EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 3, No. 1, 2010, 26-44
ISSN 1307-5543 – www.ejpam.com

Differential Subordination and Superordination of Analytic Functions Defined by an Integral Operator

M. K. Aouf[1*] and T. M. Seoudy[2]

[1] *Department of Mathematics, Faculty of Science, Mansoura 35516, Egypt*
[2] *Department of Mathematics, Faculty of Science, Fayoum 63514, Egypt*

Abstract. Differential subordination and superordination results are obtained for analytic functions in the open unit disk which are associated with the integral operator. These results are obtained by investigating appropriate classes of admissible functions. Sandwich-type results are also obtained. Some of the results established in this paper would provide extensions of those given in earlier works.

2000 Mathematics Subject Classifications: 30C45
Key Words and Phrases: Analytic function, integral operator, Hadamard product, differential subordination, superordination.

---

1. Introduction

Let $H(U)$ be the class of functions analytic in $U = \{z : z \in C \ and \ |z| < 1\}$ and $H[a,n]$ be the subclass of $H(U)$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots$, with $H_0 = H[0,1]$ and $H = H[1,1]$. Let $A(p)$ denote the class of all analytic functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad \left(p \in \mathbb{N} = \{1,2,3,\ldots\}; z \in U\right) \tag{1}$$

and let $A(1) = A$. Let f and F be members of $H(U)$. The function $f(z)$ is said to be subordinate to $F(z)$, or $F(z)$ is said to be superordinate to $f(z)$, if there exists a function $\omega(z)$ analytic in U with $\omega(0) = 0$ and $|\omega(z)| < 1 (z \in U)$, such that $f(z) = F(\omega(z))$. In such a case we write $f(z) \prec F(z)$. If F is univalent, then $f(z) \prec F(z)$ if and only if $f(0) = F(0)$ and $f(U) \subset F(U)$ (see [8] and [9].

For two functions $f(z)$ given by (1) and

$$g(z) = z^p + \sum_{n=1}^{\infty} b_{p+n} z^{p+n},$$

---

*Corresponding author.

Email addresses: `mkaouf127@yahoo.com` (M. Aouf), `tmseoudy@gmail.com` (T. Seoudy)

the Hadamard product (or convolution) of f and g is defined by

$$\left(f*g\right)(z)=z^{p}+\sum_{n=1}^{\infty}a_{p+n}b_{p+n}\,z^{p+n}=\left(g*f\right)(z).$$

Motivated essentially by Jung et al. Liu [5] and Owa [7] introduced the integral operator $Q_{\beta,p}^{\alpha}:A\left(p\right)\rightarrow A\left(p\right)$ as follows:

$$Q_{\beta,p}^{\alpha}f(z)=\binom{p+\alpha+\beta-1}{p+\beta-1}\frac{\alpha}{z^{\beta}}\int_{0}^{z}\left(1-\frac{t}{z}\right)^{\alpha-1}t^{\beta-1}f\left(t\right)dt,\quad\left(\alpha>0;\beta>-1;p\in\mathbb{N}\right),\quad(2)$$

and

$$Q_{\beta,p}^{0}f(z)=f(z),\qquad\left(\alpha=0;\beta>-1\right).$$

For $f\in A\left(p\right)$ given by (1), then from (2), we deduce that

$$Q_{\beta,p}^{\alpha}f(z)=z^{p}+\frac{\Gamma\left(\alpha+\beta+p\right)}{\Gamma\left(\beta+p\right)}\sum_{n=1}^{\infty}\frac{\Gamma\left(\beta+p+n\right)}{\Gamma\left(\alpha+\beta+p+n\right)}a_{p+n}z^{p+n}\quad\left(\alpha\geq0;\beta>-1;p\in\mathbb{N}\right).\quad(3)$$

It is easily verified from the definition (3) that (see [7])

$$z\left(Q_{\beta,p}^{\alpha}f(z)\right)'=\left(\alpha+\beta+p-1\right)Q_{\beta,p}^{\alpha-1}f(z)-\left(\alpha+\beta-1\right)Q_{\beta,p}^{\alpha}f(z).\qquad(4)$$

We note that the one-parameter family of integral operator $Q_{\beta,1}^{\alpha}f(z)=Q_{\beta}^{\alpha}$ was defined by Jung et al. [5].

To prove our results, we need the following definitions and Lemmas.

Denote by $\mathscr{F}$ the set of all functions $q(z)$ that are analytic and injective on $\bar{U}\backslash E(q)$ where

$$E(q)=\left\{\zeta\in\partial U:\lim_{z\to\zeta}q(z)=\infty\right\},$$

and are such that $q'(\zeta)\neq0$ for $\zeta\in\partial U\backslash E(q)$. Further let the subclass of $\mathscr{F}$ for which $q(0)=a$ be denoted by $\mathscr{F}(a)$, $\mathscr{F}(0)\equiv\mathscr{F}_0$ and $\mathscr{F}(1)\equiv\mathscr{F}_1$.

