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The Linear Algebra of the (r, β)-Stirling Matrices

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Abstract. This paper establishes the linear algebra of the (r, β)-Stirling matrix. Along the way, this paper derives various identities, such as its factorization and relationship to the Pascal matrix and the Stirling matrix of the second kind. Additionally, this paper develops a natural extension of the Vandermonde matrix, which can be used to study and evaluate successive power sums of arithmetic progressions.

2020 Mathematics Subject Classifications: 05, 11

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1. Introduction

The (r, β)-Stirling numbers are a generalization of the classical Stirling numbers of the second kind and r-Stirling numbers, and is denoted by, $\left\langle {n \atop k} \right\rangle_{\beta,r}$. They were introduced by Corcino in 1999 [5] by means of the following linear transformation:

$$t^n = \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle_{\beta,r} (t-r)_{\beta,k} \tag{1}$$

where

$$(t-r)_{\beta,k} = \prod_{i=0}^{k-1} (t - r - i\beta). \tag{2}$$

$(t)_{\beta,k}$ is called the generalized factorial of t with increment β, and as a convention $(t)_{\beta,k} = 0$ if $k \leq 0$. This numbers have applications in combinatorial and statistical problems. Corcino and Aldema (2002) [6] further studied the (r, β)-Stirling numbers and derived some combinatorial identities related to them. Corcino and Montero (2009) [7] also investigated the (r, β)-Stirling numbers in the context of 0-1 tableaux, which are a tool used in algebraic combinatorics.

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In this paper, we introduce and study the (r,β)-Stirling matrix and we derive several interesting identities about (r,β)-Stirling sequence. Two applications are given: we can generalize the Vandermonde matrices and evaluate successive power sums of arithmetic progressions.

2. Results

2.1. The (r,β)-Stirling Matrix

The key notions of the study are now defined.

Definition 1. The (r,β)-Stirling matrix is the $n\times n$ matrix defined by

$$S^{(\beta,r)}(n)=\left[\left\langle {n \atop k} \right\rangle_{\beta,r}\right]_{0\le i,j\le n-1}$$

Example 1. *When $n=4$, we have*

$$S^{(\beta,r)}(4)=\begin{bmatrix}1 & 0 & 0 & 0\\ r & 1 & 0 & 0\\ r^2 & \beta+2r & 1 & 0\\ r^3 & \beta^2+3\beta r+3r^2 & 3\beta+3r & 1\end{bmatrix}. \qquad (3)$$

For the following propositions, we need the generalized $n\times n$ Pascal matrix and the generalized $n\times n$ Stirling matrix of the second kind, defined by [2] and [3], respectively as:

$$P_n[x]=\left[x^{i-j}\binom{i}{j}\right]_{0\le i,j\le n-1}$$

where $P_n=P_n[1]$, and

$$S_2(n)[x]=\left[x^{i-j}S(i,j)\right]_{0\le i,j\le n-1}$$

where $S(i,j)$ is the Stirling numbers of the second kind.

The main technique used to prove the next propositions is the concept of the Riordan group introduced by Shapiro et al. [10]. This, briefly, is a group of infinite lower triangular arrays called Riordan matrices.

A pair of formal power series g and f in the ring $\mathbf{C}[[z]]$ define a Riordan matrix as $M=[m_{n,k}]_{n,k\ge 0}$, where $g(0)\neq 0$, $f(0)=0$, $f'(0)\neq 0$, and $[z_n]$ is the coefficient extraction operator. The matrix M is denoted by (g,f). Moreover, if $m_{n,k}=[\frac{z^n}{n!}]g\frac{f^k}{k!}$ then M is called the exponential Riordan matrix or e-Riordan maxtrix, denoted by $\langle g,f\rangle$. For example, the e-Riordan matrix representations of the three common e-Riordan matrices used in this paper - the Pascal matrix, Stirling matrix of the second kind, and the (r,β)-Stirling matrix:

$$P_n=\left\langle e^z,z\right\rangle,\quad S_2(n)=\left\langle 1,e^z-1\right\rangle,\quad S^{(\beta,r)}(n)=\left\langle e^{(r)z},\frac{e^{\beta z}-1}{\beta}\right\rangle.$$

The set of all e-Riordan matrices forms a group called e-Riordan group under the Riordan multiplication defined by

$$\langle g,f\rangle * \langle h,l\rangle = \langle gh(f), l(f)\rangle.$$

Proposition 1. Let P_n be the $n\times n$ generalized Pascal matrix, then

$$S^{(\beta,r)}(n) = P_n S^{(\beta,r-1)}(n)$$

Proof. Consider the e-Riordan matrix representations,

$$P_n = \left\langle e^z, z\right\rangle \qquad \text{and} \qquad S^{(\beta,r-1)}(n) = \left\langle e^{(r-1)z}, \tfrac{e^{\beta z}-1}{\beta}\right\rangle.$$

By using the e-Riordan matrix multiplication, we have

$$\begin{aligned}
P_n S^{(\beta,r-1)}(n) &= \left\langle e^z, z\right\rangle * \left\langle e^{(r-1)z}, \frac{e^{\beta z}-1}{\beta}\right\rangle \\
&= \left\langle e^z e^{(r-1)z}, \frac{e^{\beta z}-1}{\beta}\right\rangle \\
&= \left\langle e^{z+(r-1)z}, \frac{e^{\beta z}-1}{\beta}\right\rangle \\
&= \left\langle e^{z+rz-z}, \frac{e^{\beta z}-1}{\beta}\right\rangle \\
&= \left\langle e^{rz}, \frac{e^{\beta z}-1}{\beta}\right\rangle \\
&= S^{(\beta,r)}(n).
\end{aligned}$$

