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Solving n^{th}-order integro-differential equations by novel generalized hybrid transform

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Abstract. Recently, Shehu has introduced an integral transform called Shehu transform, which generalizes the two well-known integrals transforms, i.e. Laplace and Sumudu transform. In the literature, many integral transforms were used to compute the solution of integro-differential equations (IDEs). In this article, for the first time, we use Shehu transform for the computation of solution of n^{th}-order IDEs. We present a general scheme of solution for n^{th}-order IDEs. We give some examples with detailed solutions to show the appropriateness of the method. We present the accuracy, simplicity, and convergence of the proposed method through tables and graphs.

2020 Mathematics Subject Classifications: 44A10, 44A35, 44A40, 45E10, 45G10, 47G10

Key Words and Phrases: Integro-differential equations, Shehu transform, Integral transforms

1. Introduction and motivation

Volterra identified the genetic factors when studying a population growth model. He introduced a new topic in which both differential and integral operators appeared in the same equation, known as Volterra IDEs [27, 29]. IDEs have recently piqued the interest of researchers due to their wide range of applications in fields such as fluid dynamics,

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circuit analysis, epidemiology, infectious diseases, and heat flow [30]. The solution to such problems is linked to the solution of the Volterra form of IDEs. As a result, various methods for solving IDEs have been used by researchers. Laplace and Fourier introduced integral transforms, which are the most widely used in the literature and recently applied to many other integral transforms that can solve differential and integral equations [6, 14, 18]. The main difference between the Laplace from the FT, FT (Fourier transform) can only be defined on a stable system, while the Laplace transform can be defined for the system which is stable or unstable. Another integral transform is the Mellin transform, which is used in applied sciences due to its invariant property [21]. Many integral transforms introduced in the last few decades, including the Hankel's integral transform [17], Sumudu integral transform [41], Elzaki transforms [20], natural transform [26], Abdon-Kilicman integral transform [15], the Yang transform [42], and others. Some existing integral transforms, however, are incapable of solving models containing nonlinear terms. As a result, several researchers are interested in a different approach to solving real-world problems. The double LADM is used to solve linear and nonlinear PDEs in [19]. Belgacem et al. in 2017 [16] applied Natural transform (NT) and Sumudu transform (ST) to solve Stokes equation and diffusion equation of fractional order. However, since physical phenomena are almost nonlinear, we're interested in nonlinear integro-differential equations. The n^{th}-order nonlinear IDE is given by:

$$\mathbf{G}^{(n)}(x) + \mathtt{u}(x)\mathbf{G}(x) + \int_c^d \mathcal{K}(x,t)\mathbf{G}^{(p)}(t)dt = \mathtt{h}(x), \quad c < x < d, \tag{1}$$

with initial conditions $\mathbf{G}^j(0) = \beta_j$, where $\beta_j \in \mathbb{R}$ for $j = 0, 1, ..., n-1$,and $p \geq 1$.

Maitama and Zhao successfully derived an integral transform known as the Shehu transform from the classical FT in 2019 and demonstrated its accuracy, validity, and simplicity by applying it to both ODEs and PDEs [31]. Further, Adomian [22, 23] introduced a novel and efficient approach (named the Adomian decomposition method) for solving linear as well as nonlinear equations at the beginning of the 1980s. This approach quickly converges the solutions sequence to linear and nonlinear deterministic and stochastic equations. The purpose of the current study is to solve n^{th}-order IDEs by using a novel generalized transform called Hybrid Shehu transform (HST). The HST consists of approximating the solution as

$$\mathbf{G}(x) = \sum_{q=0}^{\infty} \mathbf{G}_q(x), \tag{2}$$

and for $p \geq 2$, the nonlinear term $\mathbf{N}(\mathbf{G}^{(p)})$ (if any) will decomposed by

$$\mathbf{N}(\mathbf{G}^{(p)}) = \sum_{q=0}^{\infty} \mathbb{A}_q, \tag{3}$$

where $\mathbb{A}_q$(Adomian polynomials) is defined as

$$\mathbb{A}_q = \frac{1}{\Gamma(q+1)} \frac{d^q}{d\eta^q} \left[\mathbf{N} \left(\sum_{q=0}^{\infty} \eta^q \mathbf{G}_q \right)^p \right]_{\eta=0}, \quad q = 0, 1, 2, \cdots .$$

The convergence of the series (2) and (3) in [1, 5].

2. Preliminaries

Definition 1. *[31] The Shehu transform of the function* $\mathbf{G}(x)$ *by the following integral*

$$\mathbb{S}\left[\mathbf{G}(x)\right] = \mathbf{H}(s,u) = \int_0^{\infty} \exp\left(\frac{-sx}{u}\right)\mathbf{G}(x)dx, \tag{4}$$

provided that the integral converges.

