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Upper and lower α-$\star$-continuity

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Abstract. Our main purpose is to introduce the concepts of upper and lower α-$\star$-continuous multifunctions. In particular, some characterizations of upper and lower α-$\star$-continuous multifunctions are investigated.

2020 Mathematics Subject Classifications: 54C08, 54C60

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1. Introduction

The field of mathematical science called topology is concerned with all questions directly or indirectly related to continuity. Continuity is an important concept for the study and investigation in topological spaces. This concept has been extended to the setting multifunctions and has been generalized by weaker forms of open sets. In 1965, Njåstad [21] introduced a weak form of open sets called α-sets. Mashhour et al. [19] defined a function to be α-continuous if the inverse image of each open set is an α-set and obtained several characterizations of such functions. Noiri [22] investigated the relationships between α-continuous functions and several known functions, for example, almost continuous functions, η-continuous functions, δ-continuous functions or irresolute functions. In [23], the present author introduced the concept of almost α-continuity in topological spaces as a generalization of α-continuity and almost continuity. Neubrunn [20] introduced the notion of upper (resp. lower) α-continuous multifunctions. These multifunctions are further investigated by the present authors [24]. Boonpok et al. [11] introduced and studied the notions of upper and lower (τ_1, τ_2)-precontinuous multifunctions. Viriyapong and Boonpok [26] introduced and investigated the concepts of upper and lower $(\tau_1, \tau_2)\alpha$-continuous multifunctions. Moreover, several characterizations of upper and lower $(\tau_1, \tau_2)\delta$-semicontinuous multifunctions were established in [6]. In [10], the authors investigated some characterizations of upper and lower almost weakly (τ_1, τ_2)-continuous multifunctions. Laprom

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et al. [18] introduced and studied the notions of upper and lower $\beta(\tau_1, \tau_2)$-continuous multifunctions.

The concept of ideal topological spaces was introduced and studied by Kuratowski [17] and Vaidyanathswamy [25]. Every topological space is an ideal topological space and all the results of ideal topological spaces are generalizations of the results established in topological spaces. In 1990, Janković and Hamlett [16] introduced the concept of $\mathscr{I}$-open sets in ideal topological spaces. Abd El-Monsef et al. [14] further investigated $\mathscr{I}$-open sets and $\mathscr{I}$-continuous functions. Later, several authors studied ideal topological spaces giving several convenient definitions. Some authors obtained decompositions of continuity. For instance, Açikgöz et al. [1] studied the concepts of α-$\mathscr{I}$-continuity and α-$\mathscr{I}$-openness in ideal topological spaces and investigated several characterizations of these functions. Hatir and Noiri [15] introduced the notions of semi-$\mathscr{I}$-open sets, α-$\mathscr{I}$-open sets and β-$\mathscr{I}$-open sets via idealization and using these sets obtained new decompositions of continuity. In [4], the author introduced and studied the notions of upper and lower $\star$-continuous multifunctions. Boonpok [7] investigated some characterizations of upper and lower $\beta(\star)$-continuous multifunctions. Furthermore, several characterizations of almost α-$\star$-continuous multifunctions and weakly α-$\star$-continuous multifunctions were established in [9] and [8], respectively. In this paper, we introduce the notions of upper and lower α-$\star$-continuous multifunctions. Moreover, some characterizations of upper and lower α-$\star$-continuous multifunctions are discussed.

2. Preliminaries

Throughout the present paper, spaces (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a topological space (X, τ). The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. An ideal $\mathscr{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X satisfying the following properties: (1) $A \in \mathscr{I}$ and $B \subseteq A$ imply $B \in \mathscr{I}$; (2) $A \in \mathscr{I}$ and $B \in \mathscr{I}$ imply $A \cup B \in \mathscr{I}$. A topological space (X, τ) with an ideal $\mathscr{I}$ on X is called an ideal topological space and is denoted by $(X, \tau, \mathscr{I})$. For an ideal topological space $(X, \tau, \mathscr{I})$ and a subset A of X, $A^\star(\mathscr{I})$ is defined as follows: $A^\star(\mathscr{I}) = \{x \in X : U \cap A \notin \mathscr{I}$ for every open neighbourhood U of $x\}$. In case there is no chance for confusion, $A^\star(\mathscr{I})$ is simply written as $A^\star$. In [17], $A^\star$ is called the local function of A with respect to $\mathscr{I}$ and τ and $\mathrm{Cl}^\star(A) = A^\star \cup A$ defines a Kuratowski closure operator for a topology $\tau^\star(\mathscr{I})$ finer than τ. A subset A is said to be $\star$-closed [16] if $A^\star \subseteq A$. The interior of a subset A in $(X, \tau^\star(\mathscr{I}))$ is denoted by $\mathrm{Int}^\star(A)$.