□

Example 2. Let $n=4$. Then

$$\begin{aligned}
P_4 S^{(\beta,r-1)}(4) &= \begin{bmatrix} 1 & 0 & 0 & 0\\ 1 & 1 & 0 & 0\\ 1 & 2 & 1 & 0\\ 1 & 3 & 3 & 1\end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0\\ r-1 & 1 & 0 & 0\\ (r-1)^2 & \beta+2(r-1) & 1 & 0\\ (r-1)^3 & \beta^2+3\beta(r-1)+3(r-1)^2 & 3\beta+3r & 1\end{bmatrix}\\
&= \begin{bmatrix} 1 & 0 & 0 & 0\\ r & 1 & 0 & 0\\ r^2 & \beta+2r & 1 & 0\\ r^3 & \beta^2+3\beta r+3r^2 & 3\beta+3r & 1\end{bmatrix}\\
&= S^{(\beta,r)}(4)
\end{aligned}$$

Proposition 2. Let $P_n[r-s]$ be the $n\times n$ Pascal matrix defined by

$$P_n[r-s] = \left[(r-s)^{i-j}\binom{n}{k}\right]_{0\le i,j\le n-1}.$$

Then,

$$S^{(\beta,r)}(n) = P_n[r-s]S^{(\beta,s)}(n),$$

provided that $r \geq s$.

Proof. Consider $P_n[r-s] = \langle e^{(r-s)z}, z\rangle$ and $S^{(\beta,s)}(n) = \langle e^{rz}, \frac{e^{\beta z}-1}{\beta}\rangle$. Then,

$$\begin{aligned} P_n[r-s]S^{(\beta,s)}(n) &= \left\langle e^{(r-s)z}, z\right\rangle * \left\langle e^{sz}, \frac{e^{\beta z}-1}{\beta}\right\rangle \\ &= \left\langle e^{(r-s)z}e^{sz}, \frac{e^{\beta z}-1}{\beta}\right\rangle \\ &= \left\langle e^{(r-s)z+sz}, \frac{e^{\beta z}-1}{\beta}\right\rangle \\ &= \left\langle e^{rz-sz+sz}, \frac{e^{\beta z}-1}{\beta}\right\rangle \\ &= \left\langle e^{rz}, \frac{e^{\beta z}-1}{\beta}\right\rangle \\ &= S^{(\beta,r)}(n). \end{aligned}$$

□

Example 3. Let $n = 4$. Then

$$P_4[r-s]S^{(\beta,s)}(4) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ r-s & 1 & 0 & 0 \\ (r-s)^2 & 2(r-s) & 1 & 0 \\ (r-s)^3 & 3(r-s)^2 & 3(r-s) & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ s & 1 & 0 & 0 \\ s^2 & \beta+2s & 1 & 0 \\ s^3 & \beta^2+3\beta s+3s^2 & 3\beta+3s & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ r & 1 & 0 & 0 \\ -1+2r+(r-1)^2 & b+2r & 1 & 0 \\ -2+3r+3(r-1)^2+(r-1)^3 & b^2+3b+6r-3+3b(r-1)+3(r-1)^2 & 3b+3r & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ r & 1 & 0 & 0 \\ r^2 & \beta+2r & 1 & 0 \\ r^3 & \beta^2+3\beta r+3r^2 & 3\beta+3r & 1 \end{bmatrix}$$

$$= S^{(\beta,r)}(4)$$

Proposition 3. Let $P_n[r]$ and $S_2(n)[\beta]$ be the $n \times n$ Pascal matrix and Stirling matrix of the second kind. Then,

$$S^{(\beta,r)}(n) = P_n[r]S_2(n)[\beta],$$

where $(S_2(n)[\beta]_{i,j} = \beta^{i-j}S(i,j)$ and $S(i,j)$ is a Stirling number of the second kind and $0 \leq i, j \leq n-1$.

Proof. It was previously shown that the e-Riordan matrix representations of $P_n[r]$ and $S_2(n)[\beta]$ are $\langle e^{rz}, z\rangle$ and $\langle 1, e^{\beta z} - 1\rangle$, respectively.
Using the e-Riordan multiplication, we have

$$\begin{aligned} P_n[r]S_2(n)[\beta] &= \left\langle e^{rz}, z\right\rangle * \left\langle 1, \frac{e^{\beta z}-1}{\beta}\right\rangle \\ &= \left\langle e^{rz}(1), \frac{e^{\beta z}-1}{\beta}\right\rangle \\ &= \left\langle e^{rz}, \frac{e^{\beta z}-1}{\beta}\right\rangle \\ &= S^{(\beta,r)}(n). \end{aligned}$$

□

Example 4. Let $n = 4$. Then

$$\begin{aligned} P_4[r]S_2(4)[\beta] &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ r & 1 & 0 & 0 \\ r^2 & 2r & 1 & 0 \\ r^3 & 3r^2 & 3r & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \beta & 1 & 0 \\ 0 & \beta^2 & 3\beta & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ r & 1 & 0 & 0 \\ r^2 & \beta+2r & 1 & 0 \\ r^3 & \beta^2+3\beta r+3r^2 & 3\beta+3r & 1 \end{bmatrix} \\ &= S^{(\beta,r)}(4) \end{aligned}$$