Definition 2. *[31] The Shehu transform is linear, i.e, for any constants* $\mathbf{k_1}, \mathbf{k_2} \neq 0$,*we have*

$$\mathbb{S}\left[\mathbf{k_1}\mathbf{G}(x) + \mathbf{k_2}\mathbf{J}(x)\right] = \mathbf{k_1}\mathbb{S}\left[\mathbf{G}(x)\right] + \mathbf{k_2}\mathbb{S}\left[\mathbf{J}(x)\right].$$

Definition 3. *[31] The formula for the Shehu transform of* $n^{th}-$*order derivative of* $\mathbf{G}(x)$ *is represented as:*

$$\mathbb{S}\left[\mathbf{G}^{(n)}(x)\right] = \frac{s^n}{u^n}\mathbf{G}(s,u) - \left(\frac{s}{u}\right)^{(n-1)}\mathbf{G}(0) - \left(\frac{s}{u}\right)^{(n-2)}\mathbf{G}'(0) - \cdots - \mathbf{G}^{(n-1)}(0). \tag{5}$$

3. Solution procedure using HST

The n^{th} order nonlinear IDE (1) can also written as

$$\mathbf{G}^{(n)}(x) = \mathtt{h}(x) - \mathtt{u}(x)\mathbf{G}(x) - \int_c^d \mathcal{K}(x,t)\mathbf{G}^{(p)}(t)dt, \quad c < x < d. \tag{6}$$

Applying Shehu transform to both side of the (6) and keep in mind the fact that the Convolution theorem holds for Shehu transform

$$\begin{aligned}
&\frac{s^n}{u^n}\mathbf{G}(s,u) - \left(\frac{s}{u}\right)^{(n-1)}\mathbf{G}(0) - \left(\frac{s}{u}\right)^{(n-2)}\mathbf{G}'(0) - \cdots - \mathbf{G}^{(n-1)}(0) \\
&= \mathbb{S}\left[\mathtt{h}(x)\right] - \mathbb{S}\left[\mathtt{u}(x) * \mathbf{G}(x)\right](s,u) - \mathbb{S}\left[\int_c^d \mathcal{K}(x,t)\mathbf{G}^{(p)}(t)dt\right] \\
&= \mathbb{S}\left[\mathtt{h}(x)\right] - \mathbb{S}\left[\mathtt{u}(x)\right]\mathbb{S}\left[\mathbf{G}(x)\right] - \int_c^d \mathbb{S}\left[\mathcal{K}(x,t)\right]\mathbf{G}^{(p)}(t)dt,
\end{aligned}$$

this can be reduce to

$$\mathbf{G}(s,u) = \begin{cases} \frac{u^n\left[\left(\frac{s}{u}\right)^{(n-1)}\mathbf{G}(0) - \left(\frac{s}{u}\right)^{(n-2)}\mathbf{G}'(0) - \cdots - \mathbf{G}^{(n-1)}(0)\right]}{s^n + u^n\mathbb{S}[\mathtt{u}(x)]} + \frac{u^n\mathbb{S}[\mathtt{h}(x)]}{s^n + u^n\mathbb{S}[\mathtt{u}(x)]} \\ -\frac{u^n}{s^n + u^n\mathbb{S}[\mathtt{u}(x)]}\int_c^d \mathbb{S}\left[\mathcal{K}(x,t)\right]\mathbf{G}^{(p)}(t)dt, \end{cases} \tag{7}$$

substituting Equ. (2) and (3) into Eq. (7), we get

$$\mathbb{S}\left[\sum_{q=0}^{\infty}\mathbf{G}_q(x)\right] = \frac{u^n\left[\left(\frac{s}{u}\right)^{(n-1)}\mathbf{G}(0) - \left(\frac{s}{u}\right)^{(n-2)}\mathbf{G}'(0) - \cdots - \mathbf{G}^{(n-1)}(0)\right]}{s^n + u^n\mathbb{S}\left[\mathtt{u}(x)\right]}$$

$$+\frac{u^n S\left[\mathtt{h}(x)\right]}{s^n+u^n\mathbb{S}\left[\mathtt{u}(x)\right]}-\frac{u^n}{s^n+u^n\mathbb{S}\left[\mathtt{u}(x)\right]}\int_c^d \mathbb{S}\left[\mathcal{K}(x,t)\right]\sum_{q=0}^{\infty}\mathbb{A}_q(t)dt,$$

the HST method and comparing terms gives

$$\begin{cases}\mathbb{S}\left[\mathbf{G}_0(x)\right] & =\frac{u^n\left[\left(\frac{s}{u}\right)^{(n-1)}\mathbf{G}(0)-\left(\frac{s}{u}\right)^{(n-2)}\mathbf{G}'(0)-\cdots-\mathbf{G}^{(n-1)}(0)\right]}{s^n+u^n\mathbb{S}[\mathtt{u}(x)]} \\ & +\frac{u^n\mathbb{S}[\mathtt{h}(x)]}{s^n+u^n\mathbb{S}[\mathtt{u}(x)]}\end{cases} \tag{8}$$

The general can be obtained as

$$\mathbb{S}\left[\mathbf{G}_{q+1}(x)\right]=-\frac{u^n}{s^n+u^n\mathbb{S}\left[\mathtt{u}(x)\right]}\int_c^d \mathbb{S}\left[\mathcal{K}(x,t)\right]\sum_{q=0}^{\infty}\mathbb{A}_q(t)dt, \tag{9}$$

for $q=0,1,2,\cdots$. A sufficient condition for (9) to comply is that

$$\lim_{s\to\infty}\frac{u^n}{s^n+u^n\mathbb{S}\left[\mathtt{u}(x)\right]}=0.$$

Application of the inverse Shehu Transform (8) gives $\mathbf{G}_0(x)$, and using the recursive relation (9) gives the other terms $\mathbf{G}_q(x)$, $q\geq 0$ as

$$\phi_q\left[\mathbf{G}(x)\right]=\sum_{r=0}^{q-1}\mathbf{G}_r(x),$$

with

$$\lim_{q\to\infty}\phi_q\left[\mathbf{G}(x)\right]=\mathbf{G}(x).$$

The following theorem and examples show the convergence of the proposed method.

