Hence $f_{\mathcal{I}}(x \to y) = 1 > 0$ and $g_{\mathcal{I}}(x \to y) = 0 < 1$, which imply that $x \to y \in \mathcal{I}_{(0,1)}$. Hence $\mathcal{I}_{(0,1)}$ is an ordered subalgebra of $\mathbf{X} := (X, \to, e, \leq_X)$.

Theorem 8. *If an intuitionistic fuzzy set If $\mathcal{I} := \{\langle x, f_{\mathcal{I}}, g_{\mathcal{I}}\rangle \mid x \in X\}$ in X satisfies:*

$$(\forall x, y \in X)\left(\begin{array}{c} e \leq_X x,\ e \leq_X y,\ x_{(t_1,s_1)}\, q\, \mathcal{I},\ y_{(t_2,s_2)}\, q\, \mathcal{I} \\ \Rightarrow\ (x \to y)_{(\min\{t_1,t_2\}, \max\{s_1,s_2\})}\, q\, \mathcal{I} \end{array}\right) \tag{28}$$

for all $(t_i, s_i) \in (0,1] \times [0,1)$ for $i = 1, 2$, then its intuitionistic support $\mathcal{I}_{(0,1)}$ is an ordered subalgebra of $\boldsymbol{X} := (X, \to, e, \leq_X)$.

Proof. Let $x, y \in X$ be such that $e \leq_X x$, $e \leq_X y$ and $x, y \in \mathcal{I}_{(0,1)}$. Then $(f_{\mathcal{I}}(x), g_{\mathcal{I}}(x)), (f_{\mathcal{I}}(y), g_{\mathcal{I}}(y)) \in (0,1] \times [0,1)$, and so $f_{\mathcal{I}}(x) + 1 > 1$, $g_{\mathcal{I}}(x) + 0 < 1$, $f_{\mathcal{I}}(y) + 1 > 1$, and $g_{\mathcal{I}}(y) + 0 < 1$. This shows that $x_{(1,0)}\, q\, \mathcal{I}$ and $y_{(1,0)}\, q\, \mathcal{I}$. Using (28), we have $(x \to y)_{(1,0)}\, q\, \mathcal{I}$. If $f_{\mathcal{I}}(x \to y) = 0$ or $g_{\mathcal{I}}(x \to y) = 1$, then $f_{\mathcal{I}}(x \to y) + 1 = 1$ or $g_{\mathcal{I}}(x \to y) + 1 = 2$, i.e., $(x \to y)_{(1,0)}\, \overline{q}\, \mathcal{I}$, a contradiction. Hence $f_{\mathcal{I}}(x \to y) > 0$ and $g_{\mathcal{I}}(x \to y) < 1$, that is, $x \to y \in \mathcal{I}_{(0,1)}$. Therefore $\mathcal{I}_{(0,1)}$ is an ordered subalgebra of $\mathbf{X} := (X, \to, e, \leq_X)$.

Theorem 9. *Given a nonempty subset B of X, let $\mathcal{I}^B := \{\langle x, f_{\mathcal{I}}^B, g_{\mathcal{I}}^B\rangle \mid x \in X\}$ be an intuitionistic fuzzy set in X in which $f_{\mathcal{I}}^B$ and $g_{\mathcal{I}}^B$ are given as follows:*

$$f_{\mathcal{I}}^B : X \to [0,1],\ x \mapsto \begin{cases} t_1 & \text{if } x \in B, \\ t_2 & \text{otherwise,} \end{cases}$$

and

$$g_{\mathcal{I}}^B : X \to [0,1],\ x \mapsto \begin{cases} s_1 & \text{if } x \in B, \\ s_2 & \text{otherwise,} \end{cases}$$

where $t_1 > t_2$ in $(0,1]$ and $s_1 < s_2$ in $[0,1)$. Then $\mathcal{I}^B := \{\langle x, f_{\mathcal{I}}^B, g_{\mathcal{I}}^B\rangle \mid x \in X\}$ is an intuitionistic fuzzy ordered subalgebra of $\boldsymbol{X} := (X, \to, e, \leq_X)$ if and only if B is an ordered subalgebra of $\boldsymbol{X} := (X, \to, e, \leq_X)$.

