EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 16, No. 3, 2023, 1359-1380
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Exploring the Companion of Ostrowski's Inequalities via Local Fractional Integrals

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Abstract. This paper investigates the companion of Ostrowski's inequality in the framework of fractal sets. First, a new identity related to local fractional integrals is introduced, serving as the foundation for establishing a set of inequalities applicable to functions with generalized s-convex and s-concave derivatives. An illustrative example is presented to validate the obtained results, demonstrating their accuracy. Additionally, the paper discusses several practical applications, highlighting the significance of the established inequalities. The research presented in this paper contributes to the growing field of studying functions on fractal sets, which has attracted considerable interest from scientists and engineers.

2020 Mathematics Subject Classifications: 26D10, 26D15, 26A51

Key Words and Phrases: Tow-point Newton-Cotes, generalized s-convex functions, local fractional integral, fractal set

1. Introduction and preliminaries

Convexity is a fundamental property in mathematics that appears in various fields such as optimization, convex analysis, geometry, probability theory, and finance. A function $\mathcal{J} : I \to \mathbb{R}$ is said to be convex if it satisfies the following condition

$$\mathcal{J}\left(\varkappa\kappa_1 + (1-\varkappa)\kappa_2\right) \leq \varkappa\mathcal{J}\left(\kappa_1\right) + (1-\varkappa)\mathcal{J}(\kappa_2),$$

for all $\kappa_1, \kappa_2 \in I$ and all $\varkappa \in [0,1]$.

The most famous result connected to this notion is the one called the Hermite-Hadamard inequality, which can be formulated as follows (see [22]): For a convex function $\mathcal{J}$ defined on the interval $I = [a,b]$, we have

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DOI: https://doi.org/10.29020/nybg.ejpam.v16i3.4850

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$$\mathcal{J}\left(\tfrac{a+b}{2}\right) \leq \tfrac{1}{b-a}\int_a^b \mathcal{J}(t)dt \leq \tfrac{\mathcal{J}(a)+\mathcal{J}(b)}{2}. \tag{1}$$

Several scientists have been interested in inequalities related to (1). In [14], Kirmaci established the following result connected to the left part of (1) for the class of functions whose first derivatives in absolute value are convex, known as the midpoint inequality.

$$\left|\mathcal{J}\left(\tfrac{a+b}{2}\right) - \tfrac{1}{b-a}\int_a^b \mathcal{J}(t)dt\right| \leq \tfrac{b-a}{8}\left(\left|\mathcal{J}'(a)\right| + \left|\mathcal{J}'(b)\right|\right). \tag{2}$$

This estimate holds even for the right part of inequality (1), also known as the trapezoid inequality, as was proved by Dragomir and Agarwal in [6].

$$\left|\tfrac{\mathcal{J}(a)+\mathcal{J}(b)}{2} - \tfrac{1}{b-a}\int_a^b \mathcal{J}(t)dt\right| \leq \tfrac{b-a}{8}\left(\left|\mathcal{J}'(a)\right| + \left|\mathcal{J}'(b)\right|\right). \tag{3}$$

In [11], Alomari et al. gave a companion of Ostrowski inequality for the same classe of functions which represents a generalization of the two previous results as follows

$$\begin{aligned}&\left|\tfrac{\mathcal{J}(x)+\mathcal{J}(a+b-x)}{2} - \tfrac{1}{b-a}\int_a^b \mathcal{J}(t)dt\right| \\ \leq& \tfrac{(x-a)^2}{6(b-a)}\left(\left|\mathcal{J}'(a)\right| + \left|\mathcal{J}'(b)\right|\right) + \tfrac{8(x-a)^2+3(a+b-2x)^2}{24(b-a)}\left(\left|\mathcal{J}'(x)\right| + \left|\mathcal{J}'(a+b-x)\right|\right).\end{aligned}$$

Note that both inequalities (2) and (3) can be derived from the preceding result. Specifically, the trapezoid type inequality is obtained for $x = a$, whereas midpoint inequality can be deduced by substituting $x = \frac{a+b}{2}$ and utilizing the convexity of $|\mathcal{J}'|$, i.e.,

$$\left|\mathcal{J}'\left(\tfrac{a+b}{2}\right)\right| \leq \tfrac{|\mathcal{J}'(a)|+|\mathcal{J}'(b)|}{2}.$$

On the other hand, in their paper [9], Hudzik and Maligranda explored the class of s-convex functions in the second sense. This class is defined by the following property: A function $\mathcal{J} : [0, \infty) \to \mathbb{R}$ is said to be s-convex in the second sense if the inequality

$$\mathcal{J}\left(\varkappa u + (1-\varkappa)v\right) \leq \varkappa^s \mathcal{J}(u) + (1-\varkappa)^s \mathcal{J}(v)$$

holds for all $u, v \in I$, $\varkappa \in [0,1]$, and $s \in (0,1]$.

The counterpart of the Hermite-Hadamard inequality for s-convex functions was introduced by Dragomir and Fitzpatrick in [5] in the following manner.

$$2^{s-1}\mathcal{J}\left(\tfrac{a+b}{2}\right) \leq \tfrac{1}{b-a}\int_a^b \mathcal{J}(t)dt \leq \tfrac{\mathcal{J}(a)+\mathcal{J}(b)}{s+1}. \tag{4}$$

Recently, scientists and engineers have taken a keen interest in fractal sets and fractal theory. According to Mandelbrot [8, 15], a set is considered fractal when its Hausdorff dimension exceeds its topological dimension. Recently, several studies have been conducted with the aim of extending some results related to integral inequalities to fractal calculus, using various forms of generalized convexity. Here are some references [1–4, 7, 10, 12, 13, 16–19, 23]. Yang's research in [24] focuses extensively on investigating and advancing local fractional calculus.

In their publications [24, 25], Gao-Yang-Kang proposed the concept of local fractional integral and derivative. Their definition of the fractal set of real numbers $\mathbb{R}^{\gamma}$ specifies the following properties.

If κ_1^{γ}, κ_2^{γ}, and κ_3^{γ} are within the set $\mathbb{R}^{\gamma}$, then the following statements can be made:

- $\kappa_1^{\gamma}+\kappa_2^{\gamma}$ and $\kappa_1^{\gamma}\kappa_2^{\gamma}$ belongs the set $\mathbb{R}^{\gamma}$,
- $\kappa_1^{\gamma}+\kappa_2^{\gamma}=\kappa_2^{\gamma}+\kappa_1^{\gamma}=(\kappa_1+\kappa_2)^{\gamma}=(\kappa_2+\kappa_1)^{\gamma}$,
- $\kappa_1^{\gamma}+(\kappa_2^{\gamma}+\kappa_3^{\gamma})=(\kappa_1+\kappa_2)^{\gamma}+\kappa_3^{\gamma}$,
- $\kappa_1^{\gamma}\kappa_2^{\gamma}=\kappa_2^{\gamma}\kappa_1^{\gamma}=(\kappa_1\kappa_2)^{\gamma}=(\kappa_2\kappa_1)^{\gamma}$,
- $\kappa_1^{\gamma}\left(\kappa_2^{\gamma}\kappa_3^{\gamma}\right)=\left(\kappa_1^{\gamma}\kappa_2^{\gamma}\right)\kappa_3^{\gamma}$,
- $\kappa_1^{\gamma}\left(\kappa_2^{\gamma}+\kappa_3^{\gamma}\right)=\kappa_1^{\gamma}\kappa_2^{\gamma}+\kappa_1^{\gamma}\kappa_3^{\gamma}$,
- $\kappa_1^{\gamma}+0^{\gamma}=0^{\gamma}+\kappa_1^{\gamma}=\kappa_1^{\gamma}$ and $\kappa_1^{\gamma}1^{\gamma}=1^{\gamma}\kappa_1^{\gamma}=\kappa_1^{\gamma}$.

Lemma 1 ([24])**.** *Let $C_{\gamma}([a,b])$ be the set of all local fractional continuous functions on $[a,b]$ and $D_{\gamma}([a,b])$ the set of all local fractional differentiable functions on $[a,b]$. It can then be stated that:*

(i) Suppose that $\mathcal{J}(t)=\mathcal{Q}^{(\gamma)}(t)\in C_{\gamma}[a,b]$, then we have

$$ {}_aI_b^{\gamma}\mathcal{J}(t)=\mathcal{Q}(b)-\mathcal{Q}(a). $$

(ii) Suppose that $\mathcal{J},\mathcal{Q}\in D_{\gamma}[a,b]$ and $\mathcal{J}^{(\gamma)}(t),\mathcal{Q}^{(\gamma)}(t)\in C_{\gamma}[a,b]$, then we have

$$ {}_aI_b^{\gamma}\mathcal{J}(t)\,\mathcal{Q}^{(\gamma)}(t)=\mathcal{J}(t)\,\mathcal{Q}(t)|\,_a^b-{}_aI_b^{\gamma}\mathcal{J}^{(\gamma)}(t)\,\mathcal{Q}(t). $$

Lemma 2 ([24])**.** *For $\mathcal{J}(t)=t^{k\gamma}$, we have following equations*

$$ \frac{d^{\gamma}t^{k\gamma}}{dt^{\gamma}}=\frac{\Gamma(1+k\gamma)}{\Gamma(1+(k-1)\gamma)}t^{(k-1)\gamma}, $$

$$ \frac{1}{\Gamma(1+\gamma)}\int_a^b t^{k\gamma}\,(dt)^{\gamma}=\frac{\Gamma(1+k\gamma)}{\Gamma(1+(k+1)\gamma)}\left(b^{(k+1)\gamma}-a^{(k+1)\gamma}\right),k\in\mathbb{R}. $$

Lemma 3 (Generalized Hölder's inequality [4]). *Let $\mathcal{J}, \mathcal{Q} \in C_\gamma[a,b]$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\frac{1}{\Gamma(1+\gamma)}\int_a^b |\mathcal{J}(t)\,\mathcal{Q}(t)|\,(dt)^\gamma = \left(\frac{1}{\Gamma(1+\gamma)}\int_a^b |\mathcal{J}(t)|^p\,(dt)^\gamma\right)^{\frac{1}{p}}\left(\frac{1}{\Gamma(1+\gamma)}\int_a^b |\mathcal{Q}(t)|^q\,(dt)^\gamma\right)^{\frac{1}{q}}.$$

Definition 1 ([24]). *Let $\mathcal{J} : I \subseteq \mathbb{R} \to \mathbb{R}^\gamma$. For any $\kappa_1, \kappa_2 \in I$ and $\varkappa \in [0,1]$, if*

$$\mathcal{J}(\varkappa\kappa_1 + (1-\varkappa)\kappa_2) \leq \varkappa^\gamma \mathcal{J}(\kappa_1) + (1-\varkappa)^\gamma \mathcal{J}(\kappa_2)$$

holds, then $\mathcal{J}$ is a generalized convex function on I.

More scientists have made efforts to extend the notion of convexity in order to cover a wider class of functions. One of the most interesting extensions that has emerged is the generalized s-convexity introduced in [20].

Definition 2. *Let $\mathcal{J} : I \subseteq \mathbb{R} \to \mathbb{R}^\gamma$. For any $\kappa_1, \kappa_2 \in I$ and $\varkappa \in [0,1]$, if*

$$\mathcal{J}(\varkappa\kappa_1 + (1-\varkappa)\kappa_2) \leq \varkappa^{s\gamma} \mathcal{J}(\kappa_1) + (1-\varkappa)^{s\gamma} \mathcal{J}(\kappa_2)$$

holds for some fixed $s \in (0,1]$, then $\mathcal{J}$ is a generalized s-convex function in the second sense on I.

