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Closeness Centrality of Vertices in Graphs Under Some Operations

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Abstract. In this paper, we revisit the concept of (normalized) closeness centrality of a vertex in a graph and investigate it in some graphs under some operations. Specifically, we derive formulas to compute the closeness centrality of vertices in the shadow graph, complementary prism, edge corona, and disjunction of graphs.

2020 Mathematics Subject Classifications: 05C69

Key Words and Phrases: Closeness centrality, shadow graph, edge corona, complementary prism, disjunction

1. Introduction

According to a study in [7], centrality is one of the most studied subjects in the analysis of social networks. As mentioned in [4], the concept was developed by social scientists some decades ago with the aim of quantifying an intuitive perception that some nodes or linkages are regarded central according to some criteria in many particular networks or graphs. For example, in a social network which is often represented as a graph, where each individual is represented as a vertex, the relationship between pairs of individuals are connected by edges, and the weights on the edges indicate the strength of the relationships, centrality gives a way of determining how central an individual is located in this network (see [6]). Within graph theory and network analysis, some of the common measurements of centrality pointed out in [8] are degree centrality, closeness centrality, eigen vector centrality, and betweeness centrality. One may also refer to [1], [2], and [5] for some studies in measure of centrality.

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Closeness centrality measures how close a vertex is to all other vertices in a graph. The closeness centrality of a vertex in a graph is the inverse of the average geodesic distance from the vertex to any other vertex in the graph. The greater value of closeness centrality of a vertex would mean a better position of a vertex in spreading information to other vertices (see [6]).

A study on closeness centrality can be found in [4] where the authors derived the closeness centrality of vertices of some families of graphs such as paths, cycles, fans, wheels, complete bipartite graphs, and complete split graphs. In a more recent study, Eballe et al. in [3] presented the closeness centrality of the vertices in the corona, Cartesian product and lexicographic product of graphs.

In this paper, we derive formulas of the closeness centrality of a vertex in the shadow graph, edge corona of graphs, complementary prism, and disjunction of graphs.

2. Terminology

Definition 1. Let G be a connected graph and let $u, v \in V(G)$. The *distance* between u and v, denoted by $d_G(u, v)$, is the length of a shortest path (called *u-v geodesic*) connecting u and v. Vertices u and v are adjacent or neighbors (i.e. $uv \in E(G)$) if and only if $d_G(u, v) = 1$. The *open neighborhood* of v is the set $N_G(v) = \{w \in V(G) : d_G(v, w) = 1\}$ and its *closed neighborhood* is the set $N_G[v] = N_G(v) \cup \{v\}$.

Definition 2. Let G be a connected graph. If $v \in V(G)$ and $e \in E(G)$, then the distance $d_G(v, e)$ between v and e, is given by $d_G(v, e) = \min_{x \in V(G)} \{d_G(v, x), d_G(v, x)\}$.

Definition 3. Let $G = (V(G), E(G))$ be a nontrivial connected graph of order m. If $u \in V(G)$, then the *closeness centrality* of vertex u is given by

$$\mathcal{C}_G(u) = \frac{m-1}{\mathcal{T}_G(u)},$$

where $\mathcal{T}_G(u) = \sum_{x \in V(G)} d_G(u,x)$.

Definition 4. The *shadow graph* $S(G)$ of G is the graph obtained by taking two copies of G, say G_1 and G_2, and then joining each vertex $v \in V(G_1)$ to the neighbors of $v' \in V(G_2)$, where v' is the vertex in $V(G_2)$ corresponding to v, i.e., v and v' represent the same vertex in G.

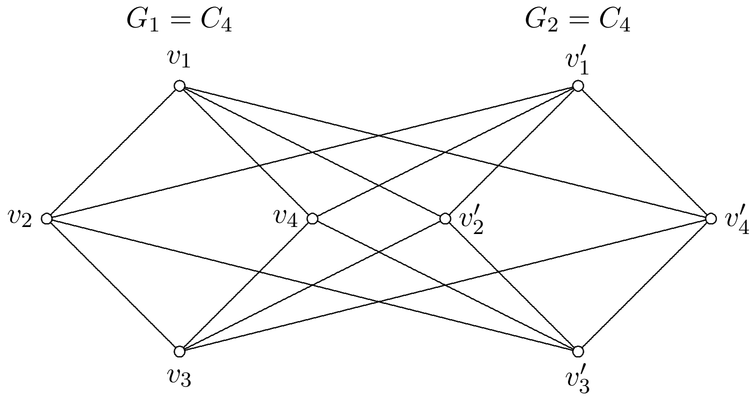

Figure 1: The shadow graph $S(C_4)$ of C_4

Definition 5. Let G and H be two graphs. The *edge corona* $G \diamond H$ of G and H is the graph obtained by taking one copy of G and $|E(G)|$ copies of H, and then joining two end-vertices of the i-th edge of G to every vertex in the i-th copy of H. For every edge $e = uv$ of G, denote by $H^e = H^{uv}$ the copy of H where vertices are joined to the vertices u and v.