A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be *semi***-*$\mathscr{I}$*-open* [12] (resp. *semi-*$\mathscr{I}$*-open* [15]) if $A \subseteq \mathrm{Cl}(\mathrm{Int}^\star(A))$ (resp. $A \subseteq \mathrm{Cl}^\star(\mathrm{Int}(A))$). The complement of a semi*-$\mathscr{I}$-open (resp. semi-$\mathscr{I}$-open) set is said to be *semi***-*$\mathscr{I}$*-closed* [12] (resp. semi-$\mathscr{I}$-closed [15]). For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the intersection of all semi-$\mathscr{I}$-closed (resp. semi*-$\mathscr{I}$-closed) sets containing A is called the *semi-*$\mathscr{I}$*-closure* [13] (resp. *semi***-*$\mathscr{I}$*-closure* [13]) of A and is denoted by $s\mathrm{Cl}_{\mathscr{I}}(A)$ (resp. $s^\star\mathrm{Cl}_{\mathscr{I}}(A)$). The union of all semi-$\mathscr{I}$-open (resp. semi*-$\mathscr{I}$-open) sets contained in A is called the *semi-*$\mathscr{I}$*-interior* (resp. *semi**-$\mathscr{I}$*-interior*) of A and is denoted by $s\mathrm{Int}_{\mathscr{I}}(A)$ (resp. $s^\star\mathrm{Int}_{\mathscr{I}}(A)$).

Lemma 1. *For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the following properties hold:*

(1) If A is an open set, then $s^\star Cl_{\mathscr{I}}(A) = Int(Cl^\star(A))$.

(2) If A is a $\star$-open set, then $sCl_{\mathscr{I}}(A) = Int^\star(Cl(A))$.

Proof. (1) Suppose that A is an open set. Then, $A \subseteq \mathrm{Int}(\mathrm{Cl}^\star(A))$ and by Lemma 13(1) of [13], we have $s^\star\mathrm{Cl}_{\mathscr{I}}(A) = A \cup \mathrm{Int}(\mathrm{Cl}^\star(A)) = \mathrm{Int}(\mathrm{Cl}^\star(A))$.

(2) Suppose that A is a $\star$-open set. Then, we have $A \subseteq \mathrm{Int}^\star(\mathrm{Cl}(A))$ and by Lemma 13(2) of [13], $s\mathrm{Cl}_{\mathscr{I}}(A) = A \cup \mathrm{Int}^\star(\mathrm{Cl}(A)) = \mathrm{Int}^\star(\mathrm{Cl}(A))$.

Recall that a subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be *α-$\star$-closed* [2] if $\mathrm{Cl}^\star(\mathrm{Int}(\mathrm{Cl}^\star(A))) \subseteq A$. The complement of an α-$\star$-closed set is said to be *α-$\star$-open.*

Proposition 1. *Let $(X, \tau, \mathscr{I})$ be an ideal topological space and $\{A_\gamma \mid \gamma \in \Gamma\}$ be a family of subsets of X. If A_γ is α-$\star$-closed for each $\gamma \in \Gamma$, then $\bigcap_{\gamma\in\Gamma} A_\gamma$ is α-$\star$-closed.*

Proof. Suppose that A_γ is α-$\star$-closed for each $\gamma \in \Gamma$. Then, we have $X - A_\gamma$ is α-$\star$-open for each $\gamma \in \Gamma$. Thus, $\bigcup_{\gamma\in\Gamma}(X - A_\gamma) = X - \bigcap_{\gamma\in\Gamma} A_\gamma$ is α-$\star$-open and hence $\bigcap_{\gamma\in\Gamma} A_\gamma$ is α-$\star$-closed.

For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the intersection of all α-$\star$-closed sets containing A is called the *α-$\star$-closure* of A and is denoted by $\star\alpha\mathrm{Cl}(A)$. The *$\alpha$-$\star$-interior* of A is defined by the union of all α-$\star$-open sets contained in A and is denoted by $\star\alpha\mathrm{Int}(A)$.