Proof. Assume that $\mathcal{I}^B := \{\langle x, f_{\mathcal{I}}^B, g_{\mathcal{I}}^B\rangle \mid x \in X\}$ is an intuitionistic fuzzy ordered subalgebra of $\mathbf{X} := (X, \to, e, \leq_X)$. Let $x, y \in X$ be such that $x, y \in B$, $e \leq_e x$ and $e \leq_e y$. Using Theorem 3, we have

$$f_{\mathcal{I}}^B(x \to y) \geq \min\{f_{\mathcal{I}}^B(x), f_{\mathcal{I}}^B(y)\} = t_1$$

and $g_{\mathcal{I}}^B(x \to y) \leq \max\{g_{\mathcal{I}}^B(x), g_{\mathcal{I}}^B(y)\} = s_1$. Hence

$$f_{\mathcal{I}}^B(x \to y) = t_1 \text{ and } g_{\mathcal{I}}^B(x \to y) = s_1,$$

and so $x \to y \in B$. Therefore B is an ordered subalgebra of $\mathbf{X} := (X, \to, e, \leq_X)$.

Conversely, suppose that B is an ordered subalgebra of $\mathbf{X} := (X, \to, e, \leq_X)$. Let $x, y \in X$ be such that $e \leq_e x$ and $e \leq_e y$. If $x \notin B$ (or $y \notin B$), then $f_{\mathcal{I}}^B(x) = t_2$ (or $f_{\mathcal{I}}^B(y) = t_2$) and $g_{\mathcal{I}}^B(x) = s_2$ (or $g_{\mathcal{I}}^B(y) = s_2$). Thus

$$f_{\mathcal{I}}^B(x \to y) \geq t_2 = \min\{f_{\mathcal{I}}^B(x), f_{\mathcal{I}}^B(y)\}$$

and $g_{\mathcal{I}}^B(x \to y) \leq s_2 = \max\{g_{\mathcal{I}}^B(x), g_{\mathcal{I}}^B(y)\}$. If $x \in B$ and $y \in B$, then $x \to y \in B$ since B is an ordered subalgebra of $\mathbf{X} := (X, \to, e, \leq_X)$. Thus

$$f_{\mathcal{I}}^B(x \to y) = t_1 = \min\{f_{\mathcal{I}}^B(x), f_{\mathcal{I}}^B(y)\}$$

and $g_{\mathcal{I}}^B(x \to y) = s_1 = \max\{g_{\mathcal{I}}^B(x), g_{\mathcal{I}}^B(y)\}$. It follows from Theorem 3 that $\mathcal{I}^B := \{\langle x, f_{\mathcal{I}}^B, g_{\mathcal{I}}^B\rangle \mid x \in X\}$ is an intuitionistic fuzzy ordered subalgebra of $\mathbf{X} := (X, \to, e, \leq_X)$.

Theorem 10. *An intuitionistic fuzzy set $\mathcal{I} := \{\langle x, f_{\mathcal{I}}, g_{\mathcal{I}}\rangle \mid x \in X\}$ in X is an intuitionistic fuzzy ordered subalgebra of $\boldsymbol{X} := (X, \to, e, \leq_X)$ if and only if its upper t-level set and lower s-level set are ordered subalgebras of $\boldsymbol{X} := (X, \to, e, \leq_X)$ for all $(t, s) \in (0,1] \times [0,1)$.*

Proof. Suppose that $\mathcal{I} := \{\langle x, f_{\mathcal{I}}, g_{\mathcal{I}}\rangle \mid x \in X\}$ in X is an intuitionistic fuzzy ordered subalgebra of $\mathbf{X} := (X, \to, e, \leq_X)$. Let $x, y, a, b \in X$ be such that $x, y \in U(f_{\mathcal{I}}, t)$ and $a, b \in$

$L(g_{\mathcal{I}}, s)$ whenever $e \leq_e x$, $e \leq_e y$, $e \leq_e a$, and $e \leq_e b$. Then $f_{\mathcal{I}}(x) \geq t$, $f_{\mathcal{I}}(y) \geq t$, $g_{\mathcal{I}}(a) \leq s$ and $g_{\mathcal{I}}(b) \leq s$. It follows from Theorem 3 that $f_{\mathcal{I}}(x \to y) \geq \min\{f_{\mathcal{I}}(x), f_{\mathcal{I}}(y)\} \geq t$ and

$$g_{\mathcal{I}}(x \to y) \leq \max\{g_{\mathcal{I}}(x), g_{\mathcal{I}}(y)\} \leq s.$$

Hence $x \to y \in U(f_{\mathcal{I}}, t)$ and $a \to b \in L(g_{\mathcal{I}}, s)$. Therefore $U(f_{\mathcal{I}}, t)$ and $L(g_{\mathcal{I}}, s)$ are ordered subalgebras of $\mathbf{X} := (X, \to, e, \leq_X)$.