Proposition 4. Let$P_n[r - r\beta]$ and $S^{(1,r)}(n)[\beta]$ be $n \times n$ matrices, where

$$(S^{(1,r)}(n)[\beta])_{ij} = \beta^{i-j} \left\langle {n \atop k} \right\rangle_{1,r}.$$

Then,

$$S^{(\beta,r)}(n) = P_n[r - r\beta]S^{(1,r)}(n)[\beta]$$

Proof. Consider the e-Riordan matrix representations

$$P_n[r - r\beta] = \left\langle e^{(r-r\beta)z}, z\right\rangle$$

and

$$S^{(1,r)}(n)[\beta] = \left[\left\langle e^{rz}, \frac{e^z-1}{1}\right\rangle\right]^{\beta} = \left\langle e^{r\beta z}, \frac{e^{\beta z}-1}{\beta}\right\rangle$$

Now,

$$\begin{aligned}
P_n[r-r\beta]S^{(1,r)}(n)[\beta] &= \left\langle e^{(r-r\beta)z}, z\right\rangle * \left\langle e^{r\beta z}, \frac{e^{\beta z}-1}{\beta}\right\rangle \\
&= \left\langle e^{(r-r\beta)z+r\beta z}, \frac{e^{\beta z}-1}{\beta}\right\rangle \\
&= \left\langle e^{rz-r\beta z+r\beta z}, \frac{e^{\beta z}-1}{\beta}\right\rangle \\
&= \left\langle e^{rz}, \frac{e^{\beta z}-1}{\beta}\right\rangle \\
&= S^{(\beta,r)}(n)
\end{aligned}$$

□

Example 5. Let $n = 4$. Then

$$\begin{aligned}
P_4[r-r\beta]S^{(1,r)}(4)[\beta] &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ (r-r\beta) & 1 & 0 & 0 \\ (r-r\beta)^2 & 2(r-r\beta) & 1 & 0 \\ (r-r\beta)^3 & 3(r-r\beta)^2 & 3(r-r\beta) & 1 \end{bmatrix} \\
&\cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ \beta r & 1 & 0 & 0 \\ (\beta r)^2 & \beta+2\beta r & 1 & 0 \\ (\beta r)^3 & \beta^2+3\beta^2 r+3(\beta r)^2 & 3\beta+3\beta r & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ r & 1 & 0 & 0 \\ r^2 & \beta+2r & 1 & 0 \\ r^3 & \beta^2+3\beta r+3r^2 & 3\beta+3r & 1 \end{bmatrix} \\
&= S^{(\beta,r)}(4).
\end{aligned}$$

2.2. Factorization of the (r, β)-Stirling Matrix

To factor the (r, β)-Stirling matrix, we need the following matrices defined by Zhang in [13]:

$$S_n[x] = \left[x^{i-j}\right]_{0\le j,i\le n-1}$$

For example, when $n = 4$,

$$S_4[x] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ x^2 & x & 1 & 0 \\ x^3 & x^2 & x & 1 \end{bmatrix}.$$

Zhang also define the $n \times n$ matrix $G_k[x]$ as

$$G_k[x] = I_{n-k} \oplus S_k[x], \quad (1 \le k \le n-1),$$

where $G_n[k] = S_n[k]$ and $\oplus$ denotes the matrix direct sum.

Proposition 5. For any integer $n, m \geq 1$ and $r \geq 0$, we have

$$S^{(\beta,r)}(n) = G_n[r]G_{n-1}[r]\cdots G_1[r]\bar{P}_{n-1}[\beta]\cdots\bar{P}_1[\beta]$$

Proof. By proposition 3, $S^{(\beta,r)} = P_n[r]S_2(n)[\beta]$.
Note that by Theorem 1 of [13],

$$P_n[r] = G_n[r]G_{n-1}[r]\cdots G_1[r].$$

Also, based on one of the results of Cheon and Kim in [3],

$$S_2(n)[\beta] = \bar{P}_{n-1}[\beta]\cdots\bar{P}_1[\beta].$$

This follows that,

$$S^{(\beta,r)}(n) = G_n[r]G_{n-1}[r]\cdots G_1[r]\bar{P}_{n-1}[\beta]\cdots\bar{P}_1[\beta].$$

□

2.3. Relationship between the (r, β)-Stirling matrix and a Generalized Vandermonde Matrix

In this section, we introduce a generalization of the Vandermonde matrix which will be useful in the study of successive power sums of arithmetic progression. To do that, we define the following matrices.