3.0.1. Convergence theorem and error estimate

Theorem 1 (Convergence of the proposed method)**.** *Let $\mathcal{H}$ be a Hilbert space and "$\mathbf{G}$" be the exact salution of the problem (6) and $\sum_{q=0}^{\infty}\mathbf{G}_q$be approximate solution of the problem (6) which is obtained by (HST), will converges to "$\mathbf{G}$" when $\exists 0\leq\alpha\leq 1,\|\mathbf{G}_{k+1}\|\leq\alpha\|\mathbf{G}\|$, $\forall\ k\ \epsilon\ Z^{+}$.*

Proof. Let we have

$$\begin{aligned}\mathbf{U_0} &= \mathbf{G}_0,\\ \mathbf{U_1} &= \mathbf{G}_0+\mathbf{G}_1,\\ \mathbf{U_2} &= \mathbf{G}_0+\mathbf{G}_1+\mathbf{G}_2,\\ &\vdots\end{aligned}$$

$$\mathbf{U_q} \quad = \quad \mathbf{G}_1 + \mathbf{G}_2 + \ldots + \mathbf{G_q},$$

and we have to show that $\{\mathbf{U_q}\}_{\mathbf{q}=0}^{\infty}$ is Cauchy Sequance in the Hilbert space "$\mathcal{H}$" .Therfore consider

$$\|\mathbf{U_{q+1}} - \mathbf{U_q}\| \quad = \quad \|\mathbf{G_{q+1}}\| \leq \alpha\|\mathbf{G_q}\| \leq \alpha^2\|\mathbf{G_{q-1}}\| \leq \ldots \leq \alpha^{\mathbf{q}+1}\|\mathbf{G}_0\|$$

But for every $\mathbf{q}, \mathbf{m} \epsilon N$, such that $\mathbf{q} \geq \mathbf{m}$, So we have

$$\begin{aligned}
\|\mathbf{U_q} - \mathbf{U_m}\| &= \|(\mathbf{U_q} - \mathbf{U_{q-1}}) + (\mathbf{U_{q-1}} - \mathbf{U_{q-2}}) + \ldots + (\mathbf{U_{m+1}} - \mathbf{U_m})\| \\
&\leq \|(\mathbf{U_q} - \mathbf{U_{q-1}})\| + \|(\mathbf{U_{q-1}} - \mathbf{U_{q-2}})\| + \ldots + \|(\mathbf{U_{m+1}} - \mathbf{U_m})\| \\
&\leq \alpha^{\mathbf{q}}\|\mathbf{G}_0\| + \alpha^{\mathbf{q}-1}\|\mathbf{G}_0\| + \ldots + \alpha^{\mathbf{q}+1}\|\mathbf{G}_0\| \\
&\leq (\alpha^{\mathbf{q}+1} + \alpha^{\mathbf{q}+2}\ldots)\|\mathbf{G}_0\| = \frac{\alpha^{\mathbf{q}+1}}{1-\alpha}\|\mathbf{G}_0\|
\end{aligned}$$

Hance

$$\lim_{\mathbf{q},\mathbf{m}\to\infty} \|\mathbf{U_n} - \mathbf{U_m}\| = 0$$

i-e $\{\mathbf{U_n}\}_{\mathbf{q}=0}^{\infty}$ is Cauchy Sequance in the Hilbert space "$\mathcal{H}$" and it implise that $\exists \mathbf{U} \epsilon \mathcal{H}$, such that $\lim_{\mathbf{q}\to\infty} \mathbf{U_q} = \mathbf{U}$, i-e $\mathbf{U} = \sum_{\mathbf{q}=0}^{\infty} \mathbf{G_q}$. This ends the proof.

Theorem 2 (Error estimate). *Let* $\sum_{i=0}^{j} \mathbf{G}_i < \infty$ *and "*$\mathbf{G}$*" be its approximate solution . Let* $\zeta > 0$ *such that* $\|\mathbf{G}_{i+1}\| \leq \zeta\|\mathbf{G}_i\|$*, then the maximum absolute error is*

$$\|\mathbf{G} - \sum_{i=0}^{j} \mathbf{G}_i\| < \frac{\zeta^{j+1}}{1-\zeta}\|\mathbf{G}_0\|.$$

Proof. Since $\sum_{i=0}^{j} \mathbf{G}_i < \infty$ this indicates that $\sum_{i=0}^{j} \mathbf{G}_i$ is finite. Consider

$$\begin{aligned}
\|\mathbf{G} - \sum_{i=0}^{j} \mathbf{G}_i\| &= \|\sum_{i=j+1}^{\infty} \mathbf{G}_i\| \\
&\leq \sum_{i=0}^{\infty} \|\mathbf{G}_i\| \\
&\leq \sum_{i=0}^{j} \zeta^j \|\mathbf{G}_0\| \\
&\leq \zeta^{j+1}(1 + \zeta + \zeta^2 + \cdots)\|\mathbf{G}_0\| \\
&\leq \frac{\zeta^{j+1}}{1-\zeta}\|\mathbf{G}_0\|.
\end{aligned}$$

This ends the proof.