Proposition 2. *For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the following properties hold:*

(1) $\star\alpha Cl(A)$ is α-$\star$-closed.

(2) A is α-$\star$-closed if and only if $A = \star\alpha Cl(A)$.

Proof. (1) Follows from Proposition 1.

(2) Follows from (1).

Lemma 2. *For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the following properties are equivalent:*

(1) A is α-$\star$-open in X;

(2) $G \subseteq A \subseteq Int^\star(Cl(G))$ for some $\star$-open set G;

(3) $G \subseteq A \subseteq sCl_{\mathscr{I}}(G)$ for some $\star$-open set G;

(4) $A \subseteq sCl_{\mathscr{I}}(Int^\star(A))$.

Proof. (1) $\Rightarrow$ (2): Suppose that A is an α-$\star$-open set. Then, $A \subseteq \text{Int}^\star(\text{Cl}(\text{Int}^\star(A)))$. Put $G = \text{Int}^\star(A)$, then G is a $\star$-open set such that $G \subseteq A \subseteq \text{Int}^\star(\text{Cl}(G))$.

(2) $\Rightarrow$ (3): This follows from Lemma 1(2).

(3) $\Rightarrow$ (4): Suppose that $G \subseteq A \subseteq s\text{Cl}_{\mathscr{I}}(G)$ for some $\star$-open set G. Then, we have $G \subseteq \text{Int}^\star(A)$ and hence $A \subseteq s\text{Cl}_{\mathscr{I}}(\text{Int}^\star(A))$.

(4) $\Rightarrow$ (1): Suppose that $A \subseteq s\text{Cl}_{\mathscr{I}}(\text{Int}^\star(A))$. Since $\text{Int}^\star(A)$ is $\star$-open in X and by Lemma 1(2), $A \subseteq \text{Int}^\star(\text{Cl}(\text{Int}^\star(A)))$. Thus, A is α-$\star$-open in X.

Lemma 3. *For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the following properties hold:*

(1) A is α-$\star$-closed in X if and only if $s\mathit{Int}_{\mathscr{I}}(\mathit{Cl}^\star(A)) \subseteq A$.

(2) $s\mathit{Int}_{\mathscr{I}}(\mathit{Cl}^\star(A)) = \mathit{Cl}^\star(\mathit{Int}(\mathit{Cl}^\star(A)))$.

(3) $\star\alpha\mathit{Cl}(A) = A \cup \mathit{Cl}^\star(\mathit{Int}(\mathit{Cl}^\star(A)))$.

(4) $\star\alpha\mathit{Int}(A) = A \cap \mathit{Int}^\star(\mathit{Cl}(\mathit{Int}^\star(A)))$.

Proof. (1) Follows from Lemma 2.

(2) Follows from Lemma 13(1) of [13].

(3) We observe that

$$\begin{aligned}\text{Cl}^\star(\text{Int}(\text{Cl}^\star(A \cup \text{Cl}^\star(\text{Int}(\text{Cl}^\star(A)))))) &\subseteq \text{Cl}^\star(\text{Int}(\text{Cl}^\star(A \cup (\text{Cl}^\star(A))))) \\ &\subseteq \text{Cl}^\star(\text{Int}(\text{Cl}^\star(A))) \\ &\subseteq A \cup \text{Cl}^\star(\text{Int}(\text{Cl}^\star(A))).\end{aligned}$$

Thus, $A \cup \text{Cl}^\star(\text{Int}(\text{Cl}^\star(A)))$ is α-$\star$-closed and hence $\star\alpha\text{Cl}(A) \subseteq A \cup \text{Cl}^\star(\text{Int}(\text{Cl}^\star(A)))$. On the other hand, since $\star\alpha\text{Cl}(A)$ is α-$\star$-closed, we have

$$\text{Cl}^\star(\text{Int}(\text{Cl}^\star(A))) \subseteq \text{Cl}^\star(\text{Int}(\text{Cl}^\star(\star\alpha\text{Cl}(A)))) \subseteq \star\alpha\text{Cl}(A)$$

and hence $A \cup \text{Cl}^\star(\text{Int}(\text{Cl}^\star(A))) \subseteq \star\alpha\text{Cl}(A)$. Thus, $\star\alpha\text{Cl}(A) = A \cup \text{Cl}^\star(\text{Int}(\text{Cl}^\star(A)))$.