Conversely, assume that the upper t-level set and lower s-level set are ordered subalgebras of $\mathbf{X} := (X, \to, e, \leq_X)$ for all $(t, s) \in (0, 1] \times [0, 1)$. Let $x, y \in X$ and $(t_1, s_1), (t_2, s_2) \in (0, 1] \times [0, 1)$ be such that $e \leq_X x$, $e \leq_X y$, $x_{(t_1,s_1)} \in \mathcal{I}$, and $y_{(t_2,s_2)} \in \mathcal{I}$. Then $f_{\mathcal{I}}(x) \geq t_1$, $g_{\mathcal{I}}(x) \leq s_1$, $f_{\mathcal{I}}(y) \geq t_2$, and $g_{\mathcal{I}}(y) \leq s_2$. Hence $x \in U(f_{\mathcal{I}}, t_1) \subseteq U(f_{\mathcal{I}}, \min\{t_1, t_2\})$, $y \in U(f_{\mathcal{I}}, t_2) \subseteq U(f_{\mathcal{I}}, \min\{t_1, t_2\})$, $x \in L(g_{\mathcal{I}}, s_1) \subseteq L(g_{\mathcal{I}}, \max\{s_1, s_2\})$, and $y \in L(g_{\mathcal{I}}, s_2) \subseteq L(g_{\mathcal{I}}, \max\{s_1, s_2\})$. Since $U(f_{\mathcal{I}}, \min\{t_1, t_2\})$ and $L(g_{\mathcal{I}}, \max\{s_1, s_2\})$ are ordered subalgebras of $\mathbf{X} := (X, \to, e, \leq_X)$ by hypothesis, it follows that

$$x \to y \in U(f_{\mathcal{I}}, \min\{t_1, t_2\}) \cap L(g_{\mathcal{I}}, \max\{s_1, s_2\}) = \mathcal{I}^{\in}_{(\min\{t_1,t_2\},\max\{s_1,s_2\})}.$$

Hence $(x \to y)_{(\min\{t_1,t_2\},\max\{s_1,s_2\})} \in \mathcal{I}$, and therefore $\mathcal{I} := \{\langle x, f_{\mathcal{I}}, g_{\mathcal{I}}\rangle \mid x \in X\}$ in X is an intuitionistic fuzzy ordered subalgebra of $\mathbf{X} := (X, \to, e, \leq_X)$.

Corollary 1. *If an intuitionistic fuzzy set $\mathcal{I} := \{\langle x, f_{\mathcal{I}}, g_{\mathcal{I}}\rangle \mid x \in X\}$ in X is an intuitionistic fuzzy ordered subalgebra of $\boldsymbol{X} := (X, \to, e, \leq_X)$, then its $\in_{(t,s)}$-level set is an ordered subalgebra of $\boldsymbol{X} := (X, \to, e, \leq_X)$ for all $(t, s) \in (0, 1] \times [0, 1)$.*

Proof. Straightforward.

We make an intuitionistic fuzzy ordered subalgebra using a collection of ordered subalgebras.

Theorem 11. *Let $\{B_t \mid t \in \Lambda \subseteq [0, 1]\}$ be a collection of ordered subalgebras of $\boldsymbol{X} := (X, \to, e, \leq_X)$ such that X is represented as the union of B_t, i.e., $X = \bigcup_{t\in\Lambda} B_t$, and*

$$(\forall t, s \in \Lambda)(t > s \;\Leftrightarrow\; B_t \subset B_s). \tag{29}$$

Then an intuitionistic fuzzy set $\mathcal{I} := \{\langle x, f_{\mathcal{I}}, g_{\mathcal{I}}\rangle \mid x \in X\}$ in X defined by

$$\begin{aligned} &f_{\mathcal{I}} : X \to [0, 1],\ x \mapsto \sup\{t \in \Lambda \mid x \in B_t\}, \\ &g_{\mathcal{I}} : X \to [0, 1],\ x \mapsto \inf\{t \in \Lambda \mid x \in B_t\} \end{aligned} \tag{30}$$

is an intuitionistic fuzzy ordered subalgebra of $\boldsymbol{X} := (X, \to, e, \leq_X)$.

Proof. According to Theorem 10, it is sufficient to show that $U(f_{\mathcal{I}}, t)$ and $L(g_{\mathcal{I}}, s)$ are ordered subalgebras of $\mathbf{X} := (X, \to, e, \leq_X)$ for every $(t, s) \in (0, 1] \times [0, 1)$. We first show that $U(f_{\mathcal{I}}, t)$ is an ordered subalgebra of $\mathbf{X} := (X, \to, e, \leq_X)$. To do that, we consider the following two cases:

(i) $t = \sup\{k \in \Lambda \mid k < t\}$,

(ii) $t \neq \sup\{k \in \Lambda \mid k < t\}$.