In [21], the authors gave the analogue of inequality (1) for generalized s-convex functions on fractal set as follows

$$\frac{2^{(s-1)\gamma}}{\Gamma(1+\gamma)}\mathcal{J}\left(\frac{a+b}{2}\right) \leq \frac{{}_aI_b^\gamma \mathcal{J}(t)}{(b-a)^\gamma} \leq \frac{\Gamma(1+s\gamma)}{\Gamma(1+(s+1)\gamma)}\left(\mathcal{J}(a) + \mathcal{J}(b)\right),\ 0 < s \leq 1. \tag{5}$$

This paper examines the companion of Ostrowski's inequality, as studied by the authors in [11], within the context of fractal sets. We start by introducing a new identity related to local fractional integrals, on the basis of which we establish several inequalities for functions possessing generalized s-convex and s-concave derivatives. The study is concluded with an example that justifies the correctness of the obtained results, as well as a few applications.

2. Main results

In order to demonstrate our results, it is necessary to present the following lemma.

Lemma 4. *Suppose $\mathcal{J} : I = [a,b] \to \mathbb{R}^\gamma$ is a differentiable function on I with $a < b$, and $\mathcal{J}^{(\gamma)} \in C_\gamma[a,b]$. Then, for all $x \in [a, \frac{a+b}{2}]$, the following equation is satisfied*

$$\begin{aligned}&\frac{\mathcal{J}(x)+\mathcal{J}(a+b-x)}{2^\gamma} - \frac{\Gamma(\gamma+1)}{(b-a)^\gamma}\,{}_aI_b^\gamma \mathcal{J}(t)\\ &= \frac{(x-a)^{2\gamma}}{(b-a)^\gamma}\left(\frac{1}{\Gamma(\gamma+1)}\int_0^1 \eta^\gamma \mathcal{J}^{(\gamma)}((1-\eta)a + \eta x)\,(d\eta)^\gamma\right.\end{aligned}$$

$$+ \frac{1}{\Gamma(\gamma+1)}\int_0^1 (\eta-1)^\gamma \mathcal{J}^{(\gamma)}\left((1-\eta)(a+b-x)+\eta b\right)(d\eta)^\gamma \Bigg)$$

$$+ \frac{(a+b-2x)^{2\gamma}}{4^\gamma (b-a)^\gamma}\left(\frac{1}{\Gamma(\gamma+1)}\int_0^1 (\eta-1)^\gamma \mathcal{J}^{(\gamma)}\left((1-\eta)x+\eta\tfrac{a+b}{2}\right)(d\eta)^\gamma\right.$$

$$\left. + \frac{1}{\Gamma(\gamma+1)}\int_0^1 \eta^\gamma \mathcal{J}^{(\gamma)}\left((1-\eta)\tfrac{a+b}{2}+\eta(a+b-x)\right)(d\eta)^\gamma\right).$$

Proof. Let

$$I = \frac{(x-a)^{2\gamma}}{(b-a)^\gamma} I_1 + \frac{(a+b-2x)^{2\gamma}}{4^\gamma(b-a)^\gamma} I_2 + \frac{(a+b-2x)^{2\gamma}}{4^\gamma(b-a)^\gamma} I_3 + \frac{(x-a)^{2\gamma}}{(b-a)^\gamma} I_4, \tag{6}$$

where

$$I_1 = \frac{1}{\Gamma(\gamma+1)}\int_0^1 \eta^\gamma \mathcal{J}^{(\gamma)}\left((1-\eta)a+\eta x\right)(d\eta)^\gamma,$$

$$I_2 = \frac{1}{\Gamma(\gamma+1)}\int_0^1 (\eta-1)^\gamma \mathcal{J}^{(\gamma)}\left((1-\eta)x+\eta\tfrac{a+b}{2}\right)(d\eta)^\gamma,$$

$$I_3 = \frac{1}{\Gamma(\gamma+1)}\int_0^1 \eta^\gamma \mathcal{J}^{(\gamma)}\left((1-\eta)\tfrac{a+b}{2}+\eta(a+b-x)\right)(d\eta)^\gamma$$

and

$$I_4 = \frac{1}{\Gamma(\gamma+1)}\int_0^1 (\eta-1)^\gamma \mathcal{J}^{(\gamma)}\left((1-\eta)(a+b-x)+\eta b\right)(d\eta)^\gamma.$$

Using Lemmas 1 and 2, we get

$$I_1 = \frac{1^\gamma}{(x-a)^\gamma}\eta^\gamma \mathcal{J}\left((1-\eta)a+\eta x\right)\Big|_{\eta=0}^{\eta=1} \tag{7}$$

$$- \frac{1^\gamma}{(x-a)^\gamma\Gamma(\gamma+1)}\int_0^1 \Gamma(\gamma+1)\mathcal{J}\left((1-\eta)a+\eta x\right)(d\eta)^\gamma$$

$$= \frac{1^\gamma}{(x-a)^\gamma}\mathcal{J}(x) - \frac{1^\gamma}{(x-a)^\gamma}\int_0^1 \mathcal{J}\left((1-\eta)a+\eta x\right)(d\varkappa)^\gamma$$

$$= \frac{1^\gamma}{(x-a)^\gamma}\mathcal{J}(x) - \frac{1^\gamma}{(x-a)^{2\gamma}}\int_a^x \mathcal{J}(\varpi)(d\varpi)^\gamma.$$

Similarly, we obtain

$$\begin{aligned} I_2 =& \tfrac{2^\gamma}{(a+b-2x)^\gamma} (\eta-1)^\gamma \mathcal{J}\left((1-\eta)x+\eta\tfrac{a+b}{2}\right)\Big|_{\eta=0}^{\eta=1} \\ &- \tfrac{2^\gamma}{(a+b-2x)^\gamma\Gamma(\gamma+1)}\int_0^1 \Gamma(\gamma+1)\mathcal{J}\left((1-\eta)x+\eta\tfrac{a+b}{2}\right)(d\eta)^\gamma \\ =& \tfrac{2^\gamma}{(a+b-2x)^\gamma}\mathcal{J}(x) - \tfrac{4^\gamma}{(a+b-2x)^{2\gamma}}\int_x^{\frac{a+b}{2}} \mathcal{J}(\varpi)(d\varpi)^\gamma, \end{aligned} \tag{8}$$

$$\begin{aligned} I_3 =& \tfrac{2^\gamma}{(a+b-2x)^\gamma}\eta^\gamma \mathcal{J}\left((1-\eta)\tfrac{a+b}{2}+\eta(a+b-x)\right)\Big|_{\eta=0}^{\eta=1} \\ &- \tfrac{2^\gamma}{(a+b-2x)^\gamma\Gamma(\gamma+1)}\int_0^1 \Gamma(\gamma+1)\mathcal{J}\left((1-\eta)\tfrac{a+b}{2}+\eta(a+b-x)\right)(d\eta)^\gamma \\ =& \tfrac{2^\gamma}{(a+b-2x)^\gamma}\mathcal{J}(a+b-x) - \tfrac{(4)^\gamma}{(a+b-2x)^{2\gamma}}\int_{\frac{a+b}{2}}^{a+b-x} \mathcal{J}(\varpi)(d\varpi)^\gamma \end{aligned} \tag{9}$$

and

$$\begin{aligned} I_4 =& \tfrac{1^\gamma}{(x-a)^\gamma} (\eta-1)^\gamma \mathcal{J}((1-\eta)(a+b-x)+\eta b)\Big|_{\eta=0}^{\eta=1} \\ &- \tfrac{1^\gamma}{(x-a)^\gamma\Gamma(\gamma+1)}\int_0^1 \Gamma(\gamma+1)\mathcal{J}((1-\eta)(a+b-x)+\eta b)(d\eta)^\gamma \\ =& \tfrac{1^\gamma}{(x-a)^\gamma}\mathcal{J}(a+b-x) - \tfrac{1^\gamma}{(x-a)^{2\gamma}}\int_{a+b-x}^{b} \mathcal{J}(\varpi)(d\varpi)^\gamma. \end{aligned} \tag{10}$$

After substituting equations (7)-(10) into equation (6), we multiply and divide the resulting equation by $\Gamma(\gamma+1)$ to obtain the desired result.

Theorem 1. *Suppose $\mathcal{J} : [a,b] \to \mathbb{R}^\gamma$ is a differentiable function on $[a,b]$ such that $\mathcal{J} \in D_\gamma[a,b]$ and $\mathcal{J}^{(\gamma)} \in C_\gamma[a,b]$ with $0 \le a < b$. If $\left|\mathcal{J}^{(\gamma)}\right|$ is generalized s-convex in the second sense on $[a,b]$, then the following inequality holds*

$$\begin{aligned} &\left|\tfrac{\mathcal{J}(x)+\mathcal{J}(a+b-x)}{2^\gamma} - \tfrac{\Gamma(\gamma+1)}{(b-a)^\gamma}\, {}_aI_b^\gamma \mathcal{J}(t)\right| \\ \le& \tfrac{(x-a)^{2\gamma}}{(b-a)^\gamma}\left(\left(\tfrac{\Gamma(1+s\gamma)}{\Gamma(1+(s+1)\gamma)} - \tfrac{\Gamma(1+(s+1)\gamma)}{\Gamma(1+(s+2)\gamma)}\right)\left(\left|\mathcal{J}^{(\gamma)}(a)\right| + \left|\mathcal{J}^{(\gamma)}(b)\right|\right)\right. \end{aligned}$$

$$+\frac{\Gamma(1+(s+1)\gamma)}{\Gamma(1+(s+2)\gamma)}\left(\left|\mathcal{J}^{(\gamma)}(x)\right|+\left|\mathcal{J}^{(\gamma)}(a+b-x)\right|\right)\Big)$$
$$+\frac{(a+b-2x)^{2\gamma}}{4^{\gamma}(b-a)^{\gamma}}\left(\frac{\Gamma(1+(s+1)\gamma)}{\Gamma(1+(s+2)\gamma)}\left(\left|\mathcal{J}^{(\gamma)}(x)\right|+\left|\mathcal{J}^{(\gamma)}(a+b-x)\right|\right)\right.$$
$$\left.+2^{\gamma}\left(\frac{\Gamma(1+s\gamma)}{\Gamma(1+(s+1)\gamma)}-\frac{\Gamma(1+(s+1)\gamma)}{\Gamma(1+(s+2)\gamma)}\right)\left|\mathcal{J}^{(\gamma)}\left(\frac{a+b}{2}\right)\right|\right).$$