Definition 2. *Let $S^{(\beta,r)}(n)$ be the (r, β)-Stirling matrix. The matrix factorial of the (r, β)-Stirling matrix, denoted by $\tilde{S}^{(\beta,r)}(n)$ is defined by*

$$\tilde{S}^{(\beta,r)}(n) := S^{(\beta,r)}(n) \cdot diag(0!, 1!, \ldots, n!). \tag{4}$$

Example 6. *Let $n = 4$. Then*

$$\begin{aligned}\tilde{S}^{(\beta,r)}(4) &= S^{(\beta,r)}(4) \cdot diag(0!, 1!, 2!, 3!) \\ &= \begin{bmatrix} 0! & 0 & 0 & 0 \\ r & 1! & 0 & 0 \\ r^2 & \beta + 2r & 2! & 0 \\ r^3 & \beta^2 + 3\beta r + 3r^2 & (3\beta + 3r)2! & 3! \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ r & 1 & 0 & 0 \\ r^2 & \beta + 2r & 2 & 0 \\ r^3 & \beta^2 + 3\beta r + 3r^2 & 6\beta + 6r & 6 \end{bmatrix}.\end{aligned}$$

Theorem 1. Let $\mathbf{V}_n^{\beta,r}(t)$ be the $n \times n$ generalized Vandermonde matrix defined by

$$\mathbf{V}_n^{\beta,r}(t) := \mathbf{V}_n^{\beta,r}(\beta t + r, \beta t + \beta + r, \beta t + 2\beta + r, \ldots, \beta t + (n-1)\beta + r)$$
$$= \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ \beta t + r & \beta t + \beta + r & \beta t + 2\beta + r & \cdots & \beta t + (n-1)\beta + r \\ (\beta t + r)^2 & (\beta t + \beta + r)^2 & (\beta t + 2\beta + r)^2 & \cdots & (\beta t + (n-1)\beta + r)^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (\beta t + r)^{n-1} & (\beta t + \beta + r)^{n-1} & (\beta t + 2\beta + r)^{n-1} & \cdots & (\beta t + (n-1)\beta + r)^{n-1} \end{bmatrix}$$

and

$$C_n^{\beta}(t) = \left[\beta^i \binom{t+j}{i}\right]_{0 \le i,j \le n-1}.$$

Then we can factor $\mathbf{V}_n^{\beta,r}(t)$ as

$$\mathbf{V}_n^{\beta,r}(t) = \tilde{S}^{(\beta,r)}(n) C_n^{\beta}(t). \tag{5}$$

Proof. Consider the Equation (1),

$$t^n = \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle_{\beta,r} (t-r)_{\beta,k}.$$

Note that we can write this as

$$t^n = \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle_{\beta,r} \binom{\frac{t-r}{\beta}}{k} \beta^k k!.$$

Replacing t by $\beta t + r$, we have

$$(\beta t + r)^n = \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle_{\beta,r} \binom{\frac{(\beta t + r) - r}{\beta}}{k} \beta^k k!$$

$$(\beta t + r)^n = \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle_{\beta,r} \binom{t}{k} \beta^k k!. \tag{6}$$

This equation (6) can be represented by the following system of matrix equation for each $n = 0, 1, 2, \ldots$

$$\mathbf{v}(t) = \tilde{S}^{r,\beta}(n) c_n(t), \tag{7}$$

where

$$\mathbf{v}(t) = [1, \beta t + r, (\beta t + r)^2, (\beta t + r)^3, \ldots, (\beta t + r)^{n-1}]$$

and

$$c_n(t) = \left[\binom{t}{0}, \beta \binom{t}{1}, \beta^2 \binom{t}{2}, \ldots, \beta^{n-1} \binom{t}{n-1}\right]$$

which is the first column of the $\mathbf{V}_n^{\beta,r}(t)$. That is,

$$\begin{bmatrix} 1 \\ \beta t + r \\ (\beta t + r)^2 \\ (\beta t + r)^3 \\ \vdots \\ (\beta t + r)^{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ r & 1 & 0 & 0 & 0 \\ r^2 & \beta + 2r & 2 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r^{n-1} & \left\langle {n-1 \atop 2} \right\rangle_{\beta,r} & \left\langle {n-1 \atop 3} \right\rangle_{\beta,r} 2! & \cdots & (n-1)! \end{bmatrix} \begin{bmatrix} \binom{t}{0} \\ \beta \binom{t}{1} \\ \beta^2 \binom{t}{2} \\ \vdots \\ \beta^{n-1} \binom{t}{n-1} \end{bmatrix}.$$

Thus, by equation (5) we can generalize that,

$$\mathbf{V}_n^{\beta,r}(t) = \tilde{S}^{(\beta,r)}(n) C_n^{\beta}(t).$$

□

Example 7. *Let $n = 4$. Then we have matrices*

$$\tilde{S}^{(\beta,r)}(4) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ r & 1 & 0 & 0 \\ r^2 & \beta + 2r & 2 & 0 \\ r^3 & \beta^2 + 3\beta r + 3r^2 & 6\beta + 6r & 6 \end{bmatrix}$$

and

$$C_4^{\beta}(t) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \beta t & \beta(t+1) & \beta(t+2) & \beta(t+3) \\ \frac{\beta^2 t(t-1)}{2} & \frac{\beta^2 (t+1)t}{2} & \frac{\beta^2 (t+2)(t+1)}{2} & \frac{\beta^2 (t+3)(t+2)}{2} \\ \frac{\beta^3 t(t-1)(t-2)}{6} & \frac{\beta^3 (t+1)t(t-1)}{6} & \frac{\beta^3 (t+2)(t+1)(t)}{6} & \frac{\beta^3 (t+3)(t+2)(t+1)}{6} \end{bmatrix}.$$