Definition 1 ([8], Definition 2.3a, p. 27)**.** *Let Ω be a set in $\mathbb{C}$, $q\in\mathscr{F}$ and n be a positive integer. The class of admissible functions $\Psi_n[\Omega,q]$, consists of those functions $\psi:\mathbb{C}^3\times U\to\mathbb{C}$ that satisfy the admissibility condition:*

$$\psi(r,s,t;z)\notin\Omega$$

whenever

$$r=q(\zeta),\ s=k\zeta q'(\zeta),\ \Re\left\{\frac{t}{s}+1\right\}\geq k\Re\left\{1+\frac{\zeta q''(\zeta)}{q'(\zeta)}\right\},$$

where $z\in U$, $\zeta\in\partial U\backslash E(q)$ and $k\geq n$. We write $\Psi_1[\Omega,q]$ as $\Psi[\Omega,q]$.

In particular when $q(z) = M\frac{Mz+a}{M+\bar{a}z}$, with $M > 0$ and $|a| < M$, then $q(U) = U_M = \{w : |w| < M\}$, $q(0) = a$, $E(q) = \varnothing$ and $q \in \mathscr{F}$. In this case, we set $\Psi_n[\Omega, M, a] = \Psi_n[\Omega, q]$, and in the special case when the set $\Omega = U_M$, the class is simply denoted by $\Psi_n[M, a]$.

Definition 2 ([9], Definition 3, p. 817). *Let Ω be a set in $\mathbb{C}$, $q(z) \in H[a, n]$ with $q'(z) \neq 0$. The class of admissible functions $\Psi'_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^3 \times \bar{U} \to \mathbb{C}$ that satisfy the admissibility condition*

$$\psi(r, s, t; \zeta) \in \Omega$$

whenever

$$r = q(z),\ s = \frac{zq'(z)}{m},\ \Re\left\{\frac{t}{s} + 1\right\} \geq \frac{1}{m}\Re\left\{1 + \frac{zq''(z)}{q'(z)}\right\},$$

where $z \in U, \zeta \in \partial U$ and $m \geq n \geq 1$. In particular, we write $\Psi'_1[\Omega, q]$ as $\Psi'[\Omega, q]$.

Lemma 1 ([8],Theorem 2.3b, p. 28). *Let $\psi \in \Psi_n\left[\Omega, q\right]$ with $q(0) = a$. If the analytic function $g(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots$ satisfies*

$$\psi(g(z), zg'(z), z^2 g''(z); z) \in \Omega,$$

then $g(z) \prec q(z)$.

Lemma 2 ([9],Theorem 1, p. 818). . *Let $\psi \in \Psi'_n[\Omega, q]$ with $q(0) = a$. If $g(z) \in \mathscr{F}(a)$ and*

$$\psi(g(z), zg'(z), z^2 g''(z); z)$$

is univalent in U then

$$\Omega \subset \{\psi(g(z), zg'(z), z^2 g''(z); z) : z \in U\},$$

implies $q(z) \prec g(z)$.

In the present investigation, the differential subordination result of Miller and Mocanu [8,Theorem 2.3b, p.28] is extended for functions associated with the integral operator $Q^{\alpha}_{\beta,p}$, and we obtain certain other related results. A similar problem for analytic functions was studied by Aghalary et al. [1], Ali et al. [2], Aouf [3], Aouf et al. [4], and Kim and Srivastava [6]. Additionally, the corresponding differential superordination problem is investigated, and several sandwich-type results are obtained.

2. Subordination Results Involving the Integral Operator

Definition 3. *Let Ω be a set in $\mathbb{C}$ and $q(z) \in \mathscr{F}_0 \cap H[0, p]$. The class of admissible functions $\Phi_Q\left[\Omega, q\right]$ consists of those functions $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the admissibility condition*

$$\phi\,(u, v, w; z) \notin \Omega$$

whenever

$$u=q(\zeta),\ v=\frac{k\zeta q'(\zeta)+(\alpha+\beta-1)q(\zeta)}{\alpha+\beta+p-1},$$

$$\begin{aligned}&\Re\left\{\frac{(\alpha+\beta+p-1)(\alpha+\beta+p-2)w-(\alpha+\beta-1)(\alpha+\beta-2)u}{(\alpha+\beta+p-1)v-(\alpha+\beta-1)u}-2(\alpha+\beta)+3\right\}\\ \geq\ & k\Re\left\{1+\frac{\zeta q''(\zeta)}{q'(\zeta)}\right\},\end{aligned}$$

where $z\in U,\ \zeta\in\partial U\backslash E(q)$, *and* $k\geq p$.