4. Applications

The proposed method for solving nth-order IDEs is demonstrated in this section with three examples. To demonstrate validity and efficiency of the results obtained using the current method, we provide comparison between exact and approximate solution. For numerical values of absolute error, we define absolute error as:

$$E_q = \left|\mathbf{G}_{\text{exact}} - \phi_q(x)\right|,$$

where $q = 0, 1, 2, 3\cdots$ represent the number of the iterations.

Example 1. *Consider the second-order IDE as*

$$\mathbf{G}''(x) = \exp(x) - x + \int_0^1 xt\mathbf{G}(t)dt, \tag{10}$$

under the initial conditions (ICs) $\mathbf{G}(0) = 1$, $\mathbf{G}'(0) = 1$.

Solution. Applying Shehu transform to (10), we have

$$\begin{aligned}
\mathbb{S}\left[\mathbf{G}''(x)\right] &= \mathbb{S}\left[\exp(x)\right] - \mathbb{S}\left[x\right] + \mathbb{S}\left[\int_0^1 xt\mathbf{G}(t)dt\right] \\
\frac{s^2}{u^2}Y(s,u) &= \frac{s}{u}\mathbf{G}(0) + \mathbf{G}'(0)\frac{u}{s-u} - \frac{u^2}{s^2} + \int_0^1 \mathbb{S}\left[x\right] t\mathbf{G}(t)dt,
\end{aligned}$$

where $\mathbb{S}\left[\mathbf{G}(x)\right](s,u) = Y(s,u)$, using ICs, we have

$$Y(s,u) = \frac{u^2}{s^2} + \frac{u}{s} + \frac{u^3}{s^2(s-u)} - \frac{u^4}{s^4} + \frac{u^4}{s^4}\int_0^1 t\mathbf{G}(t)dt,$$

putting the series solution (2) for $Y(s,u)$ in the above equation, one can get

$$Y_0(s,u) = \frac{u^2}{s^2} + \frac{u}{s} + \frac{u^3}{s^2(s-u)} - \frac{u^4}{s^4}, \tag{11}$$

and by recursive relation, we obtain

$$\mathbb{S}\left[\mathbf{G}_{q+1}(x)\right](s,u) = \frac{u^4}{s^4}\int_0^1 t\mathbf{G}_q(t)dt.$$

Now applying Shehu inverse of both side of (11) give $\mathbf{G}_0(x)$ and using the recursive relation for $n = 1, 2, 3, \ldots$, we get

$$\begin{aligned}
\mathbf{G}_0(x) &= \exp(x) - \frac{x^3}{3!}, \\
\mathbf{G}_1(x) &= \frac{29}{3!\cdot 30}x^3, \\
\mathbf{G}_2(x) &= \frac{29}{3!\cdot 30^2}x^3,
\end{aligned}$$

$$\vdots$$

$$\mathbf{G}_q(x) \qquad \frac{29}{3!30^q}x^3.$$

Thus, the desired approximate solution for $q = 1, 2, 3, \cdots$ is given by

$$\phi_q(x) = \sum_{r=0}^{q-1} \mathbf{G}_r(x) \quad = \quad \exp(x) - \frac{x^3}{6 \cdot 30^{q-1}}.$$

Hence,

$$\mathbf{G}(x) = \lim_{q\to\infty} \phi_q(x) \quad = \quad \lim_{q\to\infty} \exp(x) - \lim_{q\to\infty} \frac{x^3}{6 \cdot 30^{q-1}} = \exp(x),$$

which the exact solution of (10).

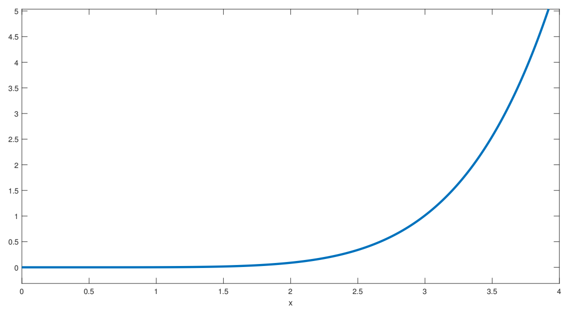

Figure 1: Solution curves of Example 1.

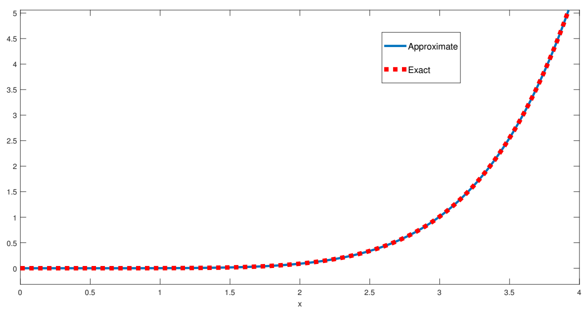

Figure 2: Comparison between approximate and exact solution of Example 1.