The first case induces

$$(\forall x \in X)\left(x \in U(f_{\mathcal{I}}, t) \Leftrightarrow (\forall k < t)(x \in B_k) \Leftrightarrow x \in \bigcap_{k<t} B_k\right).$$

Hence $U(f_{\mathcal{I}}, t) = \bigcap\limits_{k<t} B_k$ which is an ordered subalgebra of $\mathbf{X} := (X, \rightarrow, e, \leq_X)$. For the second case, let $x \in X$. If $x \in \bigcup\limits_{k \geq t} B_k$, then $x \in B_k$ for some $k \geq t$. Thus

$$f_{\mathcal{I}}(x) = \sup\{t \in \Lambda \mid x \in B_t\} \geq k \geq t,$$

and so $x \in U(f_{\mathcal{I}}, t)$. Hence $\bigcup\limits_{k\geq t} B_k \subseteq U(f_{\mathcal{I}}, t)$. If $x \notin \bigcup\limits_{k \geq t} B_k$, then $x \notin B_k$ for all $k \geq t$. Since $t \neq \sup\{k \in \Lambda \mid k < t\}$, we have $(t - \delta, t) \cap \Lambda = \emptyset$ for some $\delta > 0$. So $x \notin B_k$ for all $k > t - \delta$, which means that if $x \in B_k$, then $k \leq t - \delta$. Thus $f_{\mathcal{I}}(x) \leq t - \delta < t$, i.e., $x \notin U(f_{\mathcal{I}}, t)$. Therefore $U(f_{\mathcal{I}}, t) = \bigcup\limits_{k\geq t} B_k$ and it is an ordered subalgebra of $\mathbf{X} := (X, \rightarrow, e, \leq_X)$. By the similarly, we can verify that $L(g_{\mathcal{I}}, s)$ is an ordered subalgebra of $\mathbf{X} := (X, \rightarrow, e, \leq_X)$. Consequently, $\mathcal{I} := \{\langle x, f_{\mathcal{I}}, g_{\mathcal{I}}\rangle \mid x \in X\}$ is an intuitionistic fuzzy ordered subalgebra of $\mathbf{X} := (X, \rightarrow, e, \leq_X)$ by Theorem 10.

Theorem 12. *Given an intuitionistic fuzzy set $\mathcal{I} := \{\langle x, f_{\mathcal{I}}, g_{\mathcal{I}}\rangle \mid x \in X\}$ in X, its nonempty $\in_{(t,s)}$-level set is an ordered subalgebra of $\boldsymbol{X} := (X, \rightarrow, e, \leq_X)$ for all $(t, s) \in (0.5, 1] \times [0, 0.5)$ if and only if $\mathcal{I} := \{\langle x, f_{\mathcal{I}}, g_{\mathcal{I}}\rangle \mid x \in X\}$ satisfies:*

$$(\forall x, y \in X)\left(\begin{array}{l} e \leq_X x,\ e \leq_X y \\ \quad \Rightarrow \left\{\begin{array}{l} \max\{f_{\mathcal{I}}(x \rightarrow y), 0.5\} \geq \min\{f_{\mathcal{I}}(x), f_{\mathcal{I}}(y)\} \\ \min\{g_{\mathcal{I}}(x \rightarrow y), 0.5\} \leq \max\{g_{\mathcal{I}}(x), g_{\mathcal{I}}(y)\} \end{array}\right. \end{array}\right). \tag{31}$$

Proof. Assume that the $\in_{(t,s)}$-level set $\mathcal{I}^{\in}_{(t,s)}$ is a nonempty ordered subalgebra of $\mathbf{X} := (X, \rightarrow, e, \leq_X)$ for all $(t, s) \in (0.5, 1] \times [0, 0.5)$. If $\mathcal{I}$ does not satisfy (31), then

$$\max\{f_{\mathcal{I}}(a \rightarrow b), 0.5\} < \min\{f_{\mathcal{I}}(a), f_{\mathcal{I}}(b)\}$$

or

$$\min\{g_{\mathcal{I}}(a \rightarrow b), 0.5\} > \max\{g_{\mathcal{I}}(a), g_{\mathcal{I}}(b)\}$$

for some $a, b \in X$ with $e \leq_e a$ and $e \leq_e b$. If we put $t := \min\{f_{\mathcal{I}}(a), f_{\mathcal{I}}(b)\}$ and $s := \max\{g_{\mathcal{I}}(a), g_{\mathcal{I}}(b)\}$, then $t \in (0.5, 1]$ and $s \in [0, 0.5)$, $a, b \in \mathcal{I}^{\in}_{(t,s)}$ but $a \rightarrow b \notin \mathcal{I}^{\in}_{(t,s)}$. This is a contradiction, and so $\max\{f_{\mathcal{I}}(x \rightarrow y), 0.5\} \geq \min\{f_{\mathcal{I}}(x), f_{\mathcal{I}}(y)\}$ and $\min\{g_{\mathcal{I}}(x \rightarrow y), 0.5\} \leq \max\{g_{\mathcal{I}}(x), g_{\mathcal{I}}(y)\}$ for all $x, y \in X$ with $e \leq_e x$ and $e \leq_e y$.