Now,

$$\begin{aligned} \tilde{S}^{(\beta,r)}(4) C_4^{\beta}(t) &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ r & 1 & 0 & 0 \\ r^2 & \beta + 2r & 2 & 0 \\ r^3 & \beta^2 + 3\beta r + 3r^2 & 6\beta + 6r & 6 \end{bmatrix} \\ &\cdot \begin{bmatrix} 1 & 1 & 1 & 1 \\ \beta t & \beta(t+1) & \beta(t+2) & \beta(t+3) \\ \frac{\beta^2 t(t-1)}{2} & \frac{\beta^2 (t+1)t}{2} & \frac{\beta^2 (t+2)(t+1)}{2} & \frac{\beta^2 (t+3)(t+2)}{2} \\ \frac{\beta^3 t(t-1)(t-2)}{6} & \frac{\beta^3 (t+1)t(t-1)}{6} & \frac{\beta^3 (t+2)(t+1)(t)}{6} & \frac{\beta^3 (t+3)(t+2)(t+1)}{6} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 1 & 1 \\ \beta t + r & \beta t + \beta + r & \beta t + 2\beta + r & \beta t + 3\beta + r \\ (\beta t + r)^2 & (\beta t + \beta + r)^2 & (\beta t + 2\beta + r)^2 & (\beta t + 3\beta + r)^2 \\ (\beta t + r)^3 & (\beta t + \beta + r)^3 & (\beta t + 2\beta + r)^3 & (\beta t + 3\beta + r)^3 \end{bmatrix} \\ &= \mathbf{V}_4^{\beta,r}(t) \end{aligned}$$

Corollary 1. For any real number t, we have

$$\mathbf{V}_n^{\beta,r}(t) = \tilde{S}^{(\beta,r)}(n)\triangle_n(t)P_n^T$$

where $[\triangle_n(t)]_{ij} = \beta^i \binom{t}{i-j}$ and P_n^T is the transpose of the Pascal Matrix P_n

Proof. From Vandermonde's convolution identity,

$$\binom{m+n}{r} = \sum_{i=0}^{\binom{m}{i}\binom{n}{r-i},}$$

we can obtain the LU factorization of $C_n^\beta(t)$. That is,

$$\begin{aligned} C_n^\beta(t) &= \left[\beta^i \binom{t+j}{i}\right]_{0\leq i,j\leq n-1} \\ &= \left[\beta^i \sum_{k=0}^{i} \binom{j}{i}\binom{t}{i-k}\right]_{0\leq i,j\leq n-1} \\ &= \left[\sum_{k=0}^{i} \beta^i \binom{j}{i}\binom{t}{i-k}\right]_{0\leq i,j\leq n-1} \end{aligned}$$

Let $[\triangle_n(t)]_{ij} = \beta^i \binom{t}{i-j}$. Note the $\left[\binom{j}{i}\right]_{ij}$ is the transpose of the Pascal matrix $P_n = \left(\binom{i}{j}\right)_{ij}$. Then we can write $C_n^\beta(t)$ as

$$C_n^\beta(t) = \triangle_n(t)P_n^T.$$

By Theorem 1,

$$\mathbf{V}_n^{\beta,r}(t) = \tilde{S}^{(\beta,r)}(n)\triangle_n(t)P_n^T.$$

□

Example 8. *Let* $t = 1$, *and* $n = 4$. *Then,*

$$\tilde{S}^{(\beta,r)}(4) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ r & 1 & 0 & 0 \\ r^2 & \beta + 2r & 2 & 0 \\ r^3 & \beta^3 + 3\beta r + r^2 & 6\beta + 6r & 6 \end{bmatrix},$$

$$\triangle_4(t) = \begin{bmatrix} 1 & 0 & 0 & 0 & \\ \beta & beta & 0 & 0 & \\ 0 & \beta^2 & \beta^2 & 0 & \\ 0 & 0 & \beta^3 & \beta^3 & 0 \end{bmatrix}$$

and

$$P_4^T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Now,

$$\begin{aligned}
\tilde{S}^{(\beta,r)}(n)\triangle_n(t)P_n^T &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ r & 1 & 0 & 0 \\ r^2 & \beta+2r & 2 & 0 \\ r^3 & \beta^3+3\beta r+r^2 & 6\beta+6r & 6 \end{bmatrix} \\
&\times \begin{bmatrix} 1 & 0 & 0 & 0 \\ \beta & \beta & 0 & 0 \\ 0 & \beta^2 & \beta^2 & 0 \\ 0 & 0 & \beta^3 & \beta^3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ \beta+r & \beta & 0 & 0 \\ (\beta+r)^2 & \beta(3\beta+2r) & 2\beta^2 & 0 \\ (\beta+r)^3 & \beta(7\beta^2+9\beta r+3r^2) & 6\beta^2(2\beta+r) & 6\beta^3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 & 1 & 1 \\ \beta+r & 2\beta+r & 3\beta+r & 4\beta+r \\ (\beta+r)^2 & (2\beta+r)^2 & (3\beta+r)^2 & (4\beta+r)^2 \\ (\beta+r)^3 & (2\beta+r)^3 & (3\beta+r)^3 & (4\beta+r)^3 \\ & & & \end{bmatrix} \\
&= \mathbf{V}_4^{\beta,r}(t).
\end{aligned}$$

Corollary 2. For any real number t,

$$\mathbf{det}(\mathbf{V}_n^{\beta,r}(t)) = \prod_{k=0}^{n-1} k!\beta^k.$$

Proof. Let $\mathbf{V}_n^{\beta}(t)$ be the generalized Vandermonde matrix defined by,

$$\mathbf{V}_n^{\beta,r}(t) = \mathbf{V}_n^{\beta,r}(\beta t+r, \beta t+\beta+r, \beta t+2\beta+r, \ldots, \beta t+(n-1)\beta+r).$$