Example 2. *Consider the third-order IDE as*

$$\left\{ \mathbf{G}'''(x) = \sin(x) - x - \int_0^{\frac{\Pi}{2}} xt\mathbf{G}'(t)dt, \quad \mathbf{G}(0) = 1,\ \mathbf{G}'(0) = 0, \mathbf{G}''(0) = -1 \cdot \right. \tag{12}$$

x	E_3	E_6	E_8
0.1	1.8518E-06	6.8587E-12	7.6207E-15
0.2	1.3714E-06	5.3971E-11	6.1877E-14
0.3	5.0000E-05	1.8518E-10	2.0576E-13
0.4	1.1852E-05	4.2998E-10	4.9321E-13
0.5	2.3148E-05	9.0000E-11	9.5259E-13
0.6	4.1011E-05	1.3998E-09	1.5982E-12
0.7	6.3518E-05	2.3525E-09	2.6139E-12
0.8	9.5001E-05	3.4998E-09	3.8979E-12
0.9	1.3500E-04	5.0000E-09	5.5555E-12
1.0	1.7809E-04	6.9001E-09	9.5998E-12

Table 1: Absolute error for Example 1.

Solution. Taking the Shehu transform of (12), we obtain

$$\mathbb{S}\left[\mathbf{G}'''(x)\right] = \mathbb{S}\left[\sin(x) - x\right] - \mathbb{S}\left[\int_0^{\frac{\Pi}{2}} xt\mathbf{G}'(t)dt\right],$$

so that

$$\frac{s^3}{u^3}Y(s,u) - \frac{s^2}{u^2}\mathbf{G}(0) - \frac{s}{u}\mathbf{G}'(0) - \mathbf{G}''(0) = \frac{u^2}{s^2+u^2} - \frac{u^2}{s^2} - \frac{u^2}{s^2}\int_0^{\frac{\Pi}{2}} t\mathbf{G}'(t)dt,$$

by using the IC, we get

$$Y(s,u) = \frac{u}{s} - \frac{u^3}{s^3} + \frac{u^5}{s^3(s^2+u^2)} - \frac{u^5}{s^5} - \frac{u^5}{s^5}\int_0^{\frac{\Pi}{2}} t\mathbf{G}'(t)dt,$$

where $\mathbb{S}\left[\mathbf{G}(x)\right](s,u) = Y(s,u)$, substituting (2) for $Y(s,u)$ and comparing terms, we have

$$Y_0(s,u) = \frac{u}{s} - \frac{u^3}{s^3} + \frac{u^5}{s^3(s^2+u^2)} - \frac{u^5}{s^5} - \frac{u^5}{s^5}, \tag{13}$$

using the recursive relation we get

$$\mathbb{S}\left[\mathbf{G}_{q+1}(x)\right](s,u) = -\frac{u^5}{s^5}\int_0^{\frac{\Pi}{2}} t\mathbf{G}'_q(t)dt. \tag{14}$$

Taking the inverse Shehu transform of (13) and (14) gives :

$$\begin{aligned}
\mathbf{G}_0(x) &= \cos(x) - \frac{x^4}{4!},\\
\mathbf{G}_1(x) &= \frac{-(\pi^5+960)}{4!\cdot(960)}x^4,\\
\mathbf{G}_2(x) &= \frac{(\pi^5+960)\cdot\pi^5}{4!\cdot(960)^2}x^4,
\end{aligned}$$

$$\vdots$$

$$\mathbf{G}_q(x) = \frac{(-1)^q \cdot (\pi^5 + 960) \cdot \pi^{5(q-1)}}{4! \cdot (960)^q} x^4, \quad q = 1, 2, 3 \cdots$$

the desired solution is as follow

$$\phi_q(x) = \sum_{r=0}^{q-1} \mathbf{G}_r(x) = \cos(x) + \frac{(-1)^q \cdot \pi^{5(q-1)}}{4! \cdot (960)^{q-1}} x^4, \quad q = 1, 2, 3 \cdots$$

$$\mathbf{G}(x) = \lim_{q\to\infty} \phi_q(x) = \lim_{q\to\infty} \left(\cos(x) + \frac{(-1)^q \cdot \pi^{5(q-1)}}{4! \cdot (960)^{q-1}} x^4 \right) = \cos(x).$$

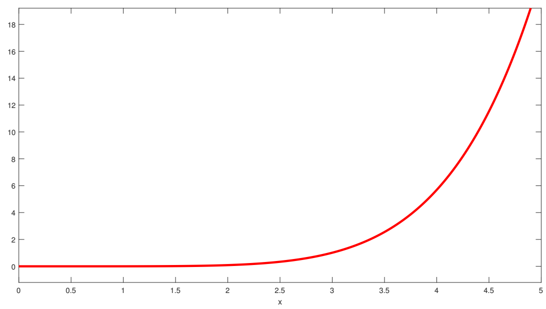

Figure 3: Solution curves of Example 2.

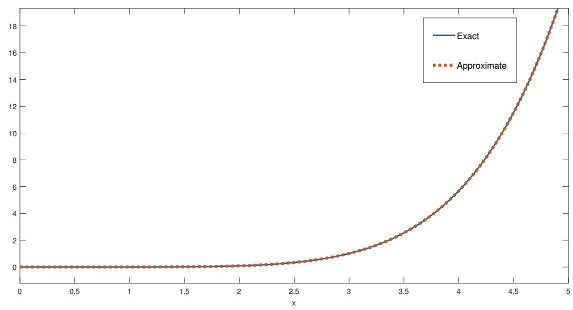

Figure 4: Comparison between exact and approximate solution Example 2.