(4) Since $\star\alpha\text{Int}(A)$ is α-$\star$-open, we have

$$\star\alpha\text{Int}(A) \subseteq \text{Int}^\star(\text{Cl}(\text{Int}^\star(\star\alpha\text{Int}(A)))) \subseteq \text{Int}^\star(\text{Cl}(\text{Int}^\star(A)))$$

and hence $\star\alpha\text{Int}(A) \subseteq A \cap \text{Int}^\star(\text{Cl}(\text{Int}^\star(A)))$. On the other hand, we have

$$\begin{aligned}A \cap \text{Int}^\star(\text{Cl}(\text{Int}^\star(A))) &\subseteq \text{Int}^\star(\text{Cl}(\text{Int}^\star(A))) \\ &= \text{Int}^\star(\text{Cl}(\text{Int}^\star(A) \cap \text{Int}^\star(\text{Cl}(\text{Int}^\star(A))))) \\ &= \text{Int}^\star(\text{Cl}(\text{Int}^\star(A \cap \text{Int}^\star(\text{Cl}(\text{Int}^\star(A)))))).\end{aligned}$$

Thus, $A \cap \text{Int}^\star(\text{Cl}(\text{Int}^\star(A)))$ is α-$\star$-open and so $A \cap \text{Int}^\star(\text{Cl}(\text{Int}^\star(A))) \subseteq \star\alpha\text{Int}(A)$. This shows that $\star\alpha\text{Int}(A) = A \cap \text{Int}^\star(\text{Cl}(\text{Int}^\star(A)))$.

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, following [3] we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and

$$F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}.$$

In particular, $F^-(y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$. For each $A \subseteq X$, $F(A) = \cup_{x\in A} F(x)$.

3. Upper and lower α-$\star$-continuous multifunctions

In this section, we introduce the notions of upper and lower α-$\star$-continuous multifunctions. Moreover, several characterizations of upper and lower α-$\star$-continuous multifunctions are discussed.

Definition 1. *A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is said to be:*

(1) upper α-$\star$-continuous at a point x of X if, for each $\star$-open set V such that $F(x) \subseteq V$, there exists an α-$\star$-open set U of X containing x such that $F(U) \subseteq V$;

(2) lower α-$\star$-continuous at a point x of X if, for each $\star$-open set V such that

$$F(x) \cap V \neq \emptyset,$$

there exists an α-$\star$-open set U of X containing x such that $F(z) \cap V \neq \emptyset$ for each $z \in U$;

(3) upper (resp. lower) α-$\star$-continuous if F is upper (resp. lower) α-$\star$-continuous at each point of X.

Theorem 1. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) F is upper α-$\star$-continuous at $x \in X$;

(2) $x \in s\mathrm{Cl}_{\mathscr{I}}(\mathrm{Int}^\star(F^+(V)))$ for every α-$\star$-open set V of Y containing $F(x)$;

(3) $x \in \star\alpha\mathrm{Int}(F^+(V))$ for every α-$\star$-open set V of Y containing $F(x)$.

Proof. (1) $\Rightarrow$ (2): Let V be any $\star$-open set of Y containing $F(x)$. Then, there exists an α-$\star$-open set U of X containing x such that $F(U) \subseteq V$; hence $x \in U \subseteq F^+(V)$. Since U is α-$\star$-open, by Lemma 2, we have $x \in U \subseteq s\mathrm{Cl}_{\mathscr{I}}(\mathrm{Int}^\star(U)) \subseteq s\mathrm{Cl}_{\mathscr{I}}(\mathrm{Int}^\star(F^+(V)))$.

(2) $\Rightarrow$ (3): Let V be any $\star$-open set of Y containing $F(x)$. Then by (2), we have $x \in s\mathrm{Cl}_{\mathscr{I}}(\mathrm{Int}^\star(F^+(V)))$ and by Lemma 1(2), $x \in \mathrm{Int}^\star(\mathrm{Cl}(\mathrm{Int}^\star(F^+(V))))$. Thus, by Lemma 3(4), $x \in \star\alpha\mathrm{Int}(F^+(V))$.