By Kalman [11], the formula for getting the determinant of a Vandermonde matris is

$$\mathbf{det}((V_n^{\beta}(t)) = \prod_{k=0}^{n-1}(t_i - t_j).$$

Now,

$$\begin{aligned}
\mathbf{det}((V_n^{\beta}(t)) &= [(\beta t + (n-1)\beta + r) - (\beta t + r)] \cdots [(\beta t + (n-1)\beta + r) - (\beta t + (n-2)\beta + r)] \\
&\times [(\beta t + (n-2)\beta + r) - (\beta t + r)] \cdots [(\beta t + (n-2)\beta + r) - (\beta t + (n-3)\beta + r)] \\
&\times [(\beta t + (n-3)\beta + r) - (\beta t + r)] \cdots [(\beta t + (n-3)\beta + r) - (\beta t + (n-4)\beta + r)] \\
&\vdots \\
&\times [(\beta t + \beta + r) - (\beta t + r)] \\
&= (n-1)\beta \cdot (n-2)\beta \cdot (n-3)\beta \cdots \beta \times (n-2)\beta \cdot (n-3)\beta \cdot (n-4)\beta \cdots \beta \\
&\times (n-3)\beta \cdot (n-4)\beta \cdot (n-5)\beta \cdots \beta \\
&\vdots \\
&\times \beta \\
&= \prod_{k=0}^{n-1} k!\beta^k.
\end{aligned}$$

□

Example 9. *Let $n = 4$. Then we have,*

$$\mathbf{V}_4^{\beta,r}(t) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \beta + r & 2\beta + r & 3\beta + r & 4\beta + r \\ (\beta + r)^2 & (2\beta + r)^2 & (3\beta + r)^2 & (4\beta + r)^2 \\ (\beta + r)^3 & (2\beta + r)^3 & (3\beta + r)^3 & (4\beta + r)^3 \\ \end{bmatrix}$$

Now,

$$\begin{aligned}
\mathbf{det}(\mathbf{V}_4^{\beta,r}(t)) &= (\beta t + 3\beta + r - (\beta t + r))(\beta t + 3\beta + r - (\beta t + \beta + r))(\beta t + 3\beta + r - (\beta t + 2\beta + r)) \\
&\quad \times (\beta t + 2\beta + r - (\beta t + r))(\beta t + 2\beta + r - (\beta t + \beta + r)) \\
&\quad \times (\beta t + \beta + r - (\beta t + r)) \\
&= (3\beta \cdot 2\beta \cdot \beta)(2\beta \cdot \beta)(\beta) \\
&= (3 \cdot 2 \cdot 1\beta^3)(2 \cdot 1\beta^2)(\beta) \\
&= (3!\beta^3)(2!\beta^2)(1!\beta) \\
&= \prod_{k=0}^{3} k!\beta^k.
\end{aligned}$$

Lemma 1. Let $L_n[\beta]$ be an $n \times n$ matrix defined by

$$[L_n[\beta]]_{1 \le i,j \le n} = \binom{j}{i-1} \beta^{i-1}(i-1)!.$$

For the $n \times n$ (r, β)-Stirling matrix $S^{(\beta,r)}(n)$,

$$\mathbf{V}_n^{\beta,r}(1) = S^{(\beta,r)}(n) L_n[\beta]^T.$$

Proof. Let $L_n[\beta]$ be an $n \times n$ matrix defined by

$$[L_n[\beta]]_{1\le i,j\le n} = \binom{j}{i-1}\beta^{i-1}(i-1)!.$$

Then,

$$[L_n[\beta]^T]_{1\le i,j\le n} = \binom{i-1}{j}\beta^j j!.$$

Now,

$$[S^{(\beta,r)}(n)L_n[\beta]^T]_{ij} = \sum_{k=0}^{i-1}\left\langle {i-1 \atop k}\right\rangle \beta^k j^k \binom{i-1}{k}.$$

By equation (5),

$$\begin{aligned}[S^{(\beta,r)}(n)L_n[\beta]^T]_{ij} &= (\beta j + r)^{i-1}\\ &= (\beta + (j-1)\beta + r)^{i-1}\\ &= [\mathbf{V}_n^{\beta,r}(1)]_{ij}.\end{aligned}$$

□

Theorem 2. For any real number t, and the generalized Pascal matrix,

$$\mathbf{V}_n^{\beta,r}(t) = P_n[\beta(t-1)]S^{(\beta,r)}(n)L_n[\beta]^T.$$

Proof. From Lemma 1,

$$\begin{aligned}P_n[\beta(t-1)]S^{(\beta,r)}(n)L_n[\beta]^T &= P_n[\beta(t-1)]\mathbf{V}_n^{\beta,r}(1)\\ &= \sum_{k=0}^{i-1}\binom{i-1}{k}(\beta(t-1))^{i-1-k}(\beta j + r)^k\\ &= (\beta(t-1) + \beta j + r)^{i-1}\\ &= (\beta t + \beta(j-1) + r)^{i-1}\\ &= [\mathbf{V}_n^{\beta,r}(x)]_{ij}.\end{aligned}$$