Proof. Using Lemma 4, properties of modulus, and the generalized s-convexity of $\left|\mathcal{J}^{(\gamma)}\right|$, we can conclude that

$$\left|\frac{\mathcal{J}(x)+\mathcal{J}(a+b-x)}{2^{\gamma}}-\frac{\Gamma(\gamma+1)}{(b-a)^{\gamma}}\,{}_aI_b^{\gamma}\mathcal{J}(t)\right|$$
$$\leq\frac{(x-a)^{2\gamma}}{(b-a)^{\gamma}}\left(\frac{1}{\Gamma(\gamma+1)}\int_0^1\eta^{\gamma}\left|\mathcal{J}^{(\gamma)}((1-\eta)a+\eta x)\right|(d\eta)^{\gamma}\right.$$
$$\left.+\frac{1}{\Gamma(\gamma+1)}\int_0^1(1-\eta)^{\gamma}\left|\mathcal{J}^{(\gamma)}((1-\eta)(a+b-x)+\eta b)\right|(d\eta)^{\gamma}\right)$$
$$+\frac{(a+b-2x)^{2\gamma}}{4^{\gamma}(b-a)^{\gamma}}\left(\frac{1}{\Gamma(\gamma+1)}\int_0^1(1-\eta)^{\gamma}\left|\mathcal{J}^{(\gamma)}\left((1-\eta)x+\eta\frac{a+b}{2}\right)\right|(d\eta)^{\gamma}\right.$$
$$\left.+\frac{1}{\Gamma(\gamma+1)}\int_0^1\eta^{\gamma}\left|\mathcal{J}^{(\gamma)}\left((1-\eta)\frac{a+b}{2}+\eta(a+b-x)\right)\right|(d\eta)^{\gamma}\right)$$
$$\leq\frac{(x-a)^{2\gamma}}{(b-a)^{\gamma}}\left(\frac{1}{\Gamma(\gamma+1)}\int_0^1\eta^{\gamma}\left((1-\eta)^{s\gamma}\left|\mathcal{J}^{(\gamma)}(a)\right|+\eta^{s\gamma}\left|\mathcal{J}^{(\gamma)}(x)\right|\right)(d\eta)^{\gamma}\right.$$
$$\left.+\frac{1}{\Gamma(\gamma+1)}\int_0^1(1-\eta)^{\gamma}\left((1-\eta)^{s\gamma}\left|\mathcal{J}^{(\gamma)}(a+b-x)\right|+\eta^{s\gamma}\left|\mathcal{J}^{(\gamma)}(b)\right|\right)(d\eta)^{\gamma}\right)$$
$$+\frac{(a+b-2x)^{2\gamma}}{4^{\gamma}(b-a)^{\gamma}}\left(\frac{1}{\Gamma(\gamma+1)}\int_0^1(1-\eta)^{\gamma}\left((1-\eta)^{s\gamma}\left|\mathcal{J}^{(\gamma)}(x)\right|+\eta^{s\gamma}\left|\mathcal{J}^{(\gamma)}\left(\frac{a+b}{2}\right)\right|\right)(d\eta)^{\gamma}\right.$$
$$\left.+\frac{1}{\Gamma(\gamma+1)}\int_0^1\eta^{\gamma}\left((1-\eta)^{s\gamma}\left|\mathcal{J}^{(\gamma)}\left(\frac{a+b}{2}\right)\right|+\eta^{s\gamma}\left|\mathcal{J}^{(\gamma)}(a+b-x)\right|\right)(d\eta)^{\gamma}\right)$$
$$=\frac{(x-a)^{2\gamma}}{(b-a)^{\gamma}}\left(\left(\frac{1}{\Gamma(\gamma+1)}\int_0^1\eta^{\gamma}(1-\eta)^{s\gamma}(d\eta)^{\gamma}\right)\left|\mathcal{J}^{(\gamma)}(a)\right|+\left(\frac{1}{\Gamma(\gamma+1)}\int_0^1\eta^{(s+1)\gamma}(d\eta)^{\gamma}\right)\left|\mathcal{J}^{(\gamma)}(x)\right|\right.$$
$$+\left(\frac{1}{\Gamma(\gamma+1)}\int_0^1(1-\eta)^{(s+1)\gamma}(d\eta)^{\gamma}\right)\left|\mathcal{J}^{(\gamma)}(a+b-x)\right|$$

$$
\begin{aligned}
&+\left(\frac{1}{\Gamma(\gamma+1)}\int_0^1 (1-\eta)^{\gamma}\,\eta^{s\gamma}\,(d\eta)^{\gamma}\right)\left|\mathcal{J}^{(\gamma)}(b)\right|\Bigg)\\
&+\frac{(a+b-2x)^{2\gamma}}{4^{\gamma}(b-a)^{\gamma}}\Bigg(\left(\frac{1}{\Gamma(\gamma+1)}\int_0^1 (1-\eta)^{(s+1)\gamma}\,(d\eta)^{\gamma}\right)\left|\mathcal{J}^{(\gamma)}(x)\right|\\
&+\left(\frac{1}{\Gamma(\gamma+1)}\int_0^1 (1-\eta)^{\gamma}\,\eta^{s\gamma}\,(d\eta)^{\gamma}+\frac{1}{\Gamma(\gamma+1)}\int_0^1 \eta^{\gamma}\,(1-\eta)^{s\gamma}\,(d\eta)^{\gamma}\right)\left|\mathcal{J}^{(\gamma)}\left(\tfrac{a+b}{2}\right)\right|\\
&+\left(\frac{1}{\Gamma(\gamma+1)}\int_0^1 \eta^{(s+1)\gamma}\,(d\eta)^{\gamma}\right)\left|\mathcal{J}^{(\gamma)}(a+b-x)\right|\Bigg)\\
=&\frac{(x-a)^{2\gamma}}{(b-a)^{\gamma}}\left(\left(\frac{\Gamma(1+s\gamma)}{\Gamma(1+(s+1)\gamma)}-\frac{\Gamma(1+(s+1)\gamma)}{\Gamma(1+(s+2)\gamma)}\right)\left(\left|\mathcal{J}^{(\gamma)}(a)\right|+\left|\mathcal{J}^{(\gamma)}(b)\right|\right)\right.\\
&\left.+\frac{\Gamma(1+(s+1)\gamma)}{\Gamma(1+(s+2)\gamma)}\left(\left|\mathcal{J}^{(\gamma)}(x)\right|+\left|\mathcal{J}^{(\gamma)}(a+b-x)\right|\right)\right)\\
&+\frac{(a+b-2x)^{2\gamma}}{4^{\gamma}(b-a)^{\gamma}}\left(\frac{\Gamma(1+(s+1)\gamma)}{\Gamma(1+(s+2)\gamma)}\left(\left|\mathcal{J}^{(\gamma)}(x)\right|+\left|\mathcal{J}^{(\gamma)}(a+b-x)\right|\right)\right.\\
&\left.+2^{\gamma}\left(\frac{\Gamma(1+s\gamma)}{\Gamma(1+(s+1)\gamma)}-\frac{\Gamma(1+(s+1)\gamma)}{\Gamma(1+(s+2)\gamma)}\right)\left|\mathcal{J}^{(\gamma)}\left(\tfrac{a+b}{2}\right)\right|\right),
\end{aligned}
$$

where we have used the facts that

$$
\begin{aligned}
\frac{1}{\Gamma(\gamma+1)}\int_0^1 \eta^{\gamma}\,(1-\eta)^{s\gamma}\,(d\eta)^{\gamma} &=\frac{1}{\Gamma(\gamma+1)}\int_0^1 (1-\eta)^{\gamma}\,\eta^{s\gamma}\,(d\eta)^{\gamma}\\
&=\frac{\Gamma(1+s\gamma)}{\Gamma(1+(s+1)\gamma)}-\frac{\Gamma(1+(s+1)\gamma)}{\Gamma(1+(s+2)\gamma)} \qquad (11)
\end{aligned}
$$

and

$$
\frac{1}{\Gamma(\gamma+1)}\int_0^1 \eta^{(s+1)\gamma}\,(d\eta)^{\gamma}=\frac{1}{\Gamma(\gamma+1)}\int_0^1 (1-\eta)^{(s+1)\gamma}\,(d\eta)^{\gamma}=\frac{\Gamma(1+(s+1)\gamma)}{\Gamma(1+(s+2)\gamma)}. \qquad (12)
$$

The proof is completed.

Corollary 1. *In Theorem 1, if we take $s=1$, we obtain*

$$
\begin{aligned}
&\left|\frac{\mathcal{J}(x)+\mathcal{J}(a+b-x)}{2^{\gamma}}-\frac{\Gamma(\gamma+1)}{(b-a)^{\gamma}}\,{}_aI_b^{\gamma}\mathcal{J}(t)\right|\\
\leq&\frac{(x-a)^{2\gamma}}{(b-a)^{\gamma}}\left(\left(\frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)}-\frac{\Gamma(1+2\gamma)}{\Gamma(1+3\gamma)}\right)\left(\left|\mathcal{J}^{(\gamma)}(a)\right|+\left|\mathcal{J}^{(\gamma)}(b)\right|\right)\right.\\
&\left.+\frac{\Gamma(1+2\gamma)}{\Gamma(1+3\gamma)}\left(\left|\mathcal{J}^{(\gamma)}(x)\right|+\left|\mathcal{J}^{(\gamma)}(a+b-x)\right|\right)\right)\\
&+\frac{(a+b-2x)^{2\gamma}}{4^{\gamma}(b-a)^{\gamma}}\left(\frac{\Gamma(1+2\gamma)}{\Gamma(1+3\gamma)}\left(\left|\mathcal{J}^{(\gamma)}(x)\right|+\left|\mathcal{J}^{(\gamma)}(a+b-x)\right|\right)\right.\\
&\left.+2^{\gamma}\left(\frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)}-\frac{\Gamma(1+2\gamma)}{\Gamma(1+3\gamma)}\right)\left|\mathcal{J}^{(\gamma)}\left(\tfrac{a+b}{2}\right)\right|\right).
\end{aligned}
$$

Corollary 2. *In Theorem 1 applying the generalized s-convexity of* $\left|\mathcal{J}^{(\gamma)}\right|$, *i.e*

$$\left|\mathcal{J}^{(\gamma)}\left(\frac{a+b}{2}\right)\right| \leq 2^{(1-s)\gamma}\frac{\Gamma(1+s\gamma)\Gamma(1+\gamma)}{\Gamma(1+(s+1)\gamma)}\left(\left|\mathcal{J}^{(\gamma)}(x)\right|+\left|\mathcal{J}^{(\gamma)}(a+b-x)\right|\right),$$

we obtain

$$\begin{aligned}&\left|\frac{\mathcal{J}(x)+\mathcal{J}(a+b-x)}{2^{\gamma}}-\frac{\Gamma(\gamma+1)}{(b-a)^{\gamma}}\,{}_aI_b^{\gamma}\mathcal{J}(t)\right|\\ \leq&\frac{(x-a)^{2\gamma}}{(b-a)^{\gamma}}\left(\frac{\Gamma(1+s\gamma)}{\Gamma(1+(s+1)\gamma)}-\frac{\Gamma(1+(s+1)\gamma)}{\Gamma(1+(s+2)\gamma)}\right)\left(\left|\mathcal{J}^{(\gamma)}(a)\right|+\left|\mathcal{J}^{(\gamma)}(b)\right|\right)\\ &+\left(\frac{(x-a)^{2\gamma}}{(b-a)^{\gamma}}\frac{\Gamma(1+(s+1)\gamma)}{\Gamma(1+(s+2)\gamma)}+\frac{(a+b-2x)^{2\gamma}}{4^{\gamma}(b-a)^{\gamma}}\left(\frac{\Gamma(1+(s+1)\gamma)}{\Gamma(1+(s+2)\gamma)}-\frac{2^{(2-s)\gamma}\Gamma(1+\gamma)\Gamma(1+s\gamma)}{\Gamma(1+(s+2)\gamma)}\right.\right.\\ &\left.\left.+\ 2^{(2-s)\gamma}\Gamma(1+\gamma)\left(\frac{\Gamma(1+s\gamma)}{\Gamma(1+(s+1)\gamma)}\right)^{2\gamma}\right)\right)\left(\left|\mathcal{J}^{(\gamma)}(x)\right|+\left|\mathcal{J}^{(\gamma)}(a+b-x)\right|\right).\end{aligned}$$