Conversely, suppose that $\mathcal{I}$ satisfies (31). Let $x, y \in X$, $t \in (0.5, 1]$ and $s \in [0, 0.5)$ be such that $e \leq_e x$, $e \leq_e y$ and $x, y \in \mathcal{I}^{\in}_{(t,s)}$. Then

$$\max\{f_{\mathcal{I}}(x \rightarrow y), 0.5\} \geq \min\{f_{\mathcal{I}}(x), f_{\mathcal{I}}(y)\} \geq t > 0.5$$

and

$$\min\{g_{\mathcal{I}}(x \to y), 0.5\} \leq \max\{g_{\mathcal{I}}(x), g_{\mathcal{I}}(y)\} \leq s < 0.5,$$

and so $f_{\mathcal{I}}(x \to y) \geq t$ and $g_{\mathcal{I}}(x \to y) \leq s$. Hence $x \to y \in \mathcal{I}^{\in}_{(t,s)}$, and therefore $\mathcal{I}^{\in}_{(t,s)}$ is an ordered subalgebra of $\mathbf{X} := (X, \to, e, \leq_X)$ for all $t \in (0.5, 1]$ and $s \in [0, 0.5)$.

Given an intuitionistic fuzzy set $\mathcal{I} := \{\langle x, f_{\mathcal{I}}, g_{\mathcal{I}}\rangle \mid x \in X\}$ and $(t, s) \in (0, 1] \times [0, 1)$, the set

$$\mathcal{I}^{q}_{(t,s)} := \{y \in X \mid y_{(t,s)}\, q\, \mathcal{I}\} \tag{32}$$

is called the $q_{(t,s)}$*-level set* of $\mathcal{I}$. It is clear that $\mathcal{I}^{q}_{(t_1,s_1)} \subseteq \mathcal{I}^{q}_{(t_2,s_2)}$ for all $(t_i, s_i) \in (0, 1] \times [0, 1)$, $i = 1, 2$, satisfying $(t_1, s_1) \ll (t_2, s_2)$, i.e., $t_1 \leq t_2$ and $s_1 \geq s_2$. We know that

$$\mathcal{I}^{q}_{(t,s)} := \{y \in X \mid y_{(t,s)}\, q\, \mathcal{I}\} = Q(f_{\mathcal{I}}, t) \cap Q(g_{\mathcal{I}}, s)$$

where $Q(f_{\mathcal{I}}, t) := \{y \in X \mid f(y) > 1 - t\}$ and $Q(g_{\mathcal{I}}, s) := \{y \in X \mid g(y) < 1 - s\}$ which are called the *upper q-level set* and the *lower q-level set* of $\mathcal{I}$ related to t and s, respectively.

Theorem 13. *If* $\mathcal{I} := \{\langle x, f_{\mathcal{I}}, g_{\mathcal{I}}\rangle \mid x \in X\}$ *is an intuitionistic fuzzy ordered subalgebra of* $\boldsymbol{X} := (X, \to, e, \leq_X)$*, then its* $q_{(t,s)}$*-level set is an ordered subalgebra of* $\boldsymbol{X} := (X, \to, e, \leq_X)$ *for all* $(t, s) \in (0, 1] \times [0, 1)$.

Proof. Assume that $\mathcal{I} := \{\langle x, f_{\mathcal{I}}, g_{\mathcal{I}}\rangle \mid x \in X\}$ is an intuitionistic fuzzy ordered subalgebra of $\mathbf{X} := (X, \to, e, \leq_X)$. Let $x, y \in X$ be such that $e \leq_e x$, $e \leq_e y$ and $x, y \in \mathcal{I}^{q}_{(t,s)}$ for all $(t, s) \in (0, 1] \times [0, 1)$. Then $x_{(t,s)}\, q\, \mathcal{I}$ and $y_{(t,s)}\, q\, \mathcal{I}$, that is, $f_{\mathcal{I}}(x) + t > 1$, $g_{\mathcal{I}}(x) + s < 1$, $f_{\mathcal{I}}(y) + t > 1$ and $g_{\mathcal{I}}(y) + s < 1$. Hence