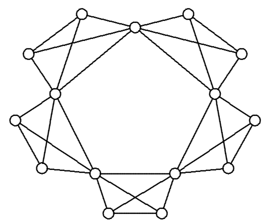

Figure 2: The edge corona $C_5 \diamond P_2$

Definition 6. The *complementary prism* of graph G, denoted by $G\overline{G}$, is the graph obtained from the disjoint union of G and $\overline{G}$ by adding the edges $v\overline{v}$, where $v \in V(G)$ and $\overline{v}$ is the vertex of $\overline{G}$ corresponding to vertex v.

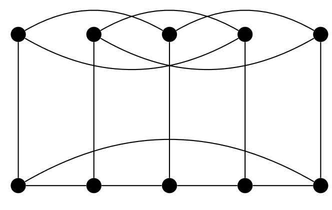

Figure 3: The complementary prism $C_5\overline{C_5}$

Definition 7. The *disjunction* of graphs G and H, denoted by $G \vee H$, is the graph with $V(G \vee H) = V(G) \times V(H)$ and $(x, p)(y, q) \in E(G \vee H)$ if and only if $xy \in E(G)$ or $pq \in E(H)$.

3. Main Results

In what follows, G_1 and G_2 are the copies of G in the shadow graph $S(G)$.

Remark 1. Let G be a graph and let $v, w \in V(G_1)$. Then

$$d_{S(G)}(v, w) = d_{G_1}(v, w) = d_{G_2}(v', w') = d_{S(G)}(v', w').$$

Lemma 1. Let G be a nontrivial connected graph. For each $p \in V(G_1)$ and $v' \in V(G_2)$,

$$d_{S(G)}(p, v') = d_{S(G)}(p', v) = \begin{cases} 2 & \text{if } p = v \\ d_{G_1}(p, v) & \text{if } p \neq v. \end{cases}$$

Proof. If $p = v$, then $p' = v'$ and $pp' \notin E(S(G))$. Let $w \in N_{G_1}(p)$. Then $wp' \in E(S(G))$. Hence, $[p, w, v']$ is a $p - v'$ geodesic in $S(G)$. This implies that $d_{S(G)}(p, v') = 2$. Next, suppose that $p \neq v$, that is, $p' \neq v'$. Let $[p_1, p_2, ..., p_k]$, where $p = p_1$ and $v = p_k$, be a p-v geodesic in G_1. Then $[p'_1, p'_2, ..., p'_k]$ is a p'-v' geodesic in G_2. Also, by definition of $S(G)$, $[p_1, p'_2, p'_3..., p'_k]$ is a p-v' geodesic in $S(G)$. Hence, $d_{S(G)}(p, v') = d_{G_1}(p, v)$. Since $d_{G_1}(p, v) = d_{G_2}(p', v')$, it follows that $d_{S(G)}(p, v') = d_{S(G)}(p', v)$. □

Consider the shadow graph $S(P_5)$ in Figure 4.

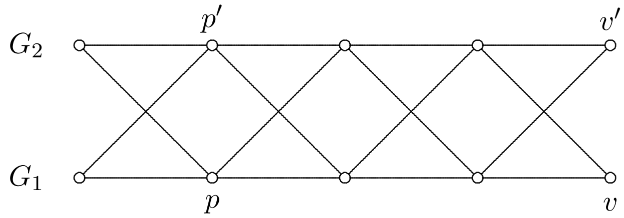

Figure 4: The shadow graph $S(P_5)$

Clearly, $d_{S(G)}(p, p') = 2$, and $d_{S(G)}(p, v) = d_{S(G)}(p, v') = d_{S(G)}(p', v') = 3$.

Theorem 1. Let G be a non-trivial connected graph. For each $p \in V(G_1)$,

$$\tau_{S(G)}(p) = 2(\tau_{G_1}(p)) + 2 \text{ and } \tau_{S(G)}(p') = 2(\tau_{G_2}(p')) + 2.$$

Proof. Let $p \in V(G_1)$. Then

$$\begin{aligned}\tau_{S(G)}(p) &= \sum_{q\in V(S(G))} d_{S(G)}(p,q)\\ &= \sum_{q\in V(G_1)} d_{S(G)}(p,q) + \sum_{v'\in V(G_2)} d_{S(G)}(p,v')\\ &= \sum_{q\in V(G_1)} d_{G_1}(p,q) + \sum_{v'\in V(G_2)-\{p'\}} d_{S(G)}(p,v') + d_{S(G)}(p,p')\\ &= \tau_{G_1}(p) + \sum_{v\in V(G_1)-\{p\}} d_{G_1}(p,v) + 2\\ &= 2\tau_{G_1}(p) + 2.\end{aligned}$$

Similarly, $\tau_{S(G)}(p') = 2\tau_{G_2}(p') + 2$. □

The next result follows from Theorem 1 and Definition 3.

Corollary 1. Let G be a non-trivial connected graph of order m and let $p \in V(G_1)$. Then

$$\mathcal{C}_{S(G)}(p) = \frac{2m-1}{2(\tau_{G_1}(p)) + 2}$$

and

$$\mathcal{C}_{S(G)}(p') = \frac{2m-1}{2(\tau_{G_2}(p')) + 2}.$$

Example 1. Consider the shadow graph $S(P_5)$ in Figure 5, where $G = P_5$.