Corollary 3. *In Corollary 2, taking* $s=1$ *we obtain*

$$\begin{aligned}&\left|\frac{\mathcal{J}(x)+\mathcal{J}(a+b-x)}{2^{\gamma}}-\frac{\Gamma(\gamma+1)}{(b-a)^{\gamma}}\,{}_aI_b^{\gamma}\mathcal{J}(t)\right|\\ \leq&\frac{(x-a)^{2\gamma}}{(b-a)^{\gamma}}\left(\frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)}-\frac{\Gamma(1+2\gamma)}{\Gamma(1+3\gamma)}\right)\left(\left|\mathcal{J}^{(\gamma)}(a)\right|+\left|\mathcal{J}^{(\gamma)}(b)\right|\right)\\ &+\left(\frac{(x-a)^{2\gamma}}{(b-a)^{\gamma}}\frac{\Gamma(1+2\gamma)}{\Gamma(1+3\gamma)}+\frac{(a+b-2x)^{2\gamma}}{4^{\gamma}(b-a)^{\gamma}}\left(\frac{\Gamma(1+2\gamma)}{\Gamma(1+3\gamma)}-\frac{2^{\gamma}(\Gamma(1+\gamma))^{2\gamma}}{\Gamma(1+3\gamma)}\right.\right.\\ &\left.\left.+\ 2^{\gamma}\Gamma(1+\gamma)\left(\frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)}\right)^{2\gamma}\right)\right)\left(\left|\mathcal{J}^{(\gamma)}(x)\right|+\left|\mathcal{J}^{(\gamma)}(a+b-x)\right|\right).\end{aligned}$$

Remark 1. *For* $\gamma=1$, *Corollary 3 will be reduces to Theorem 5 from [11].*

Corollary 4. *In Theorem 1, taking* $x=a$ *we get*

$$\begin{aligned}&\left|\frac{\mathcal{J}(a)+\mathcal{J}(b)}{2^{\gamma}}-\frac{\Gamma(\gamma+1)}{(b-a)^{\gamma}}\,{}_aI_b^{\gamma}\mathcal{J}(t)\right|\\ \leq&\frac{(b-a)^{\gamma}}{4^{\gamma}}\left(\frac{\Gamma(1+(s+1)\gamma)}{\Gamma(1+(s+2)\gamma)}\left(\left|\mathcal{J}^{(\gamma)}(a)\right|+\left|\mathcal{J}^{(\gamma)}(b)\right|\right)\right.\\ &\left.+2^{\gamma}\left(\frac{\Gamma(1+s\gamma)}{\Gamma(1+(s+1)\gamma)}-\frac{\Gamma(1+(s+1)\gamma)}{\Gamma(1+(s+2)\gamma)}\right)\left|\mathcal{J}^{(\gamma)}\left(\frac{a+b}{2}\right)\right|\right).\end{aligned}$$

Corollary 5. *In Corollary 4 using the generalized s-convexity of* $\left|\mathcal{J}^{(\gamma)}\right|$ *i.e.*

$$\left|\mathcal{J}^{(\gamma)}\left(\frac{a+b}{2}\right)\right| \leq 2^{(1-s)\gamma}\frac{\Gamma(1+s\gamma)\Gamma(1+\gamma)}{\Gamma(1+(s+1)\gamma)}\left(\left|\mathcal{J}^{(\gamma)}(a)\right|+\left|\mathcal{J}^{(\gamma)}(b)\right|\right),$$

we obtain

$$\begin{aligned}&\left|\frac{\mathcal{J}(a)+\mathcal{J}(b)}{2^{\gamma}}-\frac{\Gamma(\gamma+1)}{(b-a)^{\gamma}}\,{}_aI_b^{\gamma}\mathcal{J}(t)\right|\\ \leq&\frac{(b-a)^{\gamma}}{4^{\gamma}}\left(\frac{\Gamma(1+(s+1)\gamma)}{\Gamma(1+(s+2)\gamma)}+2^{(2-s)\gamma}\left(\frac{\Gamma(1+s\gamma)}{\Gamma(1+(s+1)\gamma)}\right)^{2\gamma}\Gamma(1+\gamma)\right.\\ &\left.-\ 2^{(2-s)\gamma}\frac{\Gamma(1+s\gamma)\Gamma(1+\gamma)}{\Gamma(1+(s+2)\gamma)}\right)\left(\left|\mathcal{J}^{(\gamma)}(a)\right|+\left|\mathcal{J}^{(\gamma)}(b)\right|\right).\end{aligned}$$

Corollary 6. *In Corollary 5, taking $s = 1$, we obtain*

$$\begin{aligned}&\left|\frac{\mathcal{J}(a)+\mathcal{J}(b)}{2^{\gamma}} - \frac{\Gamma(\gamma+1)}{(b-a)^{\gamma}}\, {}_aI_b^{\gamma}\mathcal{J}(t)\right| \\ &\leq \frac{(b-a)^{\gamma}}{4^{\gamma}}\left(\frac{\Gamma(1+2\gamma)}{\Gamma(1+3\gamma)} + 2^{\gamma}\left(\frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)}\right)^{2\gamma}\Gamma(1+\gamma) - 2^{\gamma}\frac{(\Gamma(1+\gamma))^{2\gamma}}{\Gamma(1+3\gamma)}\right)\left(\left|\mathcal{J}^{(\gamma)}(a)\right| + \left|\mathcal{J}^{(\gamma)}(b)\right|\right).\end{aligned}$$

Remark 2. *For $\gamma = 1$, Corollary 6 will be reduces to Theorem 2.2 from [6].*

Corollary 7. *In Theorem 1, taking $x = \frac{a+b}{2}$ we get*

$$\begin{aligned}&\left|\mathcal{J}\left(\frac{a+b}{2}\right) - \frac{\Gamma(\gamma+1)}{(b-a)^{\gamma}}\, {}_aI_b^{\gamma}\mathcal{J}(t)\right| \\ &\leq \frac{(b-a)^{\gamma}}{4^{\gamma}}\left(\left(\frac{\Gamma(1+s\gamma)}{\Gamma(1+(s+1)\gamma)} - \frac{\Gamma(1+(s+1)\gamma)}{\Gamma(1+(s+2)\gamma)}\right)\left(\left|\mathcal{J}^{(\gamma)}(a)\right| + \left|\mathcal{J}^{(\gamma)}(b)\right|\right)\right. \\ &\quad\left. + 2^{\gamma}\frac{\Gamma(1+(s+1)\gamma)}{\Gamma(1+(s+2)\gamma)}\left|\mathcal{J}^{(\gamma)}\left(\frac{a+b}{2}\right)\right|\right).\end{aligned}$$

Corollary 8. *In Corollary 7 using the generalized s-convexity of $\left|\mathcal{J}^{(\gamma)}\right|$*

$$\begin{aligned}&\left|\mathcal{J}\left(\frac{a+b}{2}\right) - \frac{\Gamma(\gamma+1)}{(b-a)^{\gamma}}\, {}_aI_b^{\gamma}\mathcal{J}(t)\right| \\ &\leq \frac{(b-a)^{\gamma}}{4^{\gamma}}\left(\frac{\Gamma(1+s\gamma)}{\Gamma(1+(s+1)\gamma)} - \frac{\Gamma(1+(s+1)\gamma)}{\Gamma(1+(s+2)\gamma)} + 2^{(2-s)\gamma}\frac{\Gamma(1+s\gamma)\Gamma(1+\gamma)}{\Gamma(1+(s+2)\gamma)}\right)\left(\left|\mathcal{J}^{(\gamma)}(a)\right| + \left|\mathcal{J}^{(\gamma)}(b)\right|\right).\end{aligned}$$

Corollary 9. *In Corollary 8 if we take $s = 1$ we obtain*

$$\begin{aligned}&\left|\mathcal{J}\left(\frac{a+b}{2}\right) - \frac{\Gamma(\gamma+1)}{(b-a)^{\gamma}}\, {}_aI_b^{\gamma}\mathcal{J}(t)\right| \\ &\leq \frac{(b-a)^{\gamma}}{4^{\gamma}}\left(\frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)} - \frac{\Gamma(1+2\gamma)}{\Gamma(1+3\gamma)} + 2^{\gamma}\frac{(\Gamma(1+\gamma))^{2\gamma}}{\Gamma(1+3\gamma)}\right)\left(\left|\mathcal{J}^{(\gamma)}(a)\right| + \left|\mathcal{J}^{(\gamma)}(b)\right|\right).\end{aligned}$$

Remark 3. *For $\gamma = 1$, Corollary 9 will be reduces to Theorem 2.2 from [14].*

Theorem 2. *Suppose $\mathcal{J} : [a,b] \to \mathbb{R}^{\gamma}$ is a differentiable function on $[a,b]$ such that $\mathcal{J} \in D_{\gamma}[a,b]$ and $\mathcal{J}^{(\gamma)} \in C_{\gamma}[a,b]$ with $0 \leq a < b$. If $\left|\mathcal{J}^{(\gamma)}\right|^q$ is generalized s-convex on $[a,b]$, where $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then we have*

$$\begin{aligned}&\left|\frac{\mathcal{J}(x)+\mathcal{J}(a+b-x)}{2^{\gamma}} - \frac{\Gamma(\gamma+1)}{(b-a)^{\gamma}}\, {}_aI_b^{\gamma}\mathcal{J}(t)\right| \\ &\leq \left(\frac{\Gamma(1+p\gamma)}{\Gamma(1+(p+1)\gamma)}\right)^{\frac{1}{p}}\left(\frac{\Gamma(1+s\gamma)}{\Gamma(1+(s+1)\gamma)}\right)^{\frac{1}{q}} \\ &\quad\times\left(\frac{(x-a)^{2\gamma}}{(b-a)^{\gamma}}\left(\left(\left|\mathcal{J}^{(\gamma)}(a)\right|^q + \left|\mathcal{J}^{(\gamma)}(x)\right|^q\right)^{\frac{1}{q}} + \left(\left|\mathcal{J}^{(\gamma)}(a+b-x)\right|^q + \left|\mathcal{J}^{(\gamma)}(b)\right|^q\right)^{\frac{1}{q}}\right)\right. \\ &\quad + \frac{(a+b-2x)^{2\gamma}}{4^{\gamma}(b-a)^{\gamma}}\left(\left(\left|\mathcal{J}^{(\gamma)}(x)\right|^q + \left|\mathcal{J}^{(\gamma)}\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right. \\ &\quad\left.\left. + \left(\left|\mathcal{J}^{(\gamma)}\left(\frac{a+b}{2}\right)\right|^q + \left|\mathcal{J}^{(\gamma)}(a+b-x)\right|^q\right)^{\frac{1}{q}}\right)\right).\end{aligned}$$