$$f_{\mathcal{I}}(x \to y) + t \geq \min\{f_{\mathcal{I}}(x), f_{\mathcal{I}}(y)\} + t = \min\{f_{\mathcal{I}}(x) + t, f_{\mathcal{I}}(y) + t\} > 1$$

and

$$g_{\mathcal{I}}(x \to y) + s \leq \max\{g_{\mathcal{I}}(x), g_{\mathcal{I}}(y)\} + s = \max\{g_{\mathcal{I}}(x) + s, g_{\mathcal{I}}(y) + s\} < 1,$$

and so $(x \to y)_{(t,s)}\, q\, \mathcal{I}$, i.e., $x \to y \in \mathcal{I}^{q}_{(t,s)}$. Consequently, $\mathcal{I}^{q}_{(t,s)}$ is an ordered subalgebra of $\mathbf{X} := (X, \to, e, \leq_X)$ for all $(t, s) \in (0, 1] \times [0, 1)$.

The example below describes Theorem 13.

Example 3. *Consider the OBCI-algebra* $\boldsymbol{X} := (X, \to, e, \leq_X)$ *in Example 2. Define an intuitionistic fuzzy set* $\mathcal{I} := \{\langle x, f_{\mathcal{I}}, g_{\mathcal{I}}\rangle \mid x \in X\}$ *in* X *as follows:*

$$f_{\mathcal{I}} : X \to [0, 1],\ x \mapsto \begin{cases} 0.74 & \text{if } x = 1, \\ 0.53 & \text{if } x = \frac{3}{4}, \\ 0.48 & \text{if } x = \frac{1}{2}, \\ 0.36 & \text{if } x = \frac{1}{4}, \\ 0.67 & \text{if } x = 0, \end{cases}$$

and

$$g_{\mathcal{I}} : X \to [0,1],\ x \mapsto \begin{cases} 0.24 & \text{if } x = 1, \\ 0.13 & \text{if } x = \frac{3}{4}, \\ 0.21 & \text{if } x = \frac{1}{2}, \\ 0.46 & \text{if } x = \frac{1}{4}, \\ 0.24 & \text{if } x = 0. \end{cases}$$

The upper t-level set $U(f_{\mathcal{I}}, t)$ and the lower s-level set $L(g_{\mathcal{I}}, s)$ of $\mathcal{I}$ are calculated as follows:

$$U(f_{\mathcal{I}}, t) = \begin{cases} \emptyset & \text{if } t \in (0.74, 1], \\ \{1\} & \text{if } t \in (0.67, 0.74], \\ \{1, 0\} & \text{if } t \in (0.53, 0.67], \\ \{1, 0, \frac{3}{4}\} & \text{if } t \in (0.48, 0.53], \\ \{1, 0, \frac{3}{4}, \frac{1}{2}\} & \text{if } t \in (0.36, 0.48], \\ X & \text{if } t \in (0, 0.36], \end{cases}$$

and

$$L(g_{\mathcal{I}}, s) = \begin{cases} \emptyset & \text{if } s \in [0, 0.13), \\ \{\frac{3}{4}\} & \text{if } s \in [0.13, 0.21), \\ \{\frac{3}{4}, \frac{1}{2}\} & \text{if } s \in [0.21, 0.24), \\ \{1, 0, \frac{3}{4}, \frac{1}{2}\} & \text{if } s \in [0.24, 0.46), \\ X & \text{if } s \in [0.46, 1). \end{cases}$$

It is routine to verify that $U(f_{\mathcal{I}}, t)$ and $L(g_{\mathcal{I}}, s)$ are ordered subalgebras of $\boldsymbol{X} := (X, \to, e, \leq_X)$ for all $(t, s) \in (0, 1] \times [0, 1)$. Hence $\mathcal{I} := \{\langle x, f_{\mathcal{I}}, g_{\mathcal{I}}\rangle \mid x \in X\}$ is an intuitionistic fuzzy ordered subalgebra of $\boldsymbol{X} := (X, \to, e, \leq_X)$ by Theorem 10. The upper q-level sets $Q(f_{\mathcal{I}}, t)$ related to t are provided by Tables 3.