Theorem 1. *Let* $\phi\in\Phi_Q[\Omega,q]$. *If* $f(z)\in A(p)$ *satisfies*

$$\left\{\phi\left(Q^{\alpha}_{\beta,p}f(z),Q^{\alpha-1}_{\beta,p}f(z),Q^{\alpha-2}_{\beta,p}f(z);z\right):z\in U\right\}\subset\Omega\quad(\alpha>2;\ \beta>-1;\ p\in\mathbb{N}),\tag{5}$$

then

$$Q^{\alpha}_{\beta,p}f(z)\prec q(z)\quad(z\in U).$$

Proof. Define the analytic function $g(z)$ in U by

$$g(z)=Q^{\alpha}_{\beta,p}f(z)\quad(\alpha>2;\ \beta>-1;\ p\in\mathbb{N};\ z\in U).\tag{6}$$

In view of the relation (4) from (6), we get

$$Q^{\alpha-1}_{\beta,p}f(z)=\frac{zg'(z)+(\alpha+\beta-1)g(z)}{\alpha+\beta+p-1}.\tag{7}$$

Further computations show that

$$Q^{\alpha-2}_{\beta,p}f(z)=\frac{z^2g''(z)+2(\alpha+\beta-1)zg'(z)+(\alpha+\beta-1)(\alpha+\beta-2)g(z)}{(\alpha+\beta+p-1)(\alpha+\beta+p-2)}.\tag{8}$$

Define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u=r,\ v=\frac{s+(\alpha+\beta-1)r}{\alpha+\beta+p-1},\ w=\frac{t+2(\alpha+\beta-1)s+(\alpha+\beta-1)(\alpha+\beta-2)r}{(\alpha+\beta+p-1)(\alpha+\beta+p-2)}.\tag{9}$$

Let

$$\begin{aligned}\psi(r,s,t;z)&=\phi(u,v,w;z)\\&=\phi\left(r,\frac{s+(\alpha+\beta-1)r}{\alpha+\beta+p-1},\frac{t+2(\alpha+\beta-1)s+(\alpha+\beta-1)(\alpha+\beta-2)r}{(\alpha+\beta+p-1)(\alpha+\beta+p-2)};z\right).\end{aligned}\tag{10}$$

The proof shall make use of Lemma 1. Using equations (6), (7) and (8), from (10), we obtain

$$\psi(p(z), zp'(z), z^2 p''(z); z) = \phi\left(Q^{\alpha}_{\beta,p} f(z), Q^{\alpha-1}_{\beta,p} f(z), Q^{\alpha-2}_{\beta,p} f(z); z\right) \tag{11}$$

$$\left(\alpha > 2;\ \beta > -1;\ p \in \mathbb{N}; z \in U\right).$$

Hence (5) becomes

$$\psi(p(z), zp'(z), z^2 p''(z); z) \in \Omega.$$

The proof is completed if it can be shown that the admissibility condition for $\phi \in \Phi_Q[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 1. Note that

$$\frac{t}{s} + 1 = \frac{(\alpha+\beta+p-1)(\alpha+\beta+p-2)w - (\alpha+\beta-1)(\alpha+\beta-2)u}{(\alpha+\beta+p-1)v - (\alpha+\beta-1)u} - 2(\alpha+\beta) + 3,$$

and hence $\psi \in \Psi_p[\Omega, q]$. By Lemma 1,

$$g(z) \prec q(z) \quad \text{or} \quad Q^{\alpha}_{\beta,p} f(z) \prec q(z) \quad (z \in U).$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(U)$ for some conformal mapping $h(z)$ of U onto Ω. In this case the class $\Phi_Q[h(U), q]$ is written as $\Phi_Q[h, q]$. The following result is an immediate consequence of Theorem 1.

Theorem 2. *Let $\phi \in \Phi_Q[h, q]$. If $f(z) \in A(p)$ satisfies*

$$\phi\left(Q^{\alpha}_{\beta,p} f(z), Q^{\alpha-1}_{\beta,p} f(z), Q^{\alpha-2}_{\beta,p} f(z); z\right) \prec h(z) \ \left(\alpha > 2; \beta > -1;\ p \in \mathbb{N};\ z \in U\right), \tag{12}$$

then

$$Q^{\alpha}_{\beta,p} f(z) \prec q(z) \quad (z \in U).$$

Our next result is an extension of Theorem 1 to the case where the behavior of $q(z)$ on ∂U is not known.