(3) $\Rightarrow$ (1): Let V be any $\star$-open set of Y containing $F(x)$. By (3), $x \in \star\alpha\mathrm{Int}(F^+(V))$ and so there exists an α-$\star$-open set U of X containing x such that $U \subseteq F^+(V)$; hence $F(U) \subseteq V$. This shows that F is upper α-$\star$-continuous at x.

Theorem 2. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) F is lower α-$\star$-continuous at $x \in X$;

(2) $x \in s\mathrm{Cl}_{\mathscr{I}}(\mathrm{Int}^{\star}(F^{-}(V)))$ for every α-$\star$-open set V of Y such that $F(x) \cap V \neq \emptyset$;

(3) $x \in \star\alpha\mathrm{Int}(F^{-}(V))$ for every α-$\star$-open set V of Y such that $F(x) \cap V \neq \emptyset$.

Proof. The proof is similar to that of Theorem 1.

Definition 2. *A subset N of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be a $\star$-neighbourhood (resp. α-$\star$-neighbourhood) of $x \in X$ if there exists a $\star$-open (resp. α-$\star$-open) set V of X such that $x \in V \subseteq N$.*

Theorem 3. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) F is upper α-$\star$-continuous;

(2) $F^{+}(V)$ is α-$\star$-open in X for every $\star$-open set V of Y;

(3) $F^{-}(K)$ is α-$\star$-closed in X for every $\star$-closed set K of Y;

(4) $s\mathrm{Int}_{\mathscr{I}}(\mathrm{Cl}^{\star}(F^{-}(B))) \subseteq F^{-}(\mathrm{Cl}^{\star}(B))$ for every subset B of Y;

(5) $\star\alpha\mathrm{Cl}(F^{-}(B)) \subseteq F^{-}(\mathrm{Cl}^{\star}(B))$ for every subset B of Y;

(6) for each $x \in X$ and each $\star$-neighbourhood V of $F(x)$, $F^{+}(V)$ is an α-$\star$-neighbourhood of x;

(7) for each $x \in X$ and each $\star$-neighbourhood V of $F(x)$, there exists an α-$\star$-neighbourhood U of x such that $F(U) \subseteq V$.

Proof. (1) $\Rightarrow$ (2): Let V be any $\star$-open set of Y and $x \in F^{+}(V)$. Then, $F(x) \subseteq V$. Since F is upper α-$\star$-continuous at x, there exists an α-$\star$-open set U of X containing x such that $F(U) \subseteq V$; hence $x \in U \subseteq F^{+}(V)$. By Lemma 2,

$$x \in U \subseteq s\mathrm{Cl}_{\mathscr{I}}(\mathrm{Int}^{\star}(U)) \subseteq s\mathrm{Cl}_{\mathscr{I}}(\mathrm{Int}^{\star}(F^{+}(V))).$$

Thus, $F^{+}(V) \subseteq s\mathrm{Cl}_{\mathscr{I}}(\mathrm{Int}^{\star}(F^{+}(V)))$. It follows from Lemma 2 that $F^{+}(V)$ is α-$\star$-open in X.

(2) $\Leftrightarrow$ (3): This follows from the fact that $F^{+}(Y - B) = X - F^{-}(B)$ for any subset B of Y.

(3) $\Rightarrow$ (4): Let B be any subset of Y. Then, $\mathrm{Cl}^{\star}(B)$ is $\star$-closed in Y and by (3), $F^{-}(\mathrm{Cl}^{\star}(B))$ is α-$\star$-closed in X. Thus, by Lemma 3(1),

$$s\mathrm{Int}_{\mathscr{I}}(\mathrm{Cl}^{\star}(F^{-}(B))) \subseteq s\mathrm{Int}_{\mathscr{I}}(\mathrm{Cl}^{\star}(F^{-}(\mathrm{Cl}^{\star}(B)))) \subseteq F^{-}(\mathrm{Cl}^{\star}(B)).$$

(4) ⇒ (5): Let B be any subset of Y. By (4) and Lemma 3(3),

$$\star\alpha\mathrm{Cl}(F^-(B)) = F^-(B) \cup s\mathrm{Int}_{\mathscr{I}}(\mathrm{Cl}^\star(F^-(B))) \subseteq F^-(\mathrm{Cl}^\star(B)).$$

(5) ⇒ (3): Let K be any $\star$-closed set of Y. By (5), we have

$$\star\alpha\mathrm{Cl}(F^-(K)) \subseteq F^-(\mathrm{Cl}^\star(K)) = F^-(K).$$

This shows that $F^-(K)$ is α-$\star$-closed in X.