Table 3: Calculation of $Q(f_{\mathcal{I}}, t)$

t	$1-t$	$Q(f_{\mathcal{I}}, t)$
$(0.74, 1]$	$[0, 0.26)$	X
$(0.67, 0.74]$	$[0.26, 0.33)$	X
$(0.53, 0.67]$	$[0.33, 0.47)$	X or $\{1, 0, \frac{3}{4}, \frac{1}{2}\}$
$(0.48, 0.53]$	$[0.47, 0.52)$	$\{1, 0, \frac{3}{4}, \frac{1}{2}\}$ or $\{1, 0, \frac{3}{4}\}$
$(0.36, 0.48]$	$[0.52, 0.64)$	$\{1, 0, \frac{3}{4}\}$ or $\{1, 0\}$
$(0, 0.36]$	$[0.64, 1)$	$\{1, 0\}$, $\{1\}$ or $\emptyset$

The lower q-level sets $Q(g_{\mathcal{I}}, s)$ related to s are provided by Tables 4.
We can observe that $Q(f_{\mathcal{I}}, t)$ and $Q(g_{\mathcal{I}}, s)$ are ordered subalgebras of $\boldsymbol{X} := (X, \to, e, \leq_X)$ for all $(t, s) \in (0, 1] \times [0, 1)$. Hence $\mathcal{I}^q_{(t,s)} = Q(f_{\mathcal{I}}, t) \cap Q(g_{\mathcal{I}}, s)$ is an ordered subalgebra of $\boldsymbol{X} := (X, \to, e, \leq_X)$ for all $(t, s) \in (0, 1] \times [0, 1)$, and they are displayed as follows: $\{1\}$, $\{\frac{3}{4}\}$, $\{1, 0\}$, $\{\frac{3}{4}, \frac{1}{2}\}$, $\{1, 0, \frac{3}{4}\}$, $\{1, 0, \frac{3}{4}, \frac{1}{2}\}$ and X.

Table 4: Calculation of $Q(g_{\mathcal{I}}, s)$

s	$1-s$	$Q(g_{\mathcal{I}}, s)$
$[0, 0.13)$	$(0.87, 1]$	X
$[0.13, 0.21)$	$(0.79, 0.87]$	X
$[0.21, 0.24)$	$(0.76, 0.79]$	X
$[0.24, 0.46)$	$(0.54, 0.76]$	X
$[0.46, 1)$	$(0, 0.54]$	$\{\frac{3}{4}\}$, $\{\frac{3}{4}, \frac{1}{2}\}$, $\{1, 0, \frac{3}{4}, \frac{1}{2}\}$ or X

Proposition 2. *Let $\mathcal{I} := \{\langle x, f_{\mathcal{I}}, g_{\mathcal{I}}\rangle \mid x \in X\}$ be an intuitionistic fuzzy set in X. For every $(t_i, s_i) \in (0, 0.5] \times [0.5, 1)$, $i = 1, 2$, if the $q_{(t_i,s_i)}$-level set of $\mathcal{I}$ is an ordered subalgebra of $\boldsymbol{X} := (X, \to, e, \leq_X)$, then $\mathcal{I} := \{\langle x, f_{\mathcal{I}}, g_{\mathcal{I}}\rangle \mid x \in X\}$ satisfies:*

$$(\forall x, y \in X)\left(\begin{array}{c}\left\{\begin{array}{l} x \in \mathcal{I}^q_{(t_1,s_1)},\ e \leq_e x \\ y \in \mathcal{I}^q_{(t_2,s_2)},\ e \leq_e y \end{array}\right\} \\ \Rightarrow x \to y \in \mathcal{I}^{\in}_{(\max\{t_1,t_2\},\min\{s_1,s_2\})}\end{array}\right). \tag{33}$$

Proof. Assume that the $q_{(t_i,s_i)}$-level set $\mathcal{I}^q_{(t_i,s_i)}$ is an ordered subalgebra of $\mathbf{X} := (X, \to, e, \leq_X)$ for all $(t_i, s_i) \in (0, 0.5] \times [0.5, 1)$, $i = 1, 2$, Let $x, y \in X$ and $(t_i, s_i) \in (0, 0.5] \times [0.5, 1)$ be such that $x \in \mathcal{I}^q_{(t_1,s_1)}$, $y \in \mathcal{I}^q_{(t_2,s_2)}$, $e \leq_e x$ and $e \leq_e y$. Then

$$\begin{aligned} 1 &< f_{\mathcal{I}}(x) + t_1 \leq f_{\mathcal{I}}(x) + \max\{t_1, t_2\}, \\ 1 &< f_{\mathcal{I}}(y) + t_2 \leq f_{\mathcal{I}}(y) + \max\{t_1, t_2\}, \\ 1 &> g_{\mathcal{I}}(x) + s_1 \geq g_{\mathcal{I}}(x) + \min\{s_1, s_2\}, \\ 1 &> g_{\mathcal{I}}(y) + s_2 \geq g_{\mathcal{I}}(y) + \min\{s_1, s_2\}. \end{aligned}$$