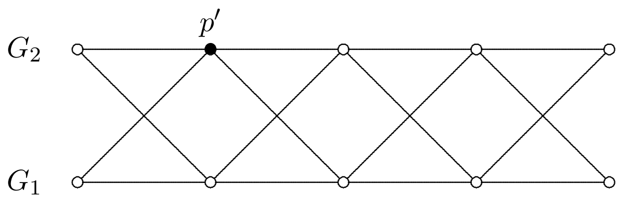

Figure 5: The shadow graph $S(P_5)$ and $p' \in V(G_2)$

Then $m = 5$ and $\tau_{G_2}(p') = 7$. Using Corollary 1, the closeness centrality of p' is

$$\begin{aligned}\mathcal{C}_{S(G)}(p') &= \frac{2m-1}{2\tau_{G_2}(p') + 2}\\ &= \frac{2(5)-1}{2(7)+2}\\ &= \frac{10-1}{14+2}\end{aligned}$$

$$= \frac{9}{16}.$$

For a non-trivial connected graph G and $v \in V(G)$, the set E_v is given by

$$E_v = \{xv \in E(G) : x \in V(G) \setminus \{v\}\}.$$

Theorem 2. Let G and H be non-trivial connected graphs. If $v \in V(G)$, then

$$\tau_{G\diamond H}(v) = \tau_G(v) + |V(H)||E(G)| + |V(H)| \sum_{e\in E(G)\setminus E_v} d_G(v, e).$$

Proof. Let $v \in V(G)$. Then $|E_v| = |N_G(v)|$ and $d_{G\diamond H}(v, a) = d_G(v, e) + 1$ for every $a \in V(H^e)$ where $e \in E(G) \setminus E_v$. Hence,

$$\begin{aligned}
\tau_{G\diamond H}(v) &= \sum_{x\in V(G)} (d_G(v, x)) + \sum_{e\in E_v} \sum_{a\in V(H^e)} d_{G\diamond H}(v, a) + \sum_{e\in E(G)\setminus E_v} \sum_{a\in V(H^e)} d_{G\diamond H}(v, a) \\
&= \tau_G(v) + |E_v||V(H)| + \sum_{e\in E(G)\setminus E_v} [(d_G(v, e) + 1)|V(H)|] \\
&= \tau_G(v) + |E_v||V(H)| + |V(H)| \sum_{e\in E(G)\setminus E_v} d_G(v, e) + |V(H)||E(G)| - |E_v||V(H)| \\
&= \tau_G(v) + |V(H)||E(G)| + |V(H)| \sum_{e\in E(G)\setminus E_v} d_G(v, e). \quad \square
\end{aligned}$$

The next result is immediate from Theorem 2.

Corollary 2. Let G and H be connected non-trivial graphs with $m = |V(G)|$, $r = |E(G)|$, and $n = |V(H)|$. If $v \in V(G)$, then

$$\mathcal{C}_{G\diamond H}(v) = \frac{m + rn - 1}{\tau_G(v) + rn + n \sum_{e\in E(G)\setminus E_v} d_G(v, e)}.$$

Example 2. Consider the edge corona of $G = C_4$, $H = P_5$, and $v \in V(G)$ in Figure 6. Then $m = |V(G)| = |E(G)| = r = 4$, $n = |V(H)| = 5$, $\tau_G(v) = 4$, $|E_v| = |E(G) \setminus E_v| = 2$ and $\sum_{e\in E(G)\setminus E_v} d_G(v, e) = 2$. Using Corollary 2, the closeness centrality of $v \in V(G)$ is

$$\begin{aligned}
\mathcal{C}_{G\diamond H} &= \frac{m + qn - 1}{\tau_G(v) + nq + n \sum_{e\in E(G)\setminus E_v} d_G(v, e)} \\
&= \frac{24 - 1}{4 + (5)(4) + (5)(2)}
\end{aligned}$$

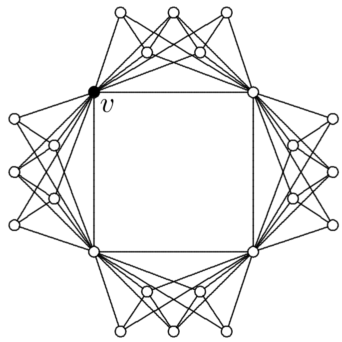

Figure 6: The edge corona graph $C_4 \diamond P_5$ and $v \in V(C_4)$

$$= \frac{23}{34}.$$

For a non-trivial connected graph G and $e = uv \in E(G)$, the sets $V_{u\leq v}, V_{v<u}, E_{u\leq v}, E_{v<u}$ are given by:

$$V_{u\leq v} = \{z \in V(G) \setminus \{u\} : d_G(z, u) \leq d_G(z, v)\}$$
$$V_{v<u} = \{y \in V(G) \setminus \{v\} : d_G(v, y) < d_G(u, y)\}$$
$$E_{u\leq v} = \{e' \in E(G) \setminus \{e\} : d_G(u, e') \leq d_G(v, e')\}$$
$$E_{v<u} = \{e' \in E(G) : d_G(v, e') < d_G(u, e')\}$$

Note that $|E_{u\leq v}| + |E_{v<u}| = |E(G)| - 1$ and $|V_{u\leq v}| + |V_{v<u}| = |V(G)| - 2$.