(2) ⇒ (6): Let $x \in X$ and V be a $\star$-neighbourhood of $F(x)$. Then, there exists a $\star$-open set G of Y such that $F(x) \subseteq G \subseteq V$. Thus, $x \in F^+(G) \subseteq F^+(V)$. By (2), $F^+(G)$ is α-$\star$-open and hence $F^+(V)$ is an α-$\star$-neighbourhood of x.

(6) ⇒ (7): Let $x \in X$ and V be a $\star$-neighbourhood of $F(x)$. By (6), we have $F^+(V)$ is an α-$\star$-neighbourhood of x. Put $U = F^+(V)$, then U is an α-$\star$-neighbourhood of x such that $F(U) \subseteq V$.

(7) ⇒ (1): Let $x \in X$ and V be any $\star$-open set of Y such that $F(x) \subseteq V$. Then, V is a $\star$-neighbourhood of $F(x)$ and so there exists an α-$\star$-neighbourhood U of x such that $F(U) \subseteq V$. Since U is an α-$\star$-neighbourhood of x, there exists an α-$\star$-open set G of X such that $x \in G \subseteq U$; hence $F(G) \subseteq V$. This shows that F is upper α-$\star$-continuous.

Theorem 4. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) F is lower α-$\star$-continuous;

(2) $F^-(V)$ is α-$\star$-open in X for every $\star$-open set V of Y;

(3) $F^+(K)$ is α-$\star$-closed in X for every $\star$-closed set K of Y;

(4) $s\mathrm{Int}_{\mathscr{I}}(\mathrm{Cl}^\star(F^+(B))) \subseteq F^+(\mathrm{Cl}^\star(B))$ for every subset B of Y;

(5) $\star\alpha\mathrm{Cl}(F^+(B)) \subseteq F^+(\mathrm{Cl}^\star(B))$ for every subset B of Y;

(6) $F(\star\alpha\mathrm{Cl}(A)) \subseteq \mathrm{Cl}^\star(F(A))$ for every subset A of X;

(7) $F(s\mathrm{Int}_{\mathscr{I}}(\mathrm{Cl}^\star(A))) \subseteq \mathrm{Cl}^\star(F(A))$ for every subset A of X;

(8) $F(\mathrm{Cl}^\star(\mathrm{Int}(\mathrm{Cl}^\star(A)))) \subseteq \mathrm{Cl}^\star(F(A))$ for every subset A of X.

Proof. The proofs except for the following are similar to the proof of Theorem 3.

(5) ⇒ (6): Let A be any subset of X. Since $A \subseteq F^+(F(A))$, we have

$$\star\alpha\mathrm{Cl}(A) \subseteq \star\alpha\mathrm{Cl}(F^+(F(A))) \subseteq F^+(\mathrm{Cl}^\star(F(A)))$$

and hence $F(\star\alpha\mathrm{Cl}(A)) \subseteq \mathrm{Cl}^\star(F(A))$.

(6) ⇒ (7): Let A be any subset of X. By (6) and Lemma 3,

$$F(s\mathrm{Int}_{\mathscr{I}}(\mathrm{Cl}^\star(A))) = F(\mathrm{Cl}^\star(\mathrm{Int}(\mathrm{Cl}^\star(A))))$$

$$\subseteq F(A \cup \mathrm{Cl}^\star(\mathrm{Int}(\mathrm{Cl}^\star(A))))$$
$$= F(\star\alpha\mathrm{Cl}(A))$$
$$\subseteq \mathrm{Cl}^\star(F(A)).$$

(7) $\Rightarrow$ (8): Let A be any subset of X. By (7) and Lemma 3(2), we have

$$F(\mathrm{Cl}^\star(\mathrm{Int}(\mathrm{Cl}^\star(A)))) = F(s\mathrm{Int}_{\mathscr{I}}(\mathrm{Cl}^\star(A))) \subseteq \mathrm{Cl}^\star(F(A)).$$