Hence $x, y \in \mathcal{I}^q_{(\max\{t_1,t_2\},\min\{s_1,s_2\})}$, and so $x \to y \in \mathcal{I}^q_{(\max\{t_1,t_2\},\min\{s_1,s_2\})}$ since $\max\{t_1, t_2\} \in (0, 0.5]$, $\min\{s_1, s_2\} \in [0.5, 1)$ and $\mathcal{I}^q_{(\max\{t_1,t_2\},\min\{s_1,s_2\})}$ is an ordered subalgebra of $\mathbf{X} := (X, \to, e, \leq_X)$. Thus

$$f_{\mathcal{I}}(x \to y) > 1 - \max\{t_1, t_2\} \geq \max\{t_1, t_2\}$$

and

$$g_{\mathcal{I}}(x \to y) < 1 - \min\{s_1, s_2\} \leq \min\{s_1, s_2\}$$

because of $\max\{t_1, t_2\} \leq 0.5$ and $\min\{s_1, s_2\} \geq 0.5$. Therefore

$$x \to y \in \mathcal{I}^{\in}_{(\max\{t_1,t_2\},\min\{s_1,s_2\})}$$

which shows that (33) is valid.

Proposition 3. *Let $\mathcal{I} := \{\langle x, f_{\mathcal{I}}, g_{\mathcal{I}}\rangle \mid x \in X\}$ be an intuitionistic fuzzy set in X. For every $(t_i, s_i) \in (0.5, 1] \times [0, 0.5)$, $i = 1, 2,$, if the $q_{(t_i,s_i)}$-level set of $\mathcal{I}$ is an ordered subalgebra* *of* $\boldsymbol{X} := (X, \rightarrow, e, \leq_X)$*, then* $\mathcal{I} := \{\langle x, f_{\mathcal{I}}, g_{\mathcal{I}}\rangle \mid x \in X\}$ *satisfies:*

$$(\forall x, y \in X)\left(\begin{array}{c}\left\{\begin{array}{l} x \in \mathcal{I}^{\in}_{(t_1,s_1)},\, e \leq_e x \\ y \in \mathcal{I}^{\in}_{(t_2,s_2)},\, e \leq_e y \end{array}\right\} \\ \Rightarrow x \rightarrow y \in \mathcal{I}^{q}_{(\max\{t_1,t_2\},\min\{s_1,s_2\})}\end{array}\right). \tag{34}$$

Proof. Suppose that the $q_{(t_i,s_i)}$-level set $\mathcal{I}^{q}_{(t_i,s_i)}$ is an ordered subalgebra of $\mathbf{X} := (X, \rightarrow, e, \leq_X)$ for all $(t_i, s_i) \in (0.5, 1] \times [0, 0.5)$, $i = 1, 2$. Let $x, y \in X$ and $(t_i, s_i) \in (0.5, 1] \times [0, 0.5)$ be such that $x \in \mathcal{I}^{\in}_{(t_1,s_1)}$, $y \in \mathcal{I}^{\in}_{(t_2,s_2)}$, $e \leq_e x$ and $e \leq_e y$. Then

$$f_{\mathcal{I}}(x) \geq t_1 > 1 - t_1,\; g_{\mathcal{I}}(x) \leq s_1 < 1 - s_1,$$
$$f_{\mathcal{I}}(y) \geq t_2 > 1 - t_2,\; g_{\mathcal{I}}(y) \leq s_2 < 1 - s_2,$$

i.e., $x_{(t_1,s_1)}\, q\, \mathcal{I}$ and $y_{(t_2,s_2)}\, q\, \mathcal{I}$. Hence $x \in \mathcal{I}^{q}_{(t_1,s_1)} \subseteq \mathcal{I}^{q}_{(\max\{t_1,t_2\},\min\{s_1,s_2\})}$ and $y \in \mathcal{I}^{q}_{(t_2,s_2)} \subseteq \mathcal{I}^{q}_{(\max\{t_1,t_2\},\min\{s_1,s_2\})}$. Since $\max\{t_1, t_2\} \in (0.5, 1]$, $\min\{s_1, s_2\} \in [0.0.5)$ and $\max\{t_1, t_2\} + \min\{s_1, s_2\} \leq 1$, it follows from the hypothesis that $\mathcal{I}^{q}_{(\max\{t_1,t_2\},\min\{s_1,s_2\})}$ is an ordered subalgebra of $\mathbf{X} := (X, \rightarrow, e, \leq_X)$. Hence $x \rightarrow y \in \mathcal{I}^{q}_{(\max\{t_1,t_2\},\min\{s_1,s_2\})}$.

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