Proof. Using Lemma 4 as well as the generalized Hölder inequality, properties of modulus, and the generalized s-convexity of $\left|\mathcal{J}^{(\gamma)}\right|^q$, we can conclude that

$$\begin{aligned}
&\left|\frac{\mathcal{J}(x)+\mathcal{J}(a+b-x)}{2^\gamma}-\frac{\Gamma(\gamma+1)}{(b-a)^\gamma}\,{}_aI_b^\gamma\mathcal{J}(t)\right| \\
&\leq \frac{(x-a)^{2\gamma}}{(b-a)^\gamma}\left(\left(\frac{1}{\Gamma(\gamma+1)}\int_0^1\eta^{p\gamma}(d\eta)^\gamma\right)^{\frac{1}{p}}\left(\frac{1}{\Gamma(\gamma+1)}\int_0^1\left|\mathcal{J}^{(\gamma)}((1-\eta)a+\eta x)\right|^q(d\eta)^\gamma\right)^{\frac{1}{q}}\right. \\
&\quad\left.+\left(\frac{1}{\Gamma(\gamma+1)}\int_0^1(1-\eta)^{p\gamma}(d\eta)^\gamma\right)^{\frac{1}{p}}\left(\frac{1}{\Gamma(\gamma+1)}\int_0^1\left|\mathcal{J}^{(\gamma)}((1-\eta)(a+b-x)+\eta b)\right|^q(d\eta)^\gamma\right)^{\frac{1}{q}}\right) \\
&\quad+\frac{(a+b-2x)^{2\gamma}}{4^\gamma(b-a)^\gamma}\left(\left(\frac{1}{\Gamma(\gamma+1)}\int_0^1(1-\eta)^{p\gamma}(d\eta)^\gamma\right)^{\frac{1}{p}}\left(\frac{1}{\Gamma(\gamma+1)}\int_0^1\left|\mathcal{J}^{(\gamma)}\left((1-\eta)x+\eta\frac{a+b}{2}\right)\right|^q(d\eta)^\gamma\right)^{\frac{1}{q}}\right. \\
&\quad\left.+\left(\frac{1}{\Gamma(\gamma+1)}\int_0^1\eta^{p\gamma}(d\eta)^\gamma\right)^{\frac{1}{p}}\left(\frac{1}{\Gamma(\gamma+1)}\int_0^1\left|\mathcal{J}^{(\gamma)}\left((1-\eta)\frac{a+b}{2}+\eta(a+b-x)\right)\right|^q(d\eta)^\gamma\right)^{\frac{1}{q}}\right) \\
&\leq \frac{(x-a)^{2\gamma}}{(b-a)^\gamma}\left(\frac{\Gamma(1+p\gamma)}{\Gamma(1+(p+1)\gamma)}\right)^{\frac{1}{p}}\left(\left(\frac{1}{\Gamma(\gamma+1)}\int_0^1\left((1-\eta)^{s\gamma}\left|\mathcal{J}^{(\gamma)}(a)\right|^q+\eta^{s\gamma}\left|\mathcal{J}^{(\gamma)}(x)\right|^q\right)(d\eta)^\gamma\right)^{\frac{1}{q}}\right. \\
&\quad\left.+\left(\frac{1}{\Gamma(\gamma+1)}\int_0^1\left((1-\eta)^{s\gamma}\left|\mathcal{J}^{(\gamma)}(a+b-x)\right|^q+\eta^{s\gamma}\left|\mathcal{J}^{(\gamma)}(b)\right|^q\right)(d\eta)^\gamma\right)^{\frac{1}{q}}\right) \\
&\quad+\frac{(a+b-2x)^{2\gamma}}{4^\gamma(b-a)^\gamma}\left(\left(\frac{1}{\Gamma(\gamma+1)}\int_0^1(1-\eta)^{p\gamma}(d\eta)^\gamma\right)^{\frac{1}{p}}\right. \\
&\quad\times\left(\frac{1}{\Gamma(\gamma+1)}\int_0^1\left((1-\eta)^{s\gamma}\left|\mathcal{J}^{(\gamma)}(x)\right|^q+\eta^{s\gamma}\left|\mathcal{J}^{(\gamma)}\left(\frac{a+b}{2}\right)\right|^q\right)(d\eta)^\gamma\right)^{\frac{1}{q}} \\
&\quad+\left(\frac{1}{\Gamma(\gamma+1)}\int_0^1\eta^{p\gamma}(d\eta)^\gamma\right)^{\frac{1}{p}}
\end{aligned}$$

$$\times\left(\frac{1}{\Gamma(\gamma+1)}\int_0^1\left((1-\eta)^{s\gamma}\left|\mathcal{J}^{(\gamma)}\left(\frac{a+b}{2}\right)\right|^q+\eta^{s\gamma}\left|\mathcal{J}^{(\gamma)}(a+b-x)\right|^q\right)(d\eta)^\gamma\right)^{\frac{1}{q}}\Bigg)$$

$$=\left(\frac{\Gamma(1+p\gamma)}{\Gamma(1+(p+1)\gamma)}\right)^{\frac{1}{p}}\left(\frac{\Gamma(1+s\gamma)}{\Gamma(1+(s+1)\gamma)}\right)^{\frac{1}{q}}$$
$$\times\left(\frac{(x-a)^{2\gamma}}{(b-a)^\gamma}\left(\left(\left|\mathcal{J}^{(\gamma)}(a)\right|^q+\left|\mathcal{J}^{(\gamma)}(x)\right|^q\right)^{\frac{1}{q}}+\left(\left|\mathcal{J}^{(\gamma)}(a+b-x)\right|^q+\left|\mathcal{J}^{(\gamma)}(b)\right|^q\right)^{\frac{1}{q}}\right)\right.$$
$$+\frac{(a+b-2x)^{2\gamma}}{4^\gamma(b-a)^\gamma}\left(\left(\left|\mathcal{J}^{(\gamma)}(x)\right|^q+\left|\mathcal{J}^{(\gamma)}\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.\left.+\left(\left|\mathcal{J}^{(\gamma)}\left(\frac{a+b}{2}\right)\right|^q+\left|\mathcal{J}^{(\gamma)}(a+b-x)\right|^q\right)^{\frac{1}{q}}\right)\right),$$

where we have used the fact that

$$\frac{1}{\Gamma(\gamma+1)}\int_0^1\eta^{p\gamma}(d\eta)^\gamma=\frac{1}{\Gamma(\gamma+1)}\int_0^1(1-\eta)^{p\gamma}(d\eta)^\gamma=\frac{\Gamma(1+p\gamma)}{\Gamma(1+(p+1)\gamma)}.$$

The proof is completed.

Corollary 10. *In Theorem 2, taking $x=a$, we obtain*

$$\left|\frac{\mathcal{J}(a)+\mathcal{J}(b)}{2^\gamma}-\frac{\Gamma(\gamma+1)}{(b-a)^\gamma}\,{}_aI_b^\gamma\mathcal{J}(t)\right|$$
$$\leq\frac{(b-a)^\gamma}{4^\gamma}\left(\frac{\Gamma(1+p\gamma)}{\Gamma(1+(p+1)\gamma)}\right)^{\frac{1}{p}}\left(\frac{\Gamma(1+s\gamma)}{\Gamma(1+(s+1)\gamma)}\right)^{\frac{1}{q}}$$
$$\times\left(\left(\left|\mathcal{J}^{(\gamma)}(a)\right|^q+\left|\mathcal{J}^{(\gamma)}\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}+\left(\left|\mathcal{J}^{(\gamma)}\left(\frac{a+b}{2}\right)\right|^q+\left|\mathcal{J}^{(\gamma)}(b)\right|^q\right)^{\frac{1}{q}}\right).$$

Corollary 11. *In Theorem 2, taking $x=\frac{a+b}{2}$, we obtain*

$$\left|\mathcal{J}\left(\frac{a+b}{2}\right)-\frac{\Gamma(\gamma+1)}{(b-a)^\gamma}\,{}_aI_b^\gamma\mathcal{J}(t)\right|$$
$$\leq\frac{(b-a)^\gamma}{4^\gamma}\left(\frac{\Gamma(1+p\gamma)}{\Gamma(1+(p+1)\gamma)}\right)^{\frac{1}{p}}\left(\frac{\Gamma(1+s\gamma)}{\Gamma(1+(s+1)\gamma)}\right)^{\frac{1}{q}}$$
$$\times\left(\left(\left|\mathcal{J}^{(\gamma)}(a)\right|^q+\left|\mathcal{J}^{(\gamma)}\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}+\left(\left|\mathcal{J}^{(\gamma)}\left(\frac{a+b}{2}\right)\right|^q+\left|\mathcal{J}^{(\gamma)}(b)\right|^q\right)^{\frac{1}{q}}\right).$$

Theorem 3. *Suppose $\mathcal{J}:[a,b]\to\mathbb{R}^\gamma$ is a differentiable function on $[a,b]$ such that $\mathcal{J}\in D_\gamma[a,b]$ and $\mathcal{J}^{(\gamma)}\in C_\gamma[a,b]$ with $0\leq a<b$. If $\left|\mathcal{J}^{(\gamma)}\right|^q$ is generalized s-convex on $[a,b]$, where $q>1$, then we have*

$$\left|\frac{\mathcal{J}(x)+\mathcal{J}(a+b-x)}{2^\gamma}-\frac{\Gamma(\gamma+1)}{(b-a)^\gamma}\,{}_aI_b^\gamma\mathcal{J}(t)\right|$$

$$\leq \left(\frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)}\right)^{1-\frac{1}{q}}$$

$$\times\left(\frac{(x-a)^{2\gamma}}{(b-a)^{\gamma}}\left(\left(\left(\frac{\Gamma(1+s\gamma)}{\Gamma(1+(s+1)\gamma)}-\frac{\Gamma(1+(s+1)\gamma)}{\Gamma(1+(s+2)\gamma)}\right)\left|\mathcal{J}^{(\gamma)}(a)\right|^{q}+\frac{\Gamma(1+(s+1)\gamma)}{\Gamma(1+(s+2)\gamma)}\left|\mathcal{J}^{(\gamma)}(x)\right|^{q}\right)^{\frac{1}{q}}\right.\right.$$

$$\left.+\left(\frac{\Gamma(1+(s+1)\gamma)}{\Gamma(1+(s+2)\gamma)}\left|\mathcal{J}^{(\gamma)}(a+b-x)\right|^{q}+\left(\frac{\Gamma(1+s\gamma)}{\Gamma(1+(s+1)\gamma)}-\frac{\Gamma(1+(s+1)\gamma)}{\Gamma(1+(s+2)\gamma)}\right)\left|\mathcal{J}^{(\gamma)}(b)\right|^{q}\right)^{\frac{1}{q}}\right)$$

$$+\frac{(a+b-2x)^{2\gamma}}{4^{\gamma}(b-a)^{\gamma}}\left(\left(\frac{\Gamma(1+(s+1)\gamma)}{\Gamma(1+(s+2)\gamma)}\left|\mathcal{J}^{(\gamma)}(x)\right|^{q}+\left(\frac{\Gamma(1+s\gamma)}{\Gamma(1+(s+1)\gamma)}-\frac{\Gamma(1+(s+1)\gamma)}{\Gamma(1+(s+2)\gamma)}\right)\left|\mathcal{J}^{(\gamma)}\left(\frac{a+b}{2}\right)\right|^{q}\right)^{\frac{1}{q}}\right.$$

$$\left.\left.+\left(\left(\frac{\Gamma(1+s\gamma)}{\Gamma(1+(s+1)\gamma)}-\frac{\Gamma(1+(s+1)\gamma)}{\Gamma(1+(s+2)\gamma)}\right)\left|\mathcal{J}^{(\gamma)}\left(\frac{a+b}{2}\right)\right|^{q}+\frac{\Gamma(1+(s+1)\gamma)}{\Gamma(1+(s+2)\gamma)}\left|\mathcal{J}^{(\gamma)}(a+b-x)\right|^{q}\right)^{\frac{1}{q}}\right)\right).$$