Theorem 3. Let G and H be connected non-trivial graphs and let $e = uv \in E(G)$. If $p \in V(H^e)$, then

$$\tau_{G\diamond H}(p) = |V(G)| - deg_H(p) + 2|V(H)||E(G)| - 2 + \sum_{x\in V_{u\leq v}} d_G(u, x) + \sum_{x\in V_{v<u}} d_G(v, x) + |V(H)| \sum_{e'\in E_{u\leq v}} d_G(u, e') + |V(H)| \sum_{e'\in E_{v<u}} d_G(v, e').$$

Proof. Let $e = uv \in E(G)$ and let $p \in V(H^e)$. Then

$$\begin{aligned}
\tau_{G\diamond H}(p) &= \sum_{q\in V(G\diamond H)} d_{G\diamond H}(p, q) \\
&= \sum_{e'\in E(G)} \sum_{q\in V(H^{e'})} d_{G\diamond H}(p, q) + \sum_{q\in V(G)} d_{G\diamond H}(p, q) \\
&= \sum_{q\in V(H^e)} d_{G\diamond H}(p, q) + \sum_{e'\neq e} \sum_{q\in V(H^{e'})} d_{G\diamond H}(p, q) + \sum_{x\in V_{u\leq v}} (1 + d_G(u, x)) \\
&\quad + \sum_{x\in V_{v<u}} (1 + d_G(v, x)) + d_{G\diamond H}(p, u) + d_{G\diamond H}(p, v)
\end{aligned}$$

$$= \sum_{q\in N_{H^e}(p)} d_{G\diamond H}(p,q) + \sum_{q\in V(H^e)\setminus N_{H^e}[p]} d_{G\diamond H}(p,q)$$
$$+ \sum_{e'\in E_{u\leq v}} \sum_{q\in V(H^{e'})} d_{G\diamond H}(p,q) + \sum_{e'\in E_{v<u}} \sum_{q\in V(H^{e'})} d_{G\diamond H}(p,q)$$
$$+ \sum_{x\in V_{u\leq v}} (1+d_G(u,x)) + \sum_{x\in V_{v<u}} (1+d_G(v,x)) + 2$$
$$= |N_H(p)| + 2(|V(H)| - |N_H[p]|) + |V(H)| \sum_{e'\in E_{u\leq v}} (2+d_G(u,e'))$$
$$+ |V(H)| \sum_{e'\in E_{v<u}} (2+d_G(v,e')) + \sum_{x\in V_{u\leq v}} (1+d_G(u,x))$$
$$+ \sum_{x\in V_{v<u}} (1+d_G(v,x)) + 2.$$

Now,

$$|V(H)| \sum_{e'\in E_{u\leq v}} (2+d_G(u,e')) = 2|V(H)||E_{u\leq v}| + |V(H)| \sum_{e'\in E_{u\leq v}} d_G(u,e'),$$
$$|V(H)| \sum_{e'\in E_{v<v}} (2+d_G(v,e')) = 2|V(H)||E_{v<u}| + |V(H)| \sum_{e'\in E_{v<u}} d_G(v,e'),$$
$$\sum_{x\in V_{u\leq v}} (1+d_G(u,x)) = |V_{u\leq v}| + \sum_{x\in V_{u\leq v}} d_G(u,x), \text{ and}$$
$$\sum_{x\in V_{v<u}} (1+d_G(v,x)) = |V_{v<u}| + \sum_{x\in V_{v<u}} d_G(u,x).$$

Hence,

$$\tau_{G\diamond H}(p) = |N_H(p)| + 2|V(H)| - 2|N_H(p)| - 2 + 2 + 2|V(H)||E_{u\leq v}|$$
$$+ |V(H)| \sum_{e'\in E_{u\leq v}} d_G(u,e') + 2|V(H)||E_{v<u}| + |V(H)| \sum_{e'\in E_{v<u}} d_G(v,e')$$
$$+ |V_{u\leq v}| + \sum_{x\in V_{u\leq v}} d_G(u,x) + |V_{v<u}| + \sum_{x\in V_{v<u}} d_G(u,x)$$
$$= 2|V(H)| - |N_H(p)| + 2|V(H)|(|E_{u\leq v}| + |E_{v<u}|)$$
$$+ (|V_{u\leq v}| + |V_{v<u}|) + \sum_{x\in V_{u\leq v}} d_G(u,x) + \sum_{x\in V_{v<u}} d_G(v,x)$$
$$+ |V(H)| \sum_{e'\in E_{u\leq v}} d_G(u,e') + |V(H)| \sum_{e'\in E_{v<u}} d_G(v,e')$$
$$= 2|V(H)| - |N_H(p)| + 2|V(H)|(|E(G)| - 1) + |V(G)| - 2$$