Corollary 1. *Let $\Omega \subset \mathbb{C}$ and let $q(z)$ be univalent in U, $q(0) = 0$. Let $\phi \in \Phi_Q[\Omega, q_\rho]$ for some $\rho \in (0,1)$ where $q_\rho(z) = q(\rho z)$. If $f(z) \in A(p)$ and*

$$\phi\left(Q^{\alpha}_{\beta,p} f(z), Q^{\alpha-1}_{\beta,p} f(z), Q^{\alpha-2}_{\beta,p} f(z); z\right) \in \Omega \quad \left(\alpha > 2;\ \beta > -1;\ p \in \mathbb{N};\ z \in U\right),$$

then

$$Q^{\alpha}_{\beta,p} f(z) \prec q(z) \quad (z \in U).$$

Proof. Theorem 1 yields $Q^{\alpha}_{\beta,p} f(z) \prec q_\rho(z)$. The result is now deduced from $q_\rho(z) \prec q(z)$.

Theorem 3. *Let $h(z)$ and $q(z)$ be univalent in U with $q(0) = 0$ and set $q_\rho(z) = q(\rho z)$ and $h_\rho(z) = h(\rho z)$. Let$\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ satisfy one of the following conditions:*

(1) $\phi \in \Phi_Q[h, q_\rho]$, for some $\rho \in (0,1)$, or

(2) there exists $\rho_0 \in (0,1)$ such that $\phi \in \Phi_Q[h_\rho, q_\rho]$, for all $\rho \in (\rho_0, 1)$.

If $f(z) \in A(p)$ satisfies (12), then

$$Q^{\alpha}_{\beta,p} f(z) \prec q(z) \quad (z \in U).$$

Proof. The proof is similar to the proof of [8, Theorem 2.3d, p.30] and is therefore omitted.

The next theorem yields the best dominant of the differential subordination (12).

Theorem 4. *Let $h(z)$ be univalent in U. Let $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$. Suppose that the differential equation*

$$\phi(q(z), zq'(z), z^2 q''(z); z) = h(z) \tag{13}$$

has a solution $q(z)$ with $q(0) = 0$ and satisfy one of the following conditions:

(1) $q(z) \in \mathscr{F}_0$ and $\phi \in \Phi_Q[h,q]$,

(2) $q(z)$ is univalent in U and $\phi \in \Phi_Q[h, q_\rho]$, for some $\rho \in (0,1)$, or

(3) $q(z)$ is univalent in U and there exists $\rho_0 \in (0,1)$ such that $\phi \in \Phi_Q[h_\rho, q_\rho]$, for all $\rho \in (\rho_0, 1)$.

If $f(z) \in A(p)$ satisfies (12), then

$$Q^{\alpha}_{\beta,p} f(z) \prec q(z) \quad (z \in U),$$

and $q(z)$ is the best dominant.

Proof. Following the same arguments in [8, Theorem 2.3e, p. 31], we deduce that $q(z)$ is a dominant from Theorems 2 and 3. Since $q(z)$ satisfies (13) it is also a solution of (12) and therefore $q(z)$ will be dominated by all dominants. Hence $q(z)$ is the best dominant.

In the particular case $q(z) = Mz$, $M > 0$, and in view of the Definition 1, the class of admissible functions $\Phi_Q[\Omega, q]$, denoted by $\Phi_Q[\Omega, M]$, is described below.

Definition 4. *Let Ω be a set in $\mathbb{C}$ and $M > 0$. The class of admissible functions $\Phi_Q[\Omega, M]$ consists of those functions $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ such that*

$$\phi\left(Me^{i\theta}, \frac{k+\alpha+\beta-1}{\alpha+\beta+p-1} Me^{i\theta}, \frac{L + \left[2(\alpha+\beta-1)k + (\alpha+\beta-1)(\alpha+\beta-2)\right] Me^{i\theta}}{(\alpha+\beta+p-1)(\alpha+\beta+p-2)}; z\right) \notin \Omega \tag{14}$$

whenever $z \in U$, $\theta \in \mathbb{R}$, $\Re\left(Le^{-i\theta}\right) \geq (k-1)kM$ for all real θ, $\alpha > 2$, $\beta > -1, p \in \mathbb{N}$ and $k \geq p$.

Corollary 2. *Let $\phi \in \Phi_Q[\Omega, M]$. If $f(z) \in A(p)$ satisfies*

$$\phi\left(Q^{\alpha}_{\beta,p} f(z), Q^{\alpha-1}_{\beta,p} f(z), Q^{\alpha-2}_{\beta,p} f(z); z\right) \in \Omega \quad \left(\alpha > 2;\ \beta > -1;\ p \in \mathbb{N};\ z \in U\right),$$

then

$$\left|Q^{\alpha}_{\beta,p} f(z)\right| < M \quad (z \in U).$$

In the special case $\Omega = q(U) = \{\omega : |\omega| < M\}$, the class $\Phi_Q[\Omega, M]$ is simply denoted by $\Phi_Q[M]$.