Example 3. *Consider the 5th order integro-differential equation as*

$$\mathbf{G}^{(5)}(x) = x + \int_0^1 (t - x)\mathbf{G}^2(t)dt,$$

with initial condition $\mathbf{G}(0) = \mathbf{G}'(0) = \mathbf{G}''(0) = \mathbf{G}'''(0) = \mathbf{G}^{(4)}(0) = 0.$

x	E_3	E_6	E_8
0.1	4.23394E-07	1.37144E-08	1.39359E-09
0.2	6.77430E-06	2.19431E-07	2.229474E-08
0.3	3.42949E-05	1.11087E-06	1.12880E-07
0.4	1.08388E-04	3.51090E-06	3.56759E-07
0.5	2.64621E-04	8.57155E-06	8.70995E-07
0.6	5.48719E-04	1.77739E-05	1.80609E-06
0.7	1.01656E-03	3.29284E-05	3.34601E-06
0.8	1.73422E-03	5.61748E-05	5.70815E-06
0.9	2.77789E-03	8.99807E-05	9.14335E-06
1.0	4.23394E-03	1.37144E-04	1.39359E-06

Table 2: Maximum error for Example 2

Solution. Applying Shehu transform, we get

$$\frac{s^5}{u^5}Y(s,u) - \frac{s^4}{u^4}\mathbf{G}(0) - \frac{s^3}{u^3}\mathbf{G}'(0) - \frac{s^2}{u^2}\mathbf{G}''(0) - \frac{s}{u}\mathbf{G}'''(0) - \mathbf{G}^{(4)}(0) = \frac{u^2}{s^2} - \frac{u^2}{s^2}\int_0^1 \mathbf{G}^2(t)dt,$$

by putting the initial condition, we get

$$Y(s,u) = \frac{u^7}{s^7} - \frac{u^7}{s^7}\int_0^1 \mathbf{G}^2(t)dt,$$

here $S[\mathbf{G}](s,u) = Y(s,u)$. Putting (2) for $Y(s,u)$ and comparing terms, we have

$$Y_0(s,u) = \frac{u^7}{s^7},$$

using the recursive relation we obtain

$$\mathbb{S}\left[\mathbf{G}_{q+1}(x)\right](s,u) = -\frac{u^7}{s^7}\int_0^1 \mathbf{G}_q^2(t)dt.$$

Now applying Shehu inverse of both side give $\mathbf{G}_0$(x) and using the recursive relation gives

$$\begin{aligned}
\mathbf{G}_0(x) &= \frac{x^6}{6!},\\
\mathbf{G}_1(x) &= -\frac{1}{(6!)^3 13}x^6,\\
\mathbf{G}_2(x) &= -\frac{1}{(6!)^7(13)^3}x^6,\\
&\vdots\\
\mathbf{G}_q(x) &= -\frac{1}{(6!)^{(2^{q+1}-1)}(13)^{(2^q-1)}}x^6,
\end{aligned}$$

thus, the desired approximate solution is:

$$\phi_q(x) = \sum_{r=0}^{q-1} \mathbf{G}_r(x) = \frac{x^6}{6!} - \frac{1}{(6!)^{(2^q-1)}(13)^{(2^{q-1}-1)}} x^6,$$

$$\mathbf{G}(x) = \lim_{q\to\infty} \phi_q(x) = \lim_{q\to\infty} \left(\frac{x^6}{6!} - \frac{1}{(6!)^{(2^q-1)}(13)^{(2^{q-1}-1)}} x^6 \right) = \frac{x^6}{6!}.$$

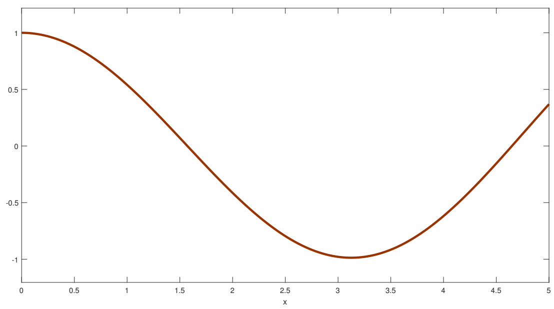

Figure 5: Solution curve of Example 2.

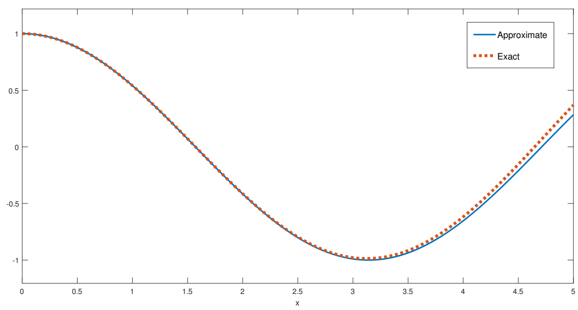

Figure 6: Comparison between approximate and exact solution of Example 3.

x	E_3	E_6	E_8
0.1	4.5377E-30	2.8563E-221	8.1172E-877
0.2	2.9041E-28	1.8280E-219	5.1950E-875
0.3	3.3056E-27	2.0822E-218	5.9174E-874
0.4	1.8586E-26	1.1699E-217	3.3248E-873
0.5	7.0902E-26	4.4630E-217	1.2683E-872
0.6	2.1171E-25	1.3326E-216	3.7871E-872
0.7	5.3386E-25	3.3604E-216	9.5498E-872
0.8	1.1895E-24	7.4877E-216	2.1278E-871
0.9	2.4115E-24	1.5179E-215	4.3138E-871
1.0	4.5377E-24	2.8563E-215	8.1172E-871

Table 3: Maximum error for Example 3.