$$+ \sum_{x \in V_{u \leq v}} d_G(u,x) + \sum_{x \in V_{v<u}} d_G(v,x) + |V(H)| \sum_{e' \in E_{u \leq v}} d_G(u,e')$$
$$+ |V(H)| \sum_{e' \in E_{v<u}} d_G(v,e')$$
$$= |V(G)| - deg_H(p) + 2|V(H)||E(G)| - 2 + \sum_{x \in V_{u \leq v}} d_G(u,x) + \sum_{x \in V_{v<u}} d_G(v,x)$$
$$+ |V(H)| \sum_{e' \in E_{u \leq v}} d_G(u,e') + |V(H)| \sum_{e' \in E_{v<u}} d_G(v,e'). \quad \square$$

The next result is a direct consequence of Theorem 3.

Corollary 3. Let G and H be connected non-trivial graphs of orders m and n, respectively, and let $r = |E(G)|$ and $e = uv \in E(G)$. If $p \in V(H^e)$, then

$$\mathcal{C}_{G \diamond H}(p) = \frac{m + rn - 1}{\tau_{G \diamond H}(p)},$$

where $\tau_{G \diamond H}(p)$ is given in Theorem 3.

Example 3. Consider $p \in V(H^e)$, where $e = uv$, in Figure 7, where $G = P_4$ and $H = P_5$. Then $m = 4$, $r = 4$, $n = 5$, $deg_H(p) = 1$, $\sum_{x \in V_{u \leq v}} d_G(u,x) = 1$, $\sum_{x \in V_{v<u}} d_G(v,x) = 1$, $\sum_{e' \in E_{u \leq v}} d_G(u,e') = 1$, and $\sum_{e' \in E_{v<u}} d_G(v,e') = 0$. By Corollary 3, the closeness centrality of $p \in H^v$ is $\mathcal{C}_{G \diamond H}(p) = \frac{23}{48}$.

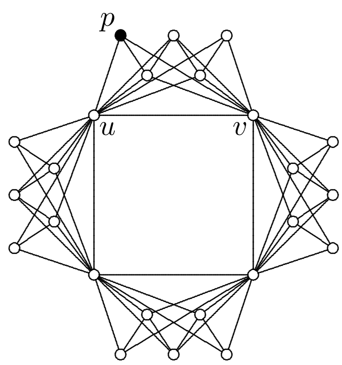

Figure 7: The edge corona graph of $C_4 \diamond P_5$ with point p in H^e

Lemma 2. Let G be a non-trivial connected graph and let $v, x \in V(G)$. Then

$$d_{G\overline{G}}(v,x) = \begin{cases} 0 & \text{if } x = v \\ 1 & \text{if } vx \in E(G) \\ 2 & \text{if } d_G(v,x) = 2 \\ 3 & \text{if } d_G(v,x) \geq 3 \end{cases}$$

Proof. Clearly, $d_{G\overline{G}}(v,x) = d_G(v,x)$ if $1 \leq d_G(v,x) \leq 2$. Suppose $d_G(v,x) \geq 3$. Then $d_{G\overline{G}}(v,x) \geq 3$. Since $\overline{v}\,\overline{x} \in E(\overline{G})$, it follows that $[v, \overline{v}, \overline{x}, x]$ is a v-x geodesic in $G\overline{G}$. Hence, $d_{G\overline{G}}(v,x) = 3$. □

Lemma 3. Let G be a non-trivial connected graph and let $v, x \in V(G)$. Then

$$d_{G\overline{G}}(v,\overline{x}) = \begin{cases} 1 & \text{if } \overline{x} = \overline{v} \\ 2 & \text{otherwise} \end{cases}$$

Proof. If $\overline{x} = \overline{v}$, then $d_{G\overline{G}}(v,\overline{v}) = 1$ by definition of $G\overline{G}$. Suppose $\overline{x} \neq \overline{v}$, i.e., $x \neq v$. If $xv \in E(G)$, then $\overline{x}v \notin E(G\overline{G})$ and $[v, x, \overline{x}]$ is a $v - \overline{x}$ geodesic in $G\overline{G}$. If $xv \notin E(G)$, then $\overline{xv} \in E(\overline{G})$. Hence, $[v, \overline{v}, \overline{x}]$ is a v-$\overline{x}$ geodesic in $G\overline{G}$. Thus $d_{G\overline{G}}(v,\overline{x}) = 2$. □

Lemma 4. Let G be a non-trivial connected graph and let $v, x \in V(G)$. Then

$$d_{G\overline{G}}(\overline{v},\overline{x}) = \begin{cases} 0 & x = v \\ 1 & \text{if } xv \notin E(G), x \neq v \\ 2 & \text{if } xv \in E(G) \text{ and } (V(G) \setminus N_G[x]) \cap (V(G) \setminus N_G[v]) \neq \varnothing, \\ & x \neq v \\ 3 & \text{if } xv \in E(G) \text{ and } (V(G) \setminus N_G[x]) \cap (V(G) \setminus N_G[v]) = \varnothing, \\ & x \neq v \end{cases}$$