Corollary 3. *Let $\phi \in \Phi_Q[M]$. If $f(z) \in A(p)$ satisfies*

$$\left|\phi\left(Q_{\beta,p}^{\alpha} f(z), Q_{\beta,p}^{\alpha-1} f(z), Q_{\beta,p}^{\alpha-2} f(z); z\right)\right| < M \qquad (\alpha > 2; \beta > -1;\ p \in \mathbb{N}; z \in U),$$

then

$$\left|Q_{\beta,p}^{\alpha} f(z)\right| < M \quad (z \in U).$$

Remark 1. *Putting $M = 1$ in the Corollary 3 we obtain the result obtained by Aouf [3, Theorem 2].*

Corollary 4. *If $k \geq p$ and $f(z) \in A(p)$ satisfies*

$$\left|Q_{\beta,p}^{\alpha-1} f(z)\right| < M \quad (\alpha > 1; \beta > -1; p \in \mathbb{N}; z \in U).$$

then

$$\left|Q_{\beta,p}^{\alpha} f(z)\right| < M \quad (z \in U).$$

Proof. This follows from Corollary 3 by taking $\phi(u, v, w; z) = v = \frac{k+\alpha+\beta-1}{\alpha+\beta+p-1} M e^{i\theta}$.

Remark 2. *For $M = 1$, Corollary 4 yields the result obtained by Aouf [3, Corollary 2].*

Definition 5. *Let Ω be a set in $\mathbb{C}$ and $q(z) \in \mathscr{F}_0 \cap H_0$. The class of admissible functions $\Phi_{Q,1}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the admissibility condition:*

$$\phi(u, v, w; z) \notin \Omega$$

whenever

$$u = q(\zeta),\ v = \frac{k\zeta q'(\zeta) + (\alpha+\beta+p-2) q(\zeta)}{\alpha+\beta+p-1},$$

$$\begin{aligned} &\Re\left\{\frac{(\alpha+\beta+p-2)\left[(\alpha+\beta+p-1) w - (\alpha+\beta+p-3) u\right]}{(\alpha+\beta+p-1) v - (\alpha+\beta+p-2) u} - 2(\alpha+\beta) + 5\right\} \\ \geq\ & k\Re\left\{1 + \frac{\zeta q''(\zeta)}{q'(\zeta)}\right\}, \end{aligned}$$

where $z \in U, \zeta \in \partial U \backslash E(q), \alpha > 2, \beta > -1; p \in \mathbb{N}$ and $k \geq 1$.

Theorem 5. *Let $\phi \in \Phi_{Q,1}[\Omega, q]$. If $f(z) \in A(p)$ satisfies*

$$\left\{\phi\left(\frac{Q_{\beta,p}^{\alpha} f(z)}{z^{p-1}}, \frac{Q_{\beta,p}^{\alpha-1} f(z)}{z^{p-1}}, \frac{Q_{\beta,p}^{\alpha-2} f(z)}{z^{p-1}}; z\right) : z \in U\right\} \subset \Omega \quad (\alpha > 2; \beta > -1), \tag{15}$$

then

$$\frac{Q_{\beta,p}^{\alpha} f(z)}{z^{p-1}} \prec q(z) \quad (z \in U).$$

Proof. Define an analytic function $g(z)$ in U by

$$g(z)=\frac{Q_{\beta,p}^{\alpha}f(z)}{z^{p-1}}\quad\left(\alpha>2;\ \beta>-1;p\in\mathbb{N};\ z\in U\right). \tag{16}$$

By making use of (4) and (16), we get

$$\frac{Q_{\beta,p}^{\alpha-1}f(z)}{z^{p-1}}=\frac{zg'(z)+\left(\alpha+\beta+p-2\right)\ g(z)}{\alpha+\beta+p-1}. \tag{17}$$

Further computations show that

$$\frac{Q_{\beta,p}^{\alpha-2}f(z)}{z^{p-1}}=\frac{z^2g''(z)+2\left(\alpha+\beta+p-2\right)zg'(z)+\left(\alpha+\beta+p-2\right)\left(\alpha+\beta+p-3\right)g(z)}{\left(\alpha+\beta+p-1\right)\left(\alpha+\beta+p-2\right)}. \tag{18}$$

Define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$\begin{aligned} u &= r,\ v=\frac{s+\left(\alpha+\beta+p-2\right)r}{\alpha+\beta+p-1},\\ w &= \frac{t+2\left(\alpha+\beta+p-2\right)s+\left(\alpha+\beta+p-2\right)\left(\alpha+\beta+p-3\right)r}{\left(\alpha+\beta+p-1\right)\left(\alpha+\beta+p-2\right)}. \end{aligned} \tag{19}$$

Let

$$\begin{aligned} \psi(r,s,t;z) &= \phi(u,v,w;z)\\ &= \phi\Big(r,\frac{s+\left(\alpha+\beta+p-2\right)r}{\alpha+\beta+p-1},\\ &\qquad \frac{t+2\left(\alpha+\beta+p-2\right)s+\left(\alpha+\beta+p-2\right)\left(\alpha+\beta+p-3\right)r}{\left(\alpha+\beta+p-1\right)\left(\alpha+\beta+p-2\right)};z\Big). \end{aligned} \tag{20}$$