5. Conclusion

In this article, we have used a more generalized novel transform called Shehu transform, which is the generalization of Sumudu and Laplace transform, to solve higher-order IDEs. We have presented a general scheme of solutions through the proposed transform. We have given few examples with a detailed solution to show the accuracy and validity of the proposed method. We have shown the convergence of the method through graphs and tables. From graphs and tables, we can say that the approximate solution is very close to the exact solution. Thus, the suggested method is more appropriate than other complex analytical methods because the proposed method is highly accurate, less computational, and fast convergent. Other numerical and analytical methods [3, 4, 9, 10, 13, 24, 25, 28, 34, 36–40] can be applied to address such types of problems and other challenging problems [2, 7, 8, 11, 12, 32, 33, 35]. In our next paper, we will use the Shehu transform to solve other types of IDEs of integer and fractional orders.

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References

[1] K Abbaoui and Y Cherruault. Convergence of adomian's method applied to differential equations. *Computers & Mathematics with Applications*, 28(5):103–109, 1994.

[2] AE Abouelregals, H Ahmad, and S-W Yao. Functionally graded piezoelectric medium exposed to a movable heat flow based on a heat equation with a memory-dependent derivative. *Materials*, 13(18):3953, 2020.

[3] M Adels, ME Ramadans, H Ahmad, and T Botmart. Sobolev-type nonlinear Hilfer fractional stochastic differential equations with noninstantaneous impulsive. *AIMS Mathematics*, 7(11):20105–20125, 2022.

[4] M Adels, NH Sweilams, MM Khaders, SM Ahmeds, H Ahmad, and T Botmart. Numerical simulation using the non-standard weighted average FDM for 2Dim variable-order Cable equation. *Results in Physics*, 39:105682, 2022.

[5] G Adomian. *Nonlinear stochastic systems theory and applications to physics*, volume 46. Springer Science & Business Media, 1988.

[6] HA Agwas, FM Ali, and A Kılıçman. A new integral transform on time scales and its applications. *Advances in Difference Equations*, 2012(1):1–14, 2012.

[7] H Ahmad and TA Khan. Variational iteration algorithm I with an auxiliary parameter for the solution of differential equations of motion for simple and damped mass–spring systems. *Noise & Vibration Worldwide*, 51(1-2):12–20, 2020.

[8] H Ahmads, TA Khan, and S-W Yao. An efficient approach for the numerical solution of fifth-order KdV equations. *Open Mathematics*, 18(1):738–748, 2020.

[9] I Ahmads, AR Seadawys, H Ahmads, P Thounthongs, and F Wang. Numerical study of multi-dimensional hyperbolic telegraph equations arising in nuclear material science via an efficient local meshless method. *International Journal of Nonlinear Sciences and Numerical Simulation*, 23(1):115–122, 2022.

[10] M Ahsans, AA Khans, S Dinibutuns, I Ahmads, H Ahmads, N Jarasthitikulchai, and W Sudsutad. The Haar wavelets based numerical solution of Reccati equation with integral boundary condition. *Thermal Science*, 27(Spec. issue 1):93–100, 2023.

[11] A Akgül and H Ahmad. Reproducing kernel method for Fangzhu's oscillator for water collection from air. *Mathematical Methods in the Applied Sciences*, 2020.

[12] R Alharbis, R Jans, S Alyobis, A Yousif, and Z Khan. Mathematical modeling and stability analysis of the dynamics of monkeypox via fractional-calculus. *Fractals*, 30(10):2240266, 2022.

[13] B Almutairis, I Ahmads, B Almohsens, H Ahmads, and DU Ozsahin. Numerical simulations of time-fractional PDEs arising in mathematics and physics using the local Meshless differential quadrature method. *Thermal Science*, 27(Spec. issue 1):263–272, 2023.

[14] A Atangana. A note on the triple laplace transform and its applications to some kind of third-order differential equation. In *Abstract and Applied Analysis*, volume 2013. Hindawi, 2013.

[15] A Atanganas and A Kilicman. A novel integral operator transform and its application to some fode and fpde with some kind of singularities. *Mathematical Problems in Engineering*, 2013, 2013.

[16] FBM Belgacems, R Silambarasans, H Zakia, and T. Mekkaoui. New and extended applications of the natural and sumudu transforms: Fractional diusion and stokes fluid flow realms. *Advances Real and Complex Analysis with Applications*, 2017.

[17] B Davies. *Integral transforms and their applications*, volume 41. Springer Science & Business Media, 2002.

[18] JM Davis, IA Gravagnes, BJ Jacksons, II Markss, J Robert J, and AA Ramos. The laplace transform on time scales revisited. *Journal of Mathematical Analysis and Applications*, 332(2):1291–1307, 2007.