Proof. Using Lemma 4 as well as the generalized power mean inequality, properties of modulus, and the generalized s-convexity of $\left|\mathcal{J}^{(\gamma)}\right|^{q}$, we can conclude that

$$\left|\frac{\mathcal{J}(x)+\mathcal{J}(a+b-x)}{2^{\gamma}}-\frac{\Gamma(\gamma+1)}{(b-a)^{\gamma}}\,{}_aI_b^{\gamma}\mathcal{J}(t)\right|$$

$$\leq\frac{(x-a)^{2\gamma}}{(b-a)^{\gamma}}\left(\left(\frac{1}{\Gamma(\gamma+1)}\int_0^1\eta^{\gamma}(d\eta)^{\gamma}\right)^{1-\frac{1}{q}}\left(\frac{1}{\Gamma(\gamma+1)}\int_0^1\eta^{\gamma}\left|\mathcal{J}^{(\gamma)}((1-\eta)a+\eta x)\right|^{q}(d\eta)^{\gamma}\right)^{\frac{1}{q}}\right.$$

$$+\left(\frac{1}{\Gamma(\gamma+1)}\int_0^1(1-\eta)^{\gamma}(d\eta)^{\gamma}\right)^{1-\frac{1}{q}}$$

$$\left.\times\left(\frac{1}{\Gamma(\gamma+1)}\int_0^1(1-\eta)^{\gamma}\left|\mathcal{J}^{(\gamma)}((1-\eta)(a+b-x)+\eta b)\right|^{q}(d\eta)^{\gamma}\right)^{\frac{1}{q}}\right)$$

$$+\frac{(a+b-2x)^{2\gamma}}{4^{\gamma}(b-a)^{\gamma}}\left(\left(\frac{1}{\Gamma(\gamma+1)}\int_0^1(1-\eta)^{\gamma}(d\eta)^{\gamma}\right)^{1-\frac{1}{q}}\right.$$

$$\times\left(\frac{1}{\Gamma(\gamma+1)}\int_0^1(1-\eta)^{\gamma}\left|\mathcal{J}^{(\gamma)}\left((1-\eta)x+\eta\frac{a+b}{2}\right)\right|^{q}(d\eta)^{\gamma}\right)^{\frac{1}{q}}$$

$$\left.+\left(\frac{1}{\Gamma(\gamma+1)}\int_0^1\eta^{\gamma}(d\eta)^{\gamma}\right)^{1-\frac{1}{q}}\left(\frac{1}{\Gamma(\gamma+1)}\int_0^1\eta^{\gamma}\left|\mathcal{J}^{(\gamma)}\left((1-\eta)\frac{a+b}{2}+\eta(a+b-x)\right)\right|^{q}(d\eta)^{\gamma}\right)^{\frac{1}{q}}\right)$$

$$\leq \left(\frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)}\right)^{1-\frac{1}{q}} \left(\frac{(x-a)^{2\gamma}}{(b-a)^{\gamma}} \left(\left(\frac{1}{\Gamma(\gamma+1)} \int_0^1 \eta^{\gamma} \left((1-\eta)^{s\gamma} \left| \mathcal{J}^{(\gamma)}(a) \right|^q + \eta^{s\gamma} \left| \mathcal{J}^{(\gamma)}(x) \right|^q \right) (d\eta)^{\gamma} \right)^{\frac{1}{q}} \right.\right.$$

$$\left.+ \left(\frac{1}{\Gamma(\gamma+1)} \int_0^1 (1-\eta)^{\gamma} \left((1-\eta)^{s\gamma} \left| \mathcal{J}^{(\gamma)}(a+b-x) \right|^q + \eta^{s\gamma} \left| \mathcal{J}^{(\gamma)}(b) \right|^q \right) (d\eta)^{\gamma} \right)^{\frac{1}{q}} \right)$$

$$+ \frac{(a+b-2x)^{2\gamma}}{4^{\gamma}(b-a)^{\gamma}} \left(\left(\frac{1}{\Gamma(\gamma+1)} \int_0^1 (1-\eta)^{\gamma} \left((1-\eta)^{s\gamma} \left| \mathcal{J}^{(\gamma)}(x) \right|^q + \eta^{s\gamma} \left| \mathcal{J}^{(\gamma)}\left(\tfrac{a+b}{2}\right) \right|^q \right) (d\eta)^{\gamma} \right)^{\frac{1}{q}} \right.$$

$$\left.\left.+ \left(\frac{1}{\Gamma(\gamma+1)} \int_0^1 \eta^{\gamma} \left((1-\eta)^{s\gamma} \left| \mathcal{J}^{(\gamma)}\left(\tfrac{a+b}{2}\right) \right|^q + \eta^{s\gamma} \left| \mathcal{J}^{(\gamma)}(a+b-x) \right|^q \right) (d\eta)^{\gamma} \right)^{\frac{1}{q}} \right)\right)$$

$$= \left(\frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)}\right)^{1-\frac{1}{q}}$$

$$\times \left(\frac{(x-a)^{2\gamma}}{(b-a)^{\gamma}} \left(\left(\left(\frac{\Gamma(1+s\gamma)}{\Gamma(1+(s+1)\gamma)} - \frac{\Gamma(1+(s+1)\gamma)}{\Gamma(1+(s+2)\gamma)} \right) \left| \mathcal{J}^{(\gamma)}(a) \right|^q + \frac{\Gamma(1+(s+1)\gamma)}{\Gamma(1+(s+2)\gamma)} \left| \mathcal{J}^{(\gamma)}(x) \right|^q \right)^{\frac{1}{q}} \right.\right.$$

$$\left.+ \left(\frac{\Gamma(1+(s+1)\gamma)}{\Gamma(1+(s+2)\gamma)} \left| \mathcal{J}^{(\gamma)}(a+b-x) \right|^q + \left(\frac{\Gamma(1+s\gamma)}{\Gamma(1+(s+1)\gamma)} - \frac{\Gamma(1+(s+1)\gamma)}{\Gamma(1+(s+2)\gamma)} \right) \left| \mathcal{J}^{(\gamma)}(b) \right|^q \right)^{\frac{1}{q}} \right)$$

$$+ \frac{(a+b-2x)^{2\gamma}}{4^{\gamma}(b-a)^{\gamma}} \left(\left(\frac{\Gamma(1+(s+1)\gamma)}{\Gamma(1+(s+2)\gamma)} \left| \mathcal{J}^{(\gamma)}(x) \right|^q + \left(\frac{\Gamma(1+s\gamma)}{\Gamma(1+(s+1)\gamma)} - \frac{\Gamma(1+(s+1)\gamma)}{\Gamma(1+(s+2)\gamma)} \right) \left| \mathcal{J}^{(\gamma)}\left(\tfrac{a+b}{2}\right) \right|^q \right)^{\frac{1}{q}} \right.$$

$$\left.\left.+ \left(\left(\frac{\Gamma(1+s\gamma)}{\Gamma(1+(s+1)\gamma)} - \frac{\Gamma(1+(s+1)\gamma)}{\Gamma(1+(s+2)\gamma)} \right) \left| \mathcal{J}^{(\gamma)}\left(\tfrac{a+b}{2}\right) \right|^q + \frac{\Gamma(1+(s+1)\gamma)}{\Gamma(1+(s+2)\gamma)} \left| \mathcal{J}^{(\gamma)}(a+b-x) \right|^q \right)^{\frac{1}{q}} \right)\right),$$

where we have used (11) and (12). The proof is completed.

Corollary 12. *In Theorem 3, taking $x = a$, we obtain*

$$\left| \frac{\mathcal{J}(a)+\mathcal{J}(b)}{2^{\gamma}} - \frac{\Gamma(\gamma+1)}{(b-a)^{\gamma}} \, {}_aI_b^{\gamma} \mathcal{J}(t) \right|$$

$$\leq \frac{(b-a)^{\gamma}}{4^{\gamma}} \left(\frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)}\right)^{1-\frac{1}{q}}$$

$$\times \left(\left(\frac{\Gamma(1+(s+1)\gamma)}{\Gamma(1+(s+2)\gamma)} \left| \mathcal{J}^{(\gamma)}(a) \right|^q + \left(\frac{\Gamma(1+s\gamma)}{\Gamma(1+(s+1)\gamma)} - \frac{\Gamma(1+(s+1)\gamma)}{\Gamma(1+(s+2)\gamma)} \right) \left| \mathcal{J}^{(\gamma)}\left(\tfrac{a+b}{2}\right) \right|^q \right)^{\frac{1}{q}} \right.$$

$$\left.+ \left(\left(\frac{\Gamma(1+s\gamma)}{\Gamma(1+(s+1)\gamma)} - \frac{\Gamma(1+(s+1)\gamma)}{\Gamma(1+(s+2)\gamma)} \right) \left| \mathcal{J}^{(\gamma)}\left(\tfrac{a+b}{2}\right) \right|^q + \frac{\Gamma(1+(s+1)\gamma)}{\Gamma(1+(s+2)\gamma)} \left| \mathcal{J}^{(\gamma)}(b) \right|^q \right)^{\frac{1}{q}} \right).$$

Corollary 13. *In Theorem 3, taking $x = \frac{a+b}{2}$, we obtain*

$$\begin{aligned}
&\left| \mathcal{J}\left(\tfrac{a+b}{2}\right) - \tfrac{\Gamma(\gamma+1)}{(b-a)^{\gamma}}\, {}_{a}I_{b}^{\gamma}\mathcal{J}(t) \right| \\
\leq &\tfrac{(b-a)^{\gamma}}{4^{\gamma}} \left(\tfrac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)}\right)^{1-\frac{1}{q}} \\
&\times \left(\left(\left(\tfrac{\Gamma(1+s\gamma)}{\Gamma(1+(s+1)\gamma)} - \tfrac{\Gamma(1+(s+1)\gamma)}{\Gamma(1+(s+2)\gamma)} \right) \left|\mathcal{J}^{(\gamma)}(a)\right|^{q} + \tfrac{\Gamma(1+(s+1)\gamma)}{\Gamma(1+(s+2)\gamma)} \left|\mathcal{J}^{(\gamma)}\left(\tfrac{a+b}{2}\right)\right|^{q} \right)^{\frac{1}{q}} \right. \\
&\left. + \left(\tfrac{\Gamma(1+(s+1)\gamma)}{\Gamma(1+(s+2)\gamma)} \left|\mathcal{J}^{(\gamma)}\left(\tfrac{a+b}{2}\right)\right|^{q} + \left(\tfrac{\Gamma(1+s\gamma)}{\Gamma(1+(s+1)\gamma)} - \tfrac{\Gamma(1+(s+1)\gamma)}{\Gamma(1+(s+2)\gamma)} \right) \left|\mathcal{J}^{(\gamma)}(b)\right|^{q} \right)^{\frac{1}{q}} \right).
\end{aligned}$$