The proof shall make use of Lemma 1. Using equations (16)-(18), and from (20), we obtain

$$\psi\left(g(z),zg'(z),z^2g''(z);z\right)=\phi\left(\frac{Q_{\beta,p}^{\alpha}f(z)}{z^{p-1}},\frac{Q_{\beta,p}^{\alpha}f(z)}{z^{p-1}},\frac{Q_{\beta,p}^{\alpha}f(z)}{z^{p-1}};z\right). \tag{21}$$

Hence (15) becomes

$$\psi\left(g(z),zg'(z),z^2g''(z);z\right)\in\Omega.$$

The proof is completed if it can be shown that the admissibility condition for $\phi\in\Phi_{Q,1}\left[\Omega,q\right]$ is equivalent to the admissibility condition for ψ as given in Definition 1. Note that

$$\frac{t}{s}+1=\frac{\left(\alpha+\beta+p-2\right)\left[\left(\alpha+\beta+p-1\right)w-\left(\alpha+\beta+p-3\right)u\right]}{\left(\alpha+\beta+p-1\right)v-\left(\alpha+\beta+p-2\right)u}-2\left(\alpha+\beta\right)+5,$$

and hence $\psi\in\Psi\left[\Omega,q\right]$. By Lemma 1,

$$g(z)\prec q(z)\quad or\quad \frac{Q_{\beta,p}^{\alpha}f(z)}{z^{p-1}}\prec q(z)\quad(z\in U).$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(U)$, for some conformal mapping $h(z)$ of U onto Ω. In this case the class $\Phi_{Q,1}\left[h(U), q\right]$ is written as $\Phi_{Q,1}\left[h, q\right]$. In the particular case $q(z) = Mz$, $M > 0$, the class of admissible functions $\Phi_{Q,1}\left[\Omega, q\right]$, denoted by $\Phi_{Q,1}\left[\Omega, M\right]$.

Proceeding similarly as in the previous section, the following result is an immediate consequence of Theorem 5.

Theorem 6. *Let $\phi \in \Phi_{Q,1}\left[h, q\right]$. If $f(z) \in A(p)$ satisfies*

$$\phi\left(\frac{Q_{\beta,p}^{\alpha} f(z)}{z^{p-1}}, \frac{Q_{\beta,p}^{\alpha-1} f(z)}{z^{p-1}}, \frac{Q_{\beta,p}^{\alpha-2} f(z)}{z^{p-1}}; z\right) \prec h(z) \quad \left(\alpha > 2; \beta > -1; p \in \mathbb{N};\ z \in U\right), \tag{22}$$

then

$$\frac{Q_{\beta,p}^{\alpha} f(z)}{z^{p-1}} \prec q(z) \quad (z \in U).$$

Definition 6. *Let Ω be a set in $\mathbb{C}$ and $M > 0$. The class of admissible functions $\Phi_{Q,1}\left[\Omega, M\right]$ consists of those functions $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ such that*

$$\phi\left(Me^{i\theta}, \frac{k+\alpha+\beta+p-2}{\alpha+\beta+p-1} Me^{i\theta}, \frac{L + (\alpha+\beta+p-2)(2k+\alpha+\beta+p-3) Me^{i\theta}}{(\alpha+\beta+p-1)(\alpha+\beta+p-2)}; z\right) \notin \Omega \tag{23}$$

whenever $z \in U$, $\theta \in \mathbb{R}$, $\Re\left(Le^{-i\theta}\right) \geq (k-1)kM$ for all real $\theta, p \in \mathbb{N}$ and $k \geq 1$.

Corollary 5. *Let $\phi \in \Phi_{Q,1}\left[\Omega, M\right]$. If $f(z) \in A(p)$ satisfies*

$$\phi\left(\frac{Q_{\beta,p}^{\alpha} f(z)}{z^{p-1}}, \frac{Q_{\beta,p}^{\alpha-1} f(z)}{z^{p-1}}, \frac{Q_{\beta,p}^{\alpha-2} f(z)}{z^{p-1}}; z\right) \in \Omega \quad \left(\alpha > 2; \beta > -1;\ p \in \mathbb{N};\ z \in U\right),$$

then

$$\left|\frac{Q_{\beta,p}^{\alpha} f(z)}{z^{p-1}}\right| < M \quad (z \in U).$$

In the special case $\Omega = \{\omega : |\omega| < M\}$, the class $\Phi_{Q,1}\left[\Omega, M\right]$ is simply denoted by $\Phi_{Q,1}\left[M\right]$.