□

Example 10. *Let* $n = 4$*. Then,*

$$P_4[\beta(t-1)] = \begin{bmatrix} 1 & 0 & 0 & 0\\ \beta(t-1) & 1 & 0 & 0\\ (\beta(t-1))^2 & 2\beta(t-1) & 1 & 0\\ (\beta(t-1))^3 & 3(\beta(t-1))^2 & 3\beta(t-1) & 1\end{bmatrix}.$$

$$S^{(\beta,r)}(4) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ r & 1 & 0 & 0 \\ r^2 & \beta + 2r & 1 & 0 \\ r^3 & \beta^3 + 3\beta r + r^2 & 3\beta + 3r & 1 \end{bmatrix},$$

and

$$L_4[\beta]^T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \beta & 2\beta & 3\beta & 4\beta \\ 0 & 2\beta^2 & 6\beta^2 & 12\beta^2 \\ 0 & 0 & 6\beta^3 & 24\beta^3 \end{bmatrix}.$$

Now,

$$\begin{aligned} P_4[\beta(t-1)]S^{(\beta,r)}(4)L_4[\beta]^T &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ \beta(t-1) & 1 & 0 & 0 \\ (\beta(t-1))^2 & 2\beta(t-1) & 1 & 0 \\ (\beta(t-1))^3 & 3(\beta(t-1))^2 & 3\beta(t-1) & 1 \end{bmatrix} \\ &\times \begin{bmatrix} 1 & 0 & 0 & 0 \\ r & 1 & 0 & 0 \\ r^2 & \beta + 2r & 1 & 0 \\ r^3 & \beta^3 + 3\beta r + r^2 & 3\beta + 3r & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ \beta & 2\beta & 3\beta & 4\beta \\ 0 & 2\beta^2 & 6\beta^2 & 12\beta^2 \\ 0 & 0 & 6\beta^3 & 24\beta^3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ \beta(t-1) & 1 & 0 & 0 \\ (\beta(t-1))^2 & 2\beta(t-1) & 1 & 0 \\ (\beta(t-1))^3 & 3(\beta(t-1))^2 & 3\beta(t-1) & 1 \end{bmatrix} \\ &\times \begin{bmatrix} 1 & 1 & 1 & 1 \\ \beta + r & 2\beta + r & 0 & 0 \\ (\beta + r)^2 & (2\beta + r)^2 & (3\beta + r)^2 & (4\beta + r)^2 \\ (\beta + r)^3 & (2\beta + r)^3 & (3\beta + r)^3 & (4\beta + r)^3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 1 & 1 \\ \beta t + r & \beta t + \beta + r & \beta t + 2\beta + r & \beta t + 3\beta + r \\ (\beta t + r)^2 & (\beta t + \beta + r)^2 & (\beta t + 2\beta + r)^2 & (\beta t + 3\beta + r)^2 \\ (\beta t + r)^3 & (\beta t + \beta + r)^3 & (\beta t + 2\beta + r)^3 & (\beta t + 3\beta + r)^3 \end{bmatrix} \\ &= \mathbf{V}_4^{\beta,r}(t). \end{aligned}$$

2.4. Successive Sum of Powers of Arithmetic Progressions

We will show that matrix $\tilde{S}^{(\beta,r)}(n)$ can be used to derive a summation formula for arithmetic progressions. The following equations defined by Bazsó and Pintér in [1], and Mezo and Ramírez in [12] will be utilized for the proof of the following theorem.

Definition 3. [1] For $k = 1, 2, \ldots, n$, the numbers $Z^k_{1,\beta,r}(l), Z^k_{2,\beta,r}(l)$ are defined by the recursive formulas:

$$Z^k_{1,\beta,r}(l) = r^k + (\beta + r)^k + (2\beta + r)^k + \cdots + ((l-1)\beta + r)^k = \sum_{j=0}^{l-1} (j\beta + r)^k, \quad (8)$$

$$Z^k_{p,\beta,r}(l) = \sum_{j=1}^{n} Z^k_{p-1,\beta,r}(l). \quad (p = 2, 3, 4, \ldots). \quad (9)$$

Note that if $p = \beta = r = 1$, we obtain the sum of powers of the first n positive integers, that is

$$Z^k_{1,1,1}(l) = 1^k + 2^k + 3^k + 4^k + \cdots + l^k$$

Definition 4. [12] For each $i = 1, 2, \ldots, n$, and for $p \geq 0$,

$$\mathbf{t}_i(p) = \left[\binom{p+i-2}{p-1}, \quad \binom{p+i-2}{p}, \quad \ldots, \quad \binom{p+i-2}{p+k-2} \right]^T \quad (10)$$

$$\mathbf{z}_i^{\beta,r}(p) = \left[\binom{p+i-2}{p-1}, \quad Z^1_{p,\beta,r}(i), \ldots, Z^{k-1}_{p-1,\beta,r}(i) \right]^T \quad (11)$$

Theorem 3. For each $p = 1, 2, 3, \ldots, n$, we have

$$\begin{aligned} &\tilde{S}^{\beta,r}(k) \left[\tbinom{n+p-1}{p}, \beta\tbinom{n+p-1}{p+1}, \ldots, \beta^{k-1}\tbinom{n+p-1}{p+k-1} \right]^T \\ &= \left[\tbinom{n+p-1}{p}, Z^1_{p,\beta,r}(n), \ldots, Z^{k-}_{p,\beta,r}(n) \right]^T \end{aligned} \quad (12)$$