Theorem 4. *Suppose $\mathcal{J} : [a,b] \to \mathbb{R}^{\gamma}$ is a differentiable function on $[a,b]$ such that $\mathcal{J} \in D_{\gamma}[a,b]$ and $\mathcal{J}^{(\gamma)} \in C_{\gamma}[a,b]$ with $0 \leq a < b$. If $\left|\mathcal{J}^{(\gamma)}\right|^{q}$ is generalized s-concave on $[a,b]$, where $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then we have*

$$\begin{aligned}
&\left| \tfrac{\mathcal{J}(x)+\mathcal{J}(a+b-x)}{2^{\gamma}} - \tfrac{\Gamma(\gamma+1)}{(b-a)^{\gamma}}\, {}_{a}I_{b}^{\gamma}\mathcal{J}(t) \right| \\
\leq &\left(\tfrac{\Gamma(1+p\gamma)}{\Gamma(1+(p+1)\gamma)}\right)^{\frac{1}{p}} \left(\tfrac{(x-a)^{2\gamma}}{(b-a)^{\gamma}} \left(\tfrac{(x-a)^{\gamma}2^{(s-1)\gamma}}{\Gamma(1+\gamma)} \right)^{\frac{1}{q}} \left(\left|\mathcal{J}^{(\gamma)}\left(\tfrac{a+x}{2}\right)\right| + \left|\mathcal{J}^{(\gamma)}\left(\tfrac{a+2b-x}{2}\right)\right| \right) \right. \\
&\left. + \tfrac{(a+b-2x)^{2\gamma}}{4^{\gamma}(b-a)^{\gamma}} \left(\tfrac{(a+b-2x)^{\gamma}2^{(s-1)\gamma}}{2^{\gamma}\Gamma(1+\gamma)} \right)^{\frac{1}{q}} \left(\left|\mathcal{J}^{(\gamma)}\left(\tfrac{a+b+2x}{4}\right)\right| + \left|\mathcal{J}^{(\gamma)}\left(\tfrac{3a+3b-2x}{4}\right)\right| \right) \right).
\end{aligned}$$

Proof. By utilizing Lemma 4, as well as the generalized Hölder's inequality, properties of the modulus function, and the generalized s-concavity of $\left|\mathcal{J}^{(\gamma)}\right|^{q}$, we have

$$\begin{aligned}
&\left| \tfrac{\mathcal{J}(x)+\mathcal{J}(a+b-x)}{2^{\gamma}} - \tfrac{\Gamma(\gamma+1)}{(b-a)^{\gamma}}\, {}_{a}I_{b}^{\gamma}\mathcal{J}(t) \right| \\
\leq &\tfrac{(x-a)^{2\gamma}}{(b-a)^{\gamma}} \left(\left(\tfrac{1}{\Gamma(\gamma+1)} \int_{0}^{1} \eta^{p\gamma} (d\eta)^{\gamma} \right)^{\frac{1}{p}} \left(\tfrac{1}{\Gamma(\gamma+1)} \int_{0}^{1} \left|\mathcal{J}^{(\gamma)}((1-\eta)a + \eta x)\right|^{q} (d\eta)^{\gamma} \right)^{\frac{1}{q}} \right. \\
&\left. + \left(\tfrac{1}{\Gamma(\gamma+1)} \int_{0}^{1} (1-\eta)^{p\gamma} (d\eta)^{\gamma} \right)^{\frac{1}{p}} \left(\tfrac{1}{\Gamma(\gamma+1)} \int_{0}^{1} \left|\mathcal{J}^{(\gamma)}((1-\eta)(a+b-x) + \eta b)\right|^{q} (d\eta)^{\gamma} \right)^{\frac{1}{q}} \right) \\
&+ \tfrac{(a+b-2x)^{2\gamma}}{4^{\gamma}(b-a)^{\gamma}} \left(\left(\tfrac{1}{\Gamma(\gamma+1)} \int_{0}^{1} (1-\eta)^{p\gamma} (d\eta)^{\gamma} \right)^{\frac{1}{p}} \right. \\
&\times \left(\tfrac{1}{\Gamma(\gamma+1)} \int_{0}^{1} \left|\mathcal{J}^{(\gamma)}\left((1-\eta)x + \eta\tfrac{a+b}{2}\right)\right|^{q} (d\eta)^{\gamma} \right)^{\frac{1}{q}}
\end{aligned}$$

$$+\left(\frac{1}{\Gamma(\gamma+1)}\int_0^1 \eta^{p\gamma}(d\eta)^\gamma\right)^{\frac{1}{p}}\left(\frac{1}{\Gamma(\gamma+1)}\int_0^1\left|\mathcal{J}^{(\gamma)}\left((1-\eta)\frac{a+b}{2}+\eta(a+b-x)\right)\right|^q(d\eta)^\gamma\right)^{\frac{1}{q}}\Bigg)$$

$$\leq\left(\frac{\Gamma(1+p\gamma)}{\Gamma(1+(p+1)\gamma)}\right)^{\frac{1}{p}}\Bigg(\frac{(x-a)^{2\gamma}}{(b-a)^\gamma}\left(\frac{(x-a)^\gamma 2^{(s-1)\gamma}}{\Gamma(1+\gamma)}\right)^{\frac{1}{q}}\left(\left|\mathcal{J}^{(\gamma)}\left(\frac{a+x}{2}\right)\right|+\left|\mathcal{J}^{(\gamma)}\left(\frac{a+2b-x}{2}\right)\right|\right)$$

$$+\frac{(a+b-2x)^{2\gamma}}{4^\gamma(b-a)^\gamma}\left(\left(\frac{(a+b-2x)^\gamma 2^{(s-1)\gamma}}{2^\gamma\Gamma(1+\gamma)}\right)^{\frac{1}{q}}\left|\mathcal{J}^{(\gamma)}\left(\frac{a+b+2x}{4}\right)\right|+\left|\mathcal{J}^{(\gamma)}\left(\frac{3a+3b-2x}{4}\right)\right|\right)\Bigg).$$

The proof is completed.

Corollary 14. *In Theorem 4, taking $x=a$, we obtain*

$$\left|\frac{\mathcal{J}(x)+\mathcal{J}(a+b-x)}{2^\gamma}-\frac{\Gamma(\gamma+1)}{(b-a)^\gamma}\,{}_aI_b^\gamma\mathcal{J}(t)\right|$$
$$\leq\frac{(b-a)^\gamma}{4^\gamma}\left(\frac{\Gamma(1+p\gamma)}{\Gamma(1+(p+1)\gamma)}\right)^{\frac{1}{p}}\left(\frac{(b-a)^\gamma}{2^{(2-s)\gamma}\Gamma(1+\gamma)}\right)^{\frac{1}{q}}\left(\left|\mathcal{J}^{(\gamma)}\left(\frac{3a+b}{4}\right)\right|+\left|\mathcal{J}^{(\gamma)}\left(\frac{a+3b}{4}\right)\right|\right).$$

Corollary 15. *In Theorem 4, taking $x=\frac{a+b}{2}$, we obtain*

$$\left|\frac{\mathcal{J}(x)+\mathcal{J}(a+b-x)}{2^\gamma}-\frac{\Gamma(\gamma+1)}{(b-a)^\gamma}\,{}_aI_b^\gamma\mathcal{J}(t)\right|$$
$$\leq\frac{(b-a)^\gamma}{4^\gamma}\left(\frac{\Gamma(1+p\gamma)}{\Gamma(1+(p+1)\gamma)}\right)^{\frac{1}{p}}\left(\frac{(b-a)^\gamma}{2^{(2-s)\gamma}\Gamma(1+\gamma)}\right)^{\frac{1}{q}}\left(\left|\mathcal{J}^{(\gamma)}\left(\frac{a+x}{2}\right)\right|+\left|\mathcal{J}^{(\gamma)}\left(\frac{a+2b-x}{2}\right)\right|\right).$$

3. Example and Applications

The purpose of this section is to verify the correctness and effectiveness of the results obtained. To achieve this, we start with an example that includes a graphical representations to demonstrate the accuracy of our results. We then provide a few applications for estimating the error of a given quadrature formula.

3.1. Example supporting our findings

In an effort to provide additional support and substantiation for the results derived in this study, we present an illustrative example that encompasses various cases and incorporates 2D and 3D graphical depictions. The purpose of this example is to demonstrate the effectiveness and accuracy of our findings. It is important to note that the figures presented herein were generated utilizing Matlab, where the color green denotes the Right Hand Side (RHS) and red signifies the Left Hand Side (LHS) of their respective inequalities.

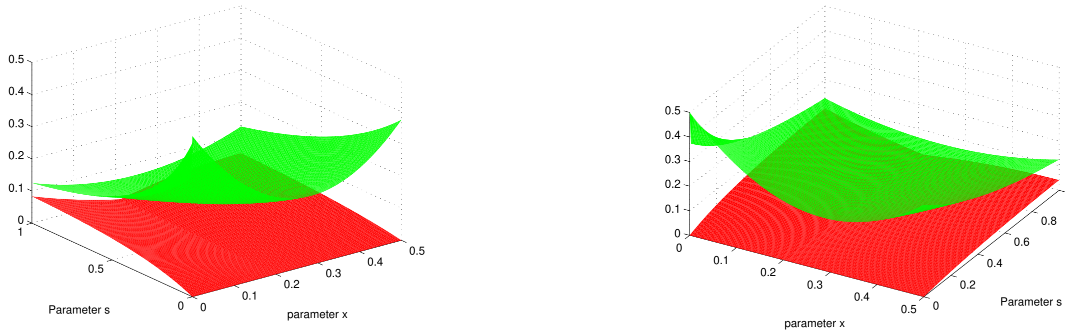
Example 1. *We present the function $\mathcal{J}:[0,1]\to\mathbb{R}^\gamma$, which is defined for a fixed value $s\in(0,1]$ as $\mathcal{J}(t)=\frac{\Gamma(1+s\gamma)}{\Gamma(1+(s+1)\gamma)}t^{(s+1)\gamma}$. The crucial aspect of this function, which underpins our investigation, is that its derivative $\left|\mathcal{J}^{(\gamma)}\right|=t^{s\gamma}$ is a generalized s-convex function.*

In the ensuing discussion, we will set $\gamma = 1$ and subsequently present the different cases in the following manner.