(8) $\Rightarrow$ (1): Let $x \in X$ and V be any $\star$-open set such that $F(x) \cap V \neq \emptyset$. Then, we have $x \in F^-(V)$. We shall show that $F^-(V)$ is α-$\star$-open in X. By the hypothesis, $F(\mathrm{Cl}^\star(\mathrm{Int}(\mathrm{Cl}^\star(F^+(Y-V))))) \subseteq \mathrm{Cl}^\star(F(F^+(Y-V))) \subseteq Y - V$ and hence

$$\mathrm{Cl}^\star(\mathrm{Int}(\mathrm{Cl}^\star(F^+(Y-V)))) \subseteq F^+(Y-V) = X - F^-(V).$$

Thus, $F^-(V) \subseteq \mathrm{Int}^\star(\mathrm{Cl}(\mathrm{Int}^\star(F^-(V))))$ and so $F^-(V)$ is α-$\star$-open in X. Put $U = F^-(V)$, then U is an α-$\star$-open set of X containing x such that $F(z) \cap V \neq \emptyset$ for every $z \in U$. This shows that F is lower α-$\star$-continuous.

Definition 3. *A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is called α-$\star$-continuous if $f^{-1}(V)$ is α-$\star$-open in X for every $\star$-open set V of Y.*

Corollary 1. *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) f is α-$\star$-continuous;

(2) $f^{-1}(K)$ is α-$\star$-closed in X for every $\star$-closed set K of Y;

(3) $sInt_{\mathscr{I}}(Cl^\star(f^{-1}(B))) \subseteq f^{-1}(Cl^\star(B))$ for any subset B of Y;

(4) $\star\alpha Cl(f^{-1}(B)) \subseteq f^{-1}(Cl^\star(B))$ for any subset B of Y;

(5) for each $x \in X$ and each $\star$-neighbourhood V of $f(x)$, $f^{-1}(V)$ is an α-$\star$-neighbourhood of x;

(6) for each $x \in X$ and each $\star$-neighbourhood V of $f(x)$, there exists an α-$\star$-neighbourhood U of x such that $f(U) \subseteq V$;

(7) $f(\star\alpha Cl(A)) \subseteq Cl^\star(f(A))$ for every subset A of X;

(8) $f(sInt_{\mathscr{I}}(Cl^\star(A))) \subseteq Cl^\star(f(A))$ for every subset A of X;

(9) $f(Cl^\star(Int(Cl^\star(A)))) \subseteq Cl^\star(f(A))$ for every subset A of X.

Definition 4. [5] *A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be:*

(1) $\star$-paracompact if every cover of A by $\star$-open sets of X is refined by a cover of A which consists of $\star$-open sets of X and is $\star$-locally finite in X;

(2) $\star$-regular if for each $x \in A$ and each $\star$-open set U of X containing x, there exists a $\star$-open set V of X such that $x \in V \subseteq Cl(V) \subseteq U$.

Lemma 4. [5] *Let A be a subset of an ideal topological space $(X, \tau, \mathscr{I})$. If A is a $\star$-regular $\star$-paracompact set of X and each $\star$-open set U containing A, then there exists a $\star$-open set V such that $A \subseteq V \subseteq Cl(V) \subseteq U$.*

A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is called *punctually $\star$-paracompact* (resp. *punctually $\star$-regular*) if for each $x \in X$, $F(x)$ is $\star$-paracompact (resp. $\star$-regular).

By $\mathrm{Cl}^\star_\alpha(F) : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, we shall denote a multifunction defined as follows: $[\mathrm{Cl}^\star_\alpha(F)](x) = \star\alpha\mathrm{Cl}_{\mathscr{J}}(F(x))$ for each $x \in X$.

Lemma 5. *If $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is punctually $\star$-regular and punctually $\star$-paracompact, then $[Cl^\star_\alpha(F)]^+(V) = F^+(V)$ for every $\star$-open set V of Y.*

Proof. Let V be any $\star$-open set of Y and $x \in [\mathrm{Cl}^\star_\alpha(F)]^+(V)$. Then, $\star\alpha\mathrm{Cl}_{\mathscr{J}}(F(x)) \subseteq V$. Thus, $F(x) \subseteq V$ and hence $x \in F^+(V)$. Therefore, $[\mathrm{Cl}^\star_\alpha(F)]^+(V) \subseteq F^+(V)$. On the other hand, let V be any $\star$-open set of Y and $x \in F^+(V)$. Then, $F(x) \subseteq V$. Since $F(x)$ is punctually $\star$-regular and punctually $\star$-paracompact, by Lemma 4, there exists a $\star$-open set G such that $F(x) \subseteq G \subseteq \mathrm{Cl}(G) \subseteq V$; hence $\star\alpha\mathrm{Cl}_{\mathscr{J}}(F(x)) \subseteq \mathrm{Cl}(G) \subseteq V$. This shows that $x \in [\mathrm{Cl}^\star_\alpha(F)]^+(V)$. Therefore, $F^+(V) \subseteq [\mathrm{Cl}^\star_\alpha(F)]^+(V)$. Consequently, we obtain $[\mathrm{Cl}^\star_\alpha(F)]^+(V) = F^+(V)$.