Proof. Clearly, $d_{G\overline{G}}(\overline{v},\overline{v}) = 0$ and $d_{G\overline{G}}(\overline{v},\overline{x}) = 1$ if $xv \notin E(G)$. Next, suppose that $xv \in E(G)$. Then $d_{G\overline{G}}(\overline{v},\overline{x}) \geq 2$. Suppose there exists $z \in (V(G) \setminus N_G[x]) \cap (V(G) \setminus N_G[v])$. Then $\bar{x}\bar{z}, \bar{z}\bar{v} \in E(\overline{G})$. Hence, $[\bar{v}, \bar{z}, \overline{x}]$ is a $\overline{v}$ - $\overline{x}$ geodesic on $G\overline{G}$. Thus, $d_G(\overline{v},\overline{x}) = 2$. Next, suppose that $(V(G) \setminus N_G[x]) \cap (V(G) \setminus N_G[v]) = \varnothing$. Suppose $[\overline{v}, p, \overline{x}]$ is a $\overline{v}$ - $\overline{x}$ geodesic in $G\overline{G}$. Suppose first that $p \in V(G)$. Then $p = v$ by definition of $G\overline{G}$. Similarly, $p = x$. This is a contradiction because $x \neq v$. Hence, $p \in V(\overline{G})$, say $p = \overline{q}$. However, since $\bar{v}\bar{q}, \bar{q}\bar{x} \in E(\overline{G})$, $q \in (V(G) \setminus N_G[v]) \cap (V(G) \setminus N_G[x])$, a contradiction. Thus, $d_{G\overline{G}}(\overline{v},\overline{x}) \geq 3$. Since $[\overline{v}, v, x, \overline{x}]$ is a $\overline{v}$ - $\overline{x}$ geodesic in $G\overline{G}$, it follows that $d_{G\overline{G}}(\overline{v},\overline{x}) = 3$. □

In what follows, $N^2_G(v) = \{w \in V(G) : d_G(v,w) = 2\}$.

Theorem 4. Let G and $\overline{G}$ be non-trivial connected graphs. If $v \in V(G)$, then

$$\tau_{G\overline{G}}(v) = 5|V(G)| - 2 \cdot deg_G(v) - |N^2_G(v)| - 4.$$

Proof. Let $v \in V(G)$.

$$
\begin{aligned}
\tau_{G\overline{G}}(v) &= \sum_{x \in V(G)} d_{G\overline{G}}(v,x) + \sum_{\overline{y} \in V(\overline{G})} d_{G\overline{G}}(v,\overline{y}) \\
&= \sum_{x \in N_G(v)} d_{G\overline{G}}(v,x) + \sum_{x \in N^2_G(v)} d_{G\overline{G}}(v,x) + \sum_{x \in V(G)-(N_G[v] \cup N^2_G(v))} d_{G\overline{G}}(v,x) \\
&+ \sum_{\overline{y} \in V(\overline{G})-\{\overline{v}\}} d_{G\overline{G}}(v,\overline{y}) + d_{G\overline{G}}(v,\overline{v}) \\
&= |N_G(v)| + 2|N^2_G(v)| + 3(|V(G)| - |N_G[v]| - |N^2_G(v)|) \\
&+ 2(|V(G)| - 1) + 1 \\
&= |N_G(v)| + 2|N^2_G(v)| + 3|V(G)| - 3N_G[v] - 3|N^2_G(v)| + 2|V(G)| \\
&- 2 + 1 \\
&= |N_G(v)| + 2|N^2_G(v)| + 3|V(G)| - 3(|N_G(v)| + 1) - 3|N^2_G(v)| \\
&+ 2|V(G)| - 1 \\
&= |N_G(v)| + 2|N^2_G(v)| + 3|V(G)| - 3|N_G(v)| - 3 - 3|N^2_G(v)| \\
&+ 2|V(G)| - 1 \\
&= 5|V(G)| - 2 \cdot deg_G(v) - |N^2_G(v)| - 4.
\end{aligned}
$$

This proves the assertion. □

Corollary 4. Let G be a non-trivial connected graph of order m such that $\overline{G}$ is connected and let $v \in V(G)$. Then

$$\mathcal{C}_{G\overline{G}}(v) = \frac{2m-1}{5|V(G)| - 2 \cdot deg_G(v) - |N^2_G(v)| - 4}.$$

Example 4. Consider $G = P_4$, $G\overline{G} = P_4\overline{P}_4$, and $v \in V(G)$ in Figure 8. Then $|V(G)| = 4$, $m = 8$, $N_G(v) = \{x\}$, and $N^2_G(v) = \{y\}$. Using Corollary 4, the closeness centrality of v is

$$
\begin{aligned}
\mathcal{C}_{G\overline{G}}(v) &= \frac{2m-1}{5|V(G)| - 2 \cdot deg_G(v) - |N^2_G(v)| - 4} \\
&= \frac{8-1}{5(4) - 2(1) - 1 - 4} \\
&= \frac{7}{20 - 2 - 1 - 4} \\
&= \frac{7}{13}.
\end{aligned}
$$