[19] H Eltayeb. A note on double laplace decomposition method and nonlinear partial differential equations. *New Trends in Mathematical Sciences*, 5(4):156–164, 2017.

[20] TM Elzaki. The new integral transform elzaki transform. *Global Journal of pure and applied mathematics*, 7(1):57–64, 2011.

[21] P Flajolets, X Gourdon, and P Dumas. Mellin transforms and asymptotics: Harmonic sums. *Theoretical computer science*, 144(1-2):3–58, 1995.

[22] JM Heris. solving the integro-differential equations using the modified laplace adomian decomposition method. *Journal of Mathematical Extension*, 6, 2012.

[23] JM Heris. solving the integro-differential equations using the modified laplace adomian decomposition method. *Journal of Mathematical Extension*, 6, 2012.

[24] H Irshads, M Shakeels, I Ahmads, H Ahmads, C Tearnbucha, and W Sudsutad. Simulation of generalized time fractional Gardner equation utilizing in plasma physics for non-linear propagation of ion-acoustic waves. *Thermal Science*, 27(Spec. issue 1):121–128, 2023.

[25] R Jans, A Khans, S Boulaarass, and SA Zubair. Dynamical behaviour and chaotic phenomena of HIV infection through fractional calculus. *Discrete Dynamics in Nature and Society*, 2022, 2022.

[26] ZH Khan and WA Khan. N-transform properties and applications. *NUST journal of engineering sciences*, 1(1):127–133, 2008.

[27] W Lederman. Handbook of applicable mathematics. vol. 5, a: Combinatorics and geometry; vol. 5, b: Combinatorics and geometry. *A Wiley-Interscience Publication*, 1985.

[28] J-F Lis, I Ahmads, H Ahmads, D Shahs, Y-M Chus, P Thounthong, and M Ayaz. Numerical solution of two-term time-fractional PDE models arising in mathematical physics using local meshless method. *Open Physics*, 18(1):1063–1072, 2020.

[29] AC Macbride. LI. G. Chambers, Integral Equations: A Short Course (International Textbook Company Limited, 1976), 198 pp.,£ 7· 00. *Proceedings of the Edinburgh Mathematical Society*, 20(4):361–362, 1977.

[30] RC MacCamy. An integro-differential equation with application in heat flow. *Quarterly of Applied Mathematics*, 35(1):1–19, 1977.

[31] S Maitama and W Zhao. New integral transform: Shehu transform a generalization of sumudu and laplace transform for solving differential equations. *arXiv preprint arXiv:1904.11370*, 2019.

[32] NA Shahs, I Ahmads, O Bazighifans, AE Abouelregal, and H Ahmad. Multistage optimal homotopy asymptotic method for the nonlinear Riccati ordinary differential equation in nonlinear physics. *Applied Mathematics*, 14(6):1009–1016, 2020.

[33] M Shakeels, I Hussains, H Ahmads, I Ahmads, P Thounthong, and Ying-Fang Y-F Zhang. Meshless technique for the solution of time-fractional partial differential equations having real-world applications. *Journal of Function Spaces*, 2020:1–17, 2020.

[34] M Shakeels, MN Khans, I Ahmads, H Ahmads, N Jarasthitikulchai, and W Sudsutad. Local meshless collocation scheme for numerical simulation of space fractional PDE. *Thermal Science*, 27(Spec. issue 1):101–109, 2023.

[35] TA Sulaimans, A Yusufs, S Abdel-Khaleks, M Bayram, and H Ahmad. Nonautonomous complex wave solutions to the (2+ 1)-dimensional variable-coefficients nonlinear Chiral Schrödinger equation. *Results in Physics*, 19:103604, 2020.

[36] T-Q Tang, Z Shahs, R Jan, and E Alzahrani. Modeling the dynamics of tumor–immune cells interactions via fractional calculus. *The European Physical Journal Plus*, 137(3):367, 2022.

[37] T-Q Tangs, Z Shahs, R Jans, W Deebani, and Meshal M Shutaywi. A robust study to conceptualize the interactions of CD4+ T-cells and human immunodeficiency virus via fractional-calculus. *Physica Scripta*, 96(12):125231, 2021.

[38] F Wangs, I Ahmads, H Ahmads, MD Alsulamis, KS Alimgeers, C Cesarano, and TA Nofal. Meshless method based on RBFs for solving three-dimensional multi-term time fractional PDEs arising in engineering phenomenons. *Journal of King Saud University-Science*, 33(8):101604, 2021.

[39] F Wangs, NA Alis, I Ahmads, H Ahmads, KM Alams, , and P Thounthong. Solution of Burgers' equation appears in fluid mechanics by multistage optimal homotopy asymptotic method. *Thermal Science*, 26(1 Part B):815–821, 2022.

[40] F Wangs, E Hous, I Ahmads, H Ahmad, and Y Gu. An efficient meshless method for hyperbolic telegraph equations in (1+ 1) dimensions. *CMES-Computer Modeling in Engineering and Sciences*, 128(2):687–98, 2021.

[41] GK Watugala. Sumudu transform: a new integral transform to solve differential equations and control engineering problems. *Integrated Education*, 24(1):35–43, 1993.

[42] X-J Yang. A new integral transform method for solving steady heat-transfer problem. *Thermal Science*, 20(suppl. 3):639–642, 2016.