Corollary 6. *Let $\phi \in \Phi_{Q,1}\left[M\right]$. If $f(z) \in A(p)$ satisfies*

$$\left|\phi\left(\frac{Q_{\beta,p}^{\alpha} f(z)}{z^{p-1}}, \frac{Q_{\beta,p}^{\alpha-1} f(z)}{z^{p-1}}, \frac{Q_{\beta,p}^{\alpha-2} f(z)}{z^{p-1}}; z\right)\right| < M \quad \left(\alpha > 2;\ \beta > -1;\ p \in \mathbb{N};\ z \in U\right),$$

then

$$\left|\frac{Q_{\beta,p}^{\alpha} f(z)}{z^{p-1}}\right| < M \quad (z \in U).$$

Corollary 7. *If $k \geq 1$ and $f(z) \in A(p)$ satisfies*

$$\left|\frac{Q_{\beta,p}^{\alpha-1} f(z)}{z^{p-1}}\right| < M \quad \left(\alpha > 1;\ \beta > -1;\ p \in \mathbb{N};\ z \in U\right).$$

then

$$\left|\frac{Q_{\beta,p}^{\alpha} f(z)}{z^{p-1}}\right| < M \quad (z \in U).$$

Proof. This follows from Corollary 6 by taking $\phi(u,v,w;z) = v = \frac{k+\alpha+\beta+p-2}{\alpha+\beta+p-1} M e^{i\theta}$.

Definition 7. *Let Ω be a set in $\mathbb{C}$ and $q(z) \in \mathscr{F}_1 \cap H$. The class of admissible functions $\Phi_{Q,2}\left[\Omega, q\right]$ consists of those functions $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the admissibility condition*

$$\phi\left(u,v,w;z\right) \notin \Omega$$

whenever

$$u = q(\zeta),\ v = \frac{1}{\alpha+\beta+p-2}\left\{-1 + \left(\alpha+\beta+p-1\right) q(\zeta) + \frac{k\zeta q'(\zeta)}{q(\zeta)}\right\},$$

$$\begin{aligned} &\Re\left\{\frac{\left[\left(\alpha+\beta+p-3\right) w - \left(\alpha+\beta+p-2\right) v + 1\right] v}{\left(\alpha+\beta+p-2\right) v - \left(\alpha+\beta+p-1\right) u + 1} + \right. \\ &\left. \left(\alpha+\beta+p-2\right) v - 2\left(\alpha+\beta+p-1\right) u + 1\right\} \\ \geq\ & k\Re\left\{1 + \frac{\zeta q''(\zeta)}{q'(\zeta)}\right\}, \end{aligned}$$

where $z \in U, \zeta \in \partial U \backslash E\left(q\right), p \in \mathbb{N}$ and $k \geq 1$.

Theorem 7. *Let $\phi \in \Phi_{Q,2}\left[\Omega, q\right]$ and $Q_{\beta,p}^{\alpha} f(z) \neq 0$. If $f(z) \in A(p)$ satisfies*

$$\left\{\phi\left(\frac{Q_{\beta,p}^{\alpha-1}(z)}{Q_{\beta,p}^{\alpha} f(z)}, \frac{Q_{\beta,p}^{\alpha-2} f(z)}{Q_{\beta,p}^{\alpha-1} f(z)}, \frac{Q_{\beta,p}^{\alpha-3} f(z)}{Q_{\beta,p}^{\alpha-2} f(z)}; z\right) : z \in U\right\} \subset \Omega \quad \left(\alpha > 3; \beta > -1; p \in \mathbb{N}\right), \tag{24}$$

then

$$\frac{Q_{\beta,p}^{\alpha-1} f(z)}{Q_{\beta,p}^{\alpha} f(z)} \prec q(z) \quad (z \in U).$$

Proof. Define an analytic function $g(z)$ in U by

$$g(z) = \frac{Q_{\beta,p}^{\alpha-1} f(z)}{Q_{\beta,p}^{\alpha} f(z)} \quad \left(\alpha > 3; \beta > -1; p \in \mathbb{N}; z \in U\right). \tag{25}$$

Using (25), we get

$$\frac{zg'(z)}{g(z)}=\frac{z\left(Q_{\beta,p}^{\alpha-1}f(z)\right)'}{Q_{\beta,p}^{\alpha-1}f(z)}-\frac{z\left(Q_{\beta,p}^{\alpha}f(z)\right)'}{Q_{\beta,p}^{\alpha}f(z)}. \tag{26}$$

By making use of (4) in (26), we get

$$\frac{Q_{\beta,p}^{\alpha-2}f(z)}{Q_{\beta,p}^{\alpha-1}f(z)}=\frac{1}{\alpha+\beta+p-2}\left\{-1+(\alpha+\beta+p-1)\,g(z)+\frac{zg'(z)}{g(z)}\right\}. \tag{27}$$