Case 1. *Applying Theorem 1 to the function under consideration yields the following result depicted in Figure 1 for $x \in \left[0, \frac{1}{2}\right]$ and $s \in (0, 1]$.*

$$\left|\frac{1}{s+1}\left(\frac{x^{s+1}+(1-x)^{s+1}}{2}-\frac{1}{s+2}\right)\right| \leq x^2\left(\frac{1}{(s+1)(s+2)}+\frac{1}{s+2}\left(x^s+(1-x)^s\right)\right) + \frac{(1-2x)^2}{4}\left(\frac{1}{s+2}\left(x^s+(1-x)^s\right)+\frac{2^{1-s}}{(s+1)(s+2)}\right).$$

Case 2. *Fixing $s = \frac{1}{2}$, we obtain the following result for x as shown in Figure 2.*



(a) View.1 (b) View.2

Figure 1: Case 1. $x \in \left[0, \frac{1}{2}\right]$ and $s \in (0, 1]$

$$\left|\frac{2}{3}\left(\frac{x^{\frac{3}{2}}+(1-x)^{\frac{3}{2}}}{2}-\frac{2}{5}\right)\right| \leq x^2\left(\frac{4}{15}+\frac{2}{5}\left(\sqrt{x}+\sqrt{1-x}\right)\right)+\frac{(1-2x)^2}{4}\left(\frac{2}{5}\left(\sqrt{x}+\sqrt{1-x}\right)+\frac{4\sqrt{2}}{15}\right).$$

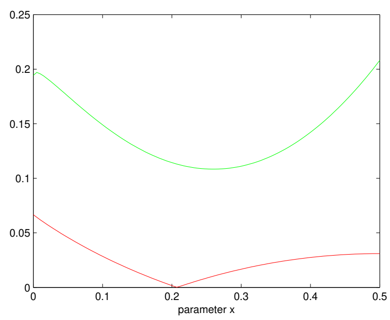


Figure 2: Case 2. $s = \frac{1}{2}$ and $x \in \left[0, \frac{1}{2}\right]$

Case 3. *Lastly, we present with respect to s the result obtained by fixing $x = 0$, as depicted in Figure 3.*

$$\left|\frac{1}{s+1}\left(\frac{1}{2}-\frac{1}{s+2}\right)\right| \leq \frac{1}{4}\left(\frac{1}{s+2}+\frac{2^{1-s}}{(s+1)(s+2)}\right).$$

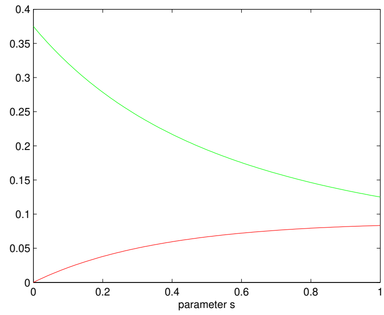


Figure 3: Case 3. $x = 0$ and $s \in (0, 1]$

3.2. Applications to quadrature formula

Let Λ be the partition of the interval $[a, b]$, $a = y_0 < y_1 < ... < y_n = b$. We consider the following quadrature rule

$$\frac{1}{\Gamma(\gamma+1)}\int_a^b \mathcal{J}(u)(du)^\gamma = \mathcal{T}(\mathcal{J},\Lambda) + \mathcal{R}(\mathcal{J},\Lambda),$$

where

$$\mathcal{T}(\mathcal{J},\Lambda) = \sum_{k=0}^{m-1}\frac{(y_{k+1}-y_k)^\gamma}{\Gamma(\gamma+1)}\left(\frac{\mathcal{J}(x)+\mathcal{J}(y_k+y_{k+1}-x)}{2^\gamma}\right)$$

and $\mathcal{R}(\mathcal{J},\Lambda)$ denotes the associated approximation error.

Proposition 1. *Suppose $m \in \mathbb{N}$ and $\mathcal{J} : [a, b] \to \mathbb{R}^\gamma$ is a differentiable function on $[a, b]$, where $0 \leq a < b$ and $\mathcal{J}^{(\gamma)} \in C_\gamma[a, b]$. If $\left|\mathcal{J}^{(\gamma)}\right|$ is a generalized s-convex function, then we have*

$$\begin{aligned}
|\mathcal{R}(\mathcal{J},\Lambda)| \leq & \sum_{k=0}^{m-1}\frac{(x-y_k)^{2\gamma}}{\Gamma(1+\gamma)}\left(\left(\frac{\Gamma(1+s\gamma)}{\Gamma(1+(s+1)\gamma)}-\frac{\Gamma(1+(s+1)\gamma)}{\Gamma(1+(s+2)\gamma)}\right)\left(\left|\mathcal{J}^{(\gamma)}(y_k)\right|+\left|\mathcal{J}^{(\gamma)}(y_{k+1})\right|\right)\right.\\
&\left.+\frac{\Gamma(1+(s+1)\gamma)}{\Gamma(1+(s+2)\gamma)}\left(\left|\mathcal{J}^{(\gamma)}(x)\right|+\left|\mathcal{J}^{(\gamma)}(y_k+y_{k+1}-x)\right|\right)\right)\\
&+\frac{(y_k+y_{k+1}-2x)^{2\gamma}}{4^\gamma\Gamma(1+\gamma)}\left(\frac{\Gamma(1+(s+1)\gamma)}{\Gamma(1+(s+2)\gamma)}\left(\left|\mathcal{J}^{(\gamma)}(x)\right|+\left|\mathcal{J}^{(\gamma)}(y_k+y_{k+1}-x)\right|\right)\right.\\
&\left.+2^\gamma\left(\frac{\Gamma(1+s\gamma)}{\Gamma(1+(s+1)\gamma)}-\frac{\Gamma(1+(s+1)\gamma)}{\Gamma(1+(s+2)\gamma)}\right)\left|\mathcal{J}^{(\gamma)}\left(\frac{y_k+y_{k+1}}{2}\right)\right|\right).
\end{aligned}$$

Proof. Applying Theorem 1 on the subintervals $[y_k, y_{k+1}]$, $(k = 0, 1, ..., m-1)$ of the partition Λ, we get

$$\begin{aligned}&\left|\frac{\mathcal{J}(x)+\mathcal{J}(y_k+y_{k+1}-x)}{2^\gamma} - \frac{\Gamma(\gamma+1)}{(y_{k+1}-y_k)^\gamma}\, {}_{y_k}I^\gamma_{y_{k+1}}\mathcal{J}(t)\right| \\ &\leq \frac{(x-y_k)^{2\gamma}}{(y_{k+1}-y_k)^\gamma}\left(\left(\frac{\Gamma(1+s\gamma)}{\Gamma(1+(s+1)\gamma)} - \frac{\Gamma(1+(s+1)\gamma)}{\Gamma(1+(s+2)\gamma)}\right)\left(\left|\mathcal{J}^{(\gamma)}(y_k)\right| + \left|\mathcal{J}^{(\gamma)}(y_{k+1})\right|\right)\right. \\ &\quad \left. + \frac{\Gamma(1+(s+1)\gamma)}{\Gamma(1+(s+2)\gamma)}\left(\left|\mathcal{J}^{(\gamma)}(x)\right| + \left|\mathcal{J}^{(\gamma)}(y_k+y_{k+1}-x)\right|\right)\right) \\ &\quad + \frac{(y_k+y_{k+1}-2x)^{2\gamma}}{4^\gamma(y_{k+1}-y_k)^\gamma}\left(\frac{\Gamma(1+(s+1)\gamma)}{\Gamma(1+(s+2)\gamma)}\left(\left|\mathcal{J}^{(\gamma)}(x)\right| + \left|\mathcal{J}^{(\gamma)}(y_k+y_{k+1}-x)\right|\right)\right. \\ &\quad \left. + 2^\gamma\left(\frac{\Gamma(1+s\gamma)}{\Gamma(1+(s+1)\gamma)} - \frac{\Gamma(1+(s+1)\gamma)}{\Gamma(1+(s+2)\gamma)}\right)\left|\mathcal{J}^{(\gamma)}\left(\frac{y_k+y_{k+1}}{2}\right)\right|\right).\end{aligned}$$

We can obtain the desired result by multiplying both sides of the inequality above by $\frac{(y_{k+1}-y_k)^\gamma}{\Gamma(1+\gamma)}$, summing the resulting inequalities for all $k = 0, 1, ..., m-1$, and then applying the triangular inequality.

Proposition 2. *Suppose $m \in \mathbb{N}$ and $\mathcal{J} : [a, b] \to \mathbb{R}^\gamma$ is a differentiable function on $[a, b]$, where $0 \leq a < b$ and $\mathcal{J}^{(\gamma)} \in C_\gamma[a, b]$. If $\left|\mathcal{J}^{(\gamma)}\right|^q$ is a generalized s-concave, where $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then we have*

$$\begin{aligned}&\left|\mathcal{R}(\mathcal{J}, \Lambda)\right| \\ &\leq \sum_{k=0}^{m-1}\left(\frac{\Gamma(1+p\gamma)}{\Gamma(1+(p+1)\gamma)}\right)^{\frac{1}{p}}\left(\frac{(x-y_k)^{2\gamma}}{\Gamma(1+\gamma)}\left(\frac{(x-y_k)^\gamma 2^{(s-1)\gamma}}{\Gamma(1+\gamma)}\right)^{\frac{1}{q}}\left(\left|\mathcal{J}^{(\gamma)}\left(\frac{y_k+x}{2}\right)\right| + \left|\mathcal{J}^{(\gamma)}\left(\frac{y_k+2y_{k+1}-x}{2}\right)\right|\right)\right. \\ &\quad \left. + \frac{(y_k+y_{k+1}-2x)^{2\gamma}}{4^\gamma\Gamma(1+\gamma)}\left(\frac{(y_k+y_{k+1}-2x)^\gamma 2^{(s-1)\gamma}}{2^\gamma\Gamma(1+\gamma)}\right)^{\frac{1}{q}}\left(\left|\mathcal{J}^{(\gamma)}\left(\frac{y_k+y_{k+1}+2x}{4}\right)\right| + \left|\mathcal{J}^{(\gamma)}\left(\frac{3y_k+3y_{k+1}-2x}{4}\right)\right|\right)\right).\end{aligned}$$

Proof. Applying Theorem 4 on the subintervals $[y_k, y_{k+1}]$, $(k = 0, 1, ..., m-1)$ of the partition Λ, we get

$$\begin{aligned}&\left|\frac{\mathcal{J}(x)+\mathcal{J}(y_k+y_{k+1}-x)}{2^\gamma} - \frac{\Gamma(\gamma+1)}{(y_{k+1}-y_k)^\gamma}\, {}_{y_k}I^\gamma_{y_{k+1}}\mathcal{J}(t)\right| \\ &\leq \left(\frac{\Gamma(1+p\gamma)}{\Gamma(1+(p+1)\gamma)}\right)^{\frac{1}{p}}\left(\frac{(x-y_k)^{2\gamma}}{(y_{k+1}-y_k)^\gamma}\left(\frac{(x-y_k)^\gamma 2^{(s-1)\gamma}}{\Gamma(1+\gamma)}\right)^{\frac{1}{q}}\left(\left|\mathcal{J}^{(\gamma)}\left(\frac{y_k+x}{2}\right)\right| + \left|\mathcal{J}^{(\gamma)}\left(\frac{y_k+2y_{k+1}-x}{2}\right)\right|\right)\right. \\ &\quad \left. + \frac{(y_k+y_{k+1}-2x)^{2\gamma}}{4^\gamma(y_{k+1}-y_k)^\gamma}\left(\frac{(y_k+y_{k+1}-2x)^\gamma 2^{(s-1)\gamma}}{2^\gamma\Gamma(1+\gamma)}\right)^{\frac{1}{q}}\left(\left|\mathcal{J}^{(\gamma)}\left(\frac{y_k+y_{k+1}+2x}{4}\right)\right| + \left|\mathcal{J}^{(\gamma)}\left(\frac{3y_k+3y_{k+1}-2x}{4}\right)\right|\right)\right).\end{aligned}$$

We can obtain the desired result by multiplying both sides of the inequality above by $\frac{(y_{k+1}-y_k)^\gamma}{\Gamma(1+\gamma)}$, summing the resulting inequalities for all $k = 0, 1, ..., m-1$, and then applying the triangular inequality.

4. Conclusion

In conclusion, fractal sets and fractal theory have generated significant interest among scientists and engineers, particularly with regards to studying the properties of functions operating on these sets using techniques of fractional calculus. This paper contributes to this area of research by examining the companion of Ostrowski's inequality within the framework of fractal sets. The introduction of a new identity related to local fractional integrals allows us to establish several inequalities for functions with generalized s-convex derivatives and s-concave derivatives. The correctness of the results is justified through an example, and a few applications are discussed. This work also opens up new horizons for the study of integral inequalities via other types of convexity and for functions of two variables. These future developments can contribute to a deeper understanding of the properties of fractal sets and the functions that operate on them.

Acknowledgements

The work of the third author was supported by DGRSDT, MESRS of Algeria. (PRFU Project A14N01EP230220230001).

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