Theorem 5. *Let $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ be punctually $\star$-regular and punctually $\star$-paracompact. Then F is upper α-$\star$-continuous if and only if*

$$Cl^\star_\alpha(F) : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$$

is upper α-$\star$-continuous.

Proof. Suppose that F is upper α-$\star$-continuous. Let $x \in X$ and V be any $\star$-open set of Y such that $\star\alpha\mathrm{Cl}_{\mathscr{J}}(F(x)) \subseteq V$. By Lemma 5, we have $x \in [\mathrm{Cl}^\star_\alpha(F)]^+(V) = F^+(V)$. Since F is upper α-$\star$-continuous, there exists an α-$\star$-open set U of X containing x such that $F(U) \subseteq V$. Sine $F(z)$ is punctually $\star$-regular and punctually $\star$-paracompact for each $z \in U$, by Lemma 4, there exists a $\star$-open set G such that $F(z) \subseteq G \subseteq \mathrm{Cl}(G) \subseteq V$. Thus, $\star\alpha\mathrm{Cl}_{\mathscr{J}}(F(z)) \subseteq \mathrm{Cl}(G) \subseteq V$ and hence $\star\alpha\mathrm{Cl}_{\mathscr{J}}(F(U)) \subseteq V$. This shows that $\mathrm{Cl}^\star_\alpha(F)$ is upper α-$\star$-continuous.

Conversely, suppose that $\mathrm{Cl}^\star_\alpha(F)$ is upper α-$\star$-continuous. Let $x \in X$ and V be any $\star$-open set of Y such that $F(x) \subseteq V$. By Lemma 5, we have $x \in F^+(V) = [\mathrm{Cl}^\star_\alpha(F)]^+(V)$ and hence $\star\alpha\mathrm{Cl}_{\mathscr{J}}(F(x)) \subseteq V$. Since $\mathrm{Cl}^\star_\alpha(F)$ is upper α-$\star$-continuous, there exists an α-$\star$-open set U of X containing x such that $\star\alpha\mathrm{Cl}_{\mathscr{J}}(F(U)) \subseteq V$; hence $F(U) \subseteq V$. This shows that F is upper α-$\star$-continuous.

Lemma 6. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, it follows that for each $\star$-open set V of Y $[Cl^\star_\alpha(F)]^-(V) = F^-(V)$.*

Proof. Suppose that V is any $\star$-open set of Y. Let $x \in [\mathrm{Cl}^{\star}_{\alpha}(F)]^{-}(V)$. Then, we have $\star\alpha\mathrm{Cl}_{\mathscr{J}}(F(x)) \cap V \neq \emptyset$ and hence $F(x) \cap V \neq \emptyset$. Thus, $x \in F^{-}(V)$. This shows that $[\mathrm{Cl}^{\star}_{\alpha}(F)]^{-}(V) \subseteq F^{-}(V)$. On the other hand, let $x \in F^{-}(V)$. Then,

$$\emptyset \neq F(x) \cap V \subseteq \star\alpha\mathrm{Cl}_{\mathscr{J}}(F(x)) \cap V.$$

Therefore, $x \in [\mathrm{Cl}^{\star}_{\alpha}(F)]^{-}(V)$. Thus, $F^{-}(V) \subseteq [\mathrm{Cl}^{\star}_{\alpha}(F)]^{-}(V)$ and hence $[\mathrm{Cl}^{\star}_{\alpha}(F)]^{-}(V) = F^{-}(V)$.

Theorem 6. *A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is lower α-$\star$-continuous if and only if $Cl^{\star}_{\alpha}(F) : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is lower α-$\star$-continuous.*

Proof. By utilizing Lemma 6, this can be proved similarly to that of Theorem 5.

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