Further computations show that

$$\begin{aligned}\frac{Q_{\beta,p}^{\alpha-3}f(z)}{Q_{\beta,p}^{\alpha-2}f(z)} &= \frac{1}{\alpha+\beta+p-2}\left\{-2+(\alpha+\beta+p-1)\,g(z)+\frac{zg'(z)}{g(z)}\right.\\ &\quad\left.+\frac{(\alpha+\beta+p-1)\,zg'(z)+\frac{zg'(z)}{g(z)}+\frac{z^2g''(z)}{g(z)}-\left(\frac{zg'(z)}{g(z)}\right)^2}{-1+(\alpha+\beta+p-1)\,g(z)+\frac{zg'(z)}{g(z)}}\right\}.\end{aligned} \tag{28}$$

Define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$\begin{aligned} u &= r,\\ v &= \frac{1}{\alpha+\beta+p-2}\left\{-1+(\alpha+\beta+p-1)\,r+\frac{s}{r}\right\},\\ w &= \frac{1}{\alpha+\beta+p-2}\left\{-2+(\alpha+\beta+p-1)\,r+\frac{s}{r}+\frac{(\alpha+\beta+p-1)\,s+\frac{s}{r}+\frac{l}{r}-\left(\frac{s}{r}\right)^2}{-1+(\alpha+\beta+p-1)\,r+\frac{s}{r}}\right\}.\end{aligned} \tag{29}$$

Let

$$\begin{aligned}\psi(r,s,t;z) &= \phi(u,v,w;z)\\ &= \phi\left(r,\frac{1}{\alpha+\beta+p-2}\left\{-1+(\alpha+\beta+p-1)\,r+\frac{s}{r}\right\},\frac{1}{\alpha+\beta+p-2}\right.\\ &\quad\left.\left\{-2+(\alpha+\beta+p-1)\,r+\frac{s}{r}+\frac{(\alpha+\beta+p-1)\,s+\frac{s}{r}+\frac{l}{r}-\left(\frac{s}{r}\right)^2}{-1+(\alpha+\beta+p-1)\,r+\frac{s}{r}}\right\};z\right).\end{aligned} \tag{30}$$

The proof shall make use of Lemma 1. Using equations (25), (27) and (28), from (30), we obtain

$$\psi(p(z),zp'(z),z^2p''(z);z)=\phi\left(\frac{Q_{\beta,p}^{\alpha-1}(z)}{Q_{\beta,p}^{\alpha}f(z)},\frac{Q_{\beta,p}^{\alpha-2}f(z)}{Q_{\beta,p}^{\alpha-1}f(z)},\frac{Q_{\beta,p}^{\alpha-3}f(z)}{Q_{\beta,p}^{\alpha-2}f(z)};z\right). \tag{31}$$

Hence (24) becomes

$$\psi(p(z), zp'(z), z^2 p''(z); z) \in \Omega.$$

The proof is completed if it can be shown that the admissibility condition for $\phi \in \Phi_{I,2}\left[\Omega, q\right]$ is equivalent to the admissibility condition for ψ as given in Definition 1. Note that

$$\begin{aligned} \frac{t}{s}+1 &= \frac{\left[(\alpha+\beta+p-3)\,w-(\alpha+\beta+p-2)\,v+1\right]v}{(\alpha+\beta+p-2)\,v-(\alpha+\beta+p-1)\,u+1} \\ &+ \quad (\alpha+\beta+p-2)\,v-2\,(\alpha+\beta+p-1)\,u+1, \end{aligned}$$

and hence $\psi \in \Psi\left[\Omega, q\right]$. By Lemma 1,

$$g(z) \prec q(z) \;\; or \quad \frac{Q_{\beta,p}^{\alpha-1}(z)}{Q_{\beta,p}^{\alpha} f(z)} \prec q(z) \quad (z \in U).$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(U)$, for some conformal mapping $h(z)$ of U onto Ω. In this case the class $\Phi_{Q,2}\left[h(U), q\right]$ is written as $\Phi_{Q,2}\left[h, q\right]$. In the particular case $q(z) = Mz$, $M > 0$, the class of admissible functions $\Phi_{Q,2}\left[\Omega, q\right]$ becomes the class $\Phi_{Q,2}\left[\Omega, M\right]$.

Proceeding similarly as in the previous section, the following result is an immediate consequence of Theorem 7.

Theorem 8. *Let $\phi \in \Phi_{Q,2}\left[h, q\right]$. If $f(z) \in A(p)$ satisfies*

$$\phi\left(\frac{Q_{\beta,p}^{\alpha-1}(z)}{Q_{\beta,p}^{\alpha} f(z)}, \frac{Q_{\beta,p}^{\alpha-2} f(z)}{Q_{\beta,p}^{\alpha-1} f(z)}, \frac{Q_{\beta,p}^{\alpha-3} f