Proof. Let n and k be positive integers such that $n \geq k$ and $p = 1, 2, 3, \ldots, n$. Now, we will prove equation (3.7) by induction on $n + p$.
Note that the sum of the entries in the second row of $\mathbf{V}_n^{\beta,r}(0)$ is

$$\begin{aligned} r + (\beta + r) + (2\beta + r) + \cdots + ((n-1)\beta + r) &= r^1 + (\beta + r)^1 + (2\beta + r)^1 + \cdots + ((n-1)\beta + r)^1 \\ &= Z^k_{1,\beta,r}(n). \end{aligned}$$

Now, substituting $t = 0$ to equation (12), we have

$$\begin{aligned} \mathbf{V}_n^{\beta,r}(0) &= \tilde{S}^{\beta,r} C_n^{\beta}(0) \\ &= \tilde{S}^{\beta,r} \left[\tbinom{n}{1}, \beta\tbinom{n}{2}, \ldots, \beta^{k-1}\tbinom{n-1}{k-1} \right]^T \\ &= \left[\tbinom{n-1}{1}, Z^1_{1,\beta,r}(n), \ldots, Z^{k-1}_{1,\beta,r}(n) \right]^T. \end{aligned}$$

Thus, equation (12) is true for $p = 1$.
Consider $p \geq 2$, and supposed the result is true for all $i \leq n + p$. Using the identity

$$\binom{n+1}{k+1} = \sum_{l-0}^{n} \binom{l}{k},$$

and equations (10) and (11), by induction we have,

$$\begin{aligned}\tilde{S}^{\beta,r}(k)\mathbf{t}_i(p+1) &= \tilde{S}^{\beta,r}(k)(t_1(p)+t_2(p)+\cdots+t_n(p))\\ &= Z_1^{\beta,r}(p)+Z_2^{\beta,r}(p)+\cdots+Z_n^{\beta,r}(p)\\ &= Z_n^{\beta,r}(p+1).\end{aligned}$$

Thus, equation (12) follows. □

Example 11. *Equation (12) in Theorem 3 yields nice formulas to sums of powers of integers.*
For example, if $p=1$, and $k=3$, we obtain

$$\begin{bmatrix}1 & 0 & 0 & 0\\ r & 1 & 0 & 0\\ r^2 & \beta+2r & 2 & 0\\ r^3 & \beta^3+3\beta r+3r^2 & 6\beta+6r & 6\end{bmatrix}\begin{bmatrix}\binom{n}{1}\\ \beta\binom{n}{2}\\ \beta^2\binom{n}{3}\\ \beta^3\binom{n}{4}\end{bmatrix}$$
$$=\begin{bmatrix}n\\ \frac{1}{2}n(\beta n-\beta+2r\\ \frac{1}{6}n(2\beta^2n^2-3\beta^2n+\beta^2+6\beta nr-6\beta r+6r^2)\\ \frac{1}{4}n(\beta n-\beta+2r)(\beta^2n^2-\beta^2n+2\beta nr-2\beta r+2r^2)\end{bmatrix}$$
$$=[n, Z^1_{1,\beta,r}(n), Z^2_{1,\beta,r}(n), Z^3_{1,\beta,r}(n)]^T$$

If $p=2$, and $k=3$, we obtain

$$\begin{bmatrix}1 & 0 & 0 & 0\\ r & 1 & 0 & 0\\ r^2 & \beta+2r & 2 & 0\\ r^3 & \beta^3+3\beta r+3r^2 & 6\beta+6r & 6\end{bmatrix}\begin{bmatrix}\binom{n+1}{2}\\ \beta\binom{n+1}{3}\\ \beta^2\binom{n+1}{4}\\ \beta^3\binom{n+1}{5}\end{bmatrix}$$
$$=\begin{bmatrix}\frac{1}{2}n(n+1)\\ \frac{1}{6}n(n+1)(\beta n-\beta+3r)\\ \frac{1}{12}n(n+1)(\beta^2n^2-\beta^2n+4\beta nr+6r^2)\\ \frac{1}{60}n(n+1)(3\beta^3n^3-3\beta^3n^2-2\beta^3n+2\beta^3+15\beta^2n^2r-15\beta^2nr+30\beta nr^2-30\beta r^2+30r^3)\end{bmatrix}$$
$$=[*, Z^1_{2,\beta,r}(n), Z^2_{2,\beta,r}(n), Z^3_{2,\beta,r}(n)]^T.$$

Note that $Z^1_{2,m,r}(n), k=1,2,3,$ expresses

$$r^k+(r^k+(\beta+r)^k)+\cdots+(r^k+(\beta+r)^k+(2\beta+r)^k)+\cdots+((n-1)\beta+r)^k)=\sum_{l=1}^{n}\sum_{j=0}^{l-1}(j\beta+r)^k.$$

Corollary 3. For each $p=1,2,\ldots,n$, we have

$$\sum_{n_{p-1}=1}^{n_p}\sum_{n_{p-2}=1}^{n_p-1}\cdots\sum_{i=0}^{n_1-1}(i\beta+r)^k=\sum_{i=0}^{k}i!\left\langle {k \atop i}\right\rangle_{\beta,r}\binom{n+p-1}{p+i}=Z^k_{p,\beta,r}(n),$$

where $n_p=n$.

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