For a non-trivial connected graph G and $v \in V(G)$, the sets $N^0_G(V)$ and $N^1_G(v)$ are given below:

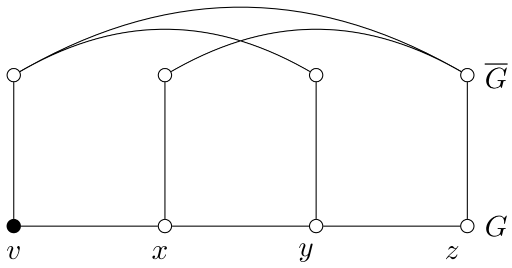

Figure 8: The complementary prism $P_4\overline{P}_4$ and $v \in V(P_4)$

$$N_G^0(v) = \{x \in N_G(v) : (V(G) \setminus N_G[x]) \cap (V(G) \setminus N_G[v]) \neq \varnothing\} \text{ and}$$
$$N_G^1(v) = \{x \in N_G(v) : (V(G) \setminus N_G[x]) \cap (V(G) \setminus N_G[v]) = \varnothing\}.$$

Clearly, $N_G(v) = N_G^0(v) \cup N_G^1(v)$.

Theorem 5. Let G and $\overline{G}$ be non-trivial connected graphs and let $v \in V(G)$. Then

$$\tau_{G\overline{G}}(\overline{v}) = 3|V(G) + deg_G(v) + |N_G^1(v)| - 2.$$

Proof.

$$\begin{aligned}
\tau_{G\overline{G}}(\overline{v}) &= \sum_{x \in V(G)} d_{G\overline{G}}(\overline{v}, x) + \sum_{\overline{x} \in V(\overline{G}) - \{\overline{v}\}} d_{G\overline{G}}(\overline{v}, \overline{x}) \\
&= \sum_{x \in V(G) - \{v\}} d_{G\overline{G}}(\overline{v}, x) + d_{G\overline{G}}(\overline{v}, v) + \sum_{x \in V(G) - N_G[v]} d_{G\overline{G}}(\overline{v}, \overline{x}) \\
&+ \sum_{x \in N_G^0(v)} d_{G\overline{G}}(\overline{v}, \overline{x}) + \sum_{x \in N_G^1(v)} d_{G\overline{G}}(\overline{v}, \overline{x}) \\
&= 2(|V(G)| - 1) + 1 + (|V(G)| - N_G[v]) + 2|N_G^0(v)| + 3|N_G^1(v)| \\
&= 2|V(G)| - 2 + 1 + |V(G)| - |N_G(v)| - 1 + 2(|N_G^0(v)| + |N_G^1(v)|) \\
&+ |N_G^1(v)| \\
&= 3|V(G)| + |N_G(v)| + |N_G^1(v)| - 2 \\
&= 3|V(G)| + deg_G(v) + |N_G^1(v)| - 2.
\end{aligned}$$

□

Corollary 5. Let G be a non-trivial connected graph of order m such that $\overline{G}$ is connected and let $v \in V(G)$. Then

$$\mathcal{C}_{G\overline{G}}(\overline{v}) = \frac{2m - 1}{3|V(G)| + deg_G(v) + |N_G^1(v)| - 2}.$$

Example 5. Let $G = P_4$ and consider $G\overline{G} = P_4\overline{P}_4$ and $\overline{v} \in V(\overline{G})$ in Figure 9. Then $N_G(v) = \{x\}$ and $N^1_G(v) = \varnothing$. Hence $|V(G)| = 4$, $m = 8$, $|N_G(v)| = deg_G(v) = 1$ and $|N^1_G(v)| = 0$. Using Corollary 5, we have

$$\begin{aligned}\mathcal{C}_{G\overline{G}}(\overline{v}) &= \frac{2m-1}{3|V(G)| + deg_G(v) + |N^1_G(v)| - 2} \\ &= \frac{8-1}{3(4)+1+0-2} \\ &= \frac{7}{11}.\end{aligned}$$

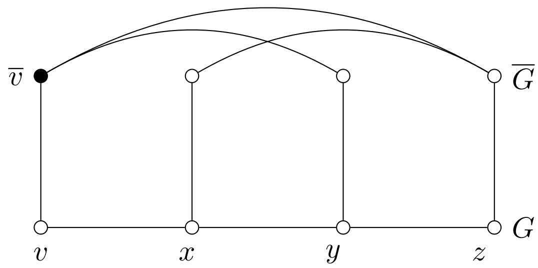

Figure 9: The complementary prism $P_4\overline{P}_4$ and $\overline{v} \in V(\overline{P_4})$

Lemma 5. Let G and H be a non-trivial connected graphs and let $(x,p),(y,q) \in V(G \vee H)$. Then

$$d_{G\vee H}((x,p),(y,q)) = \begin{cases} 0 & \text{if } x = y \text{ and } p = q \\ 1 & \text{if } xy \in E(G) \text{ or } pq \in E(H) \\ 2 & \text{otherwise} \end{cases}$$

Proof. Clearly, $d_{G\vee H}((x,p),(y,q)) = 0$ if $x = y$ and $p = q$. Also, by definition, $d_{G\vee H}((x,p),(y,q)) = 1$ if $xy \in E(G)$ $pq \in E(H)$. Suppose $x \neq y$, $p \neq q$, $xy \notin E(G)$, and $pq \notin E(H)$. Let $z \in N_G(y)$ and $r \in N_H(p)$. Then $(x,p)(z,r),(z,r)(y,q) \in E(G \vee H)$. This implies that $[(x,p),(z,r),(y,q)]$ is an (x,p)-(y,q) geodesic in $G \vee H$. Thus, $d_{G\vee H}((x,p),(y,q)) = 2$. □

Theorem 6. Let G and H be connected non-trivial graphs of orders m and n, respectively, and let $(x,p) \in E(G \vee H)$. Then

$$\tau_{G\vee H}((x,p)) = 2mn + deg_G(x)deg_H(p) - n \cdot deg_G(x) - m \cdot deg_H(p) - 2.$$

Proof. By Definition 3 and Lemma 5,

$$\tau_{G\vee H}((x,p)) = \sum_{q\in V(H)} \sum_{y\in N_G(x)} d_{G\vee H}(x,p)(y,q) + \sum_{q\in N_H(p)} \sum_{y\in V(G)\setminus N_G(x)} d_{G\vee H}(x,p)(y,q)$$

$$
\begin{aligned}
&+ \sum_{q\in V(H)\setminus N_H[p]} d_{G\vee H}(x,p)(x,q) + \sum_{q\in V(H)\setminus N_H(p)} \sum_{y\in V(G)\setminus N_G[x]} d_{G\vee H}(x,p)(y,q) \\
&= |V(H)||N_G(x)| + (|V(G)| - |N_G(x)|)|N_H(p)| + 2(|V(H)| - |N_H(p)| - 1) \\
&\quad + 2(|V(H)| - |N_H(p)|)(|V(G)| - |N_G(x)| - 1) \\
&= n|N_G(x)| + (m - |N_G(x)|)|N_H(p)| + 2(n - |N_H(p)| - 1) \\
&\quad + 2(n - |N_H(p)|)(m - |N_G(x)| - 1) \\
&= n|N_G(x)| + m|N_H(p)| - |N_G(x)||N_H(p)| - 2|N_H(p)| - 2n - 2 \\
&\quad + 2mn - 2n|N_G(x)| - 2m|N_H(p)| + 2|N_G(x)||N_H(p)| - 2n + 2|N_H(p)| \\
&= 2mn + |N_G(x)||N_H(p)| - n|N_G(x)| - m|N_H(p)| - 2 \\
&= 2mn + deg_G(x)deg_H(p) - n\cdot deg_G(x) - m\cdot deg_H(p) - 2. \quad \square
\end{aligned}
$$

Corollary 6. Let G and H be connected non-trivial graphs of orders m and n, respectively, and let $(x,p)\in V(G\vee H)$. Then

$$\mathcal{C}_{G\vee H}((x,p)) = \frac{mn-1}{2mn + deg_G(x)deg_H(p) - n\cdot deg_G(x) - m\cdot deg_H(p) - 2}.$$

Example 6. Let $G = P_3 = [x,y,z]$, $H = P_3 = [q,p,s]$, and $(x,p)\in V(G\vee H)$. By Corollary 6 with $mn = 9$, $deg_G(x) = 1$, and $deg_H(p) = 2$, we find that $\mathcal{C}_{G\vee H}((x,p)) = \frac{8}{9}$.

4. Conclusion

The study revisited the concept of closeness centrality of a vertex in a graph. Formulas that can be used to determine the closeness centrality of vertices in the shadow graph, edge corona of graphs, complementary prism, and disjunction of graphs were generated. The parameter can be studied further for other graphs under some other binary operations.

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References

[1] S Borgatti. Centrality and network flow. *Social Networks*, 27:55–71, 2005.

[2] M Van den Heuvel and O Sporns. Network hubs in the human brain. *Trends in the Cognitive Sciences*, 17(12):683–696, 2013.

[3] R. Eballe, C. M. Balingit, I. Cabahug Jr., A. Flores, S. Lumpayao, B. Peñalosa, G. Tampipi, and C. Villarta. Closeness centrality in graph products. *Advances and Applications in Discrete Mathematics*, 39(1):29–41, 2023.

[4] R Eballe and I Cabahug. Closeness centrality of some graph families. *International Journal of Contemporary Mathematical Sciences*, 16(4):127–134, 2021.

[5] J Golbeck. Network structure and measures. *Analyzing the Social Web*, pages 25–44, 2013.

[6] W Chen K Okamoto and X Li. Ranking of closeness centrality for large-scale social networks. *International workshop on frontiers in algorithmics*, pages 186–195, 2008.

[7] Y Rochat. Closeness centrality extended to unconnected graphs: The harmonic centrality index. 2009.

[8] J Zhang and Y Luo. Degree centrality, betweenness centrality, and closeness centrality in social network. *2nd International Conference on Modelling, Simulation and Applied Mathematics*, pages 300–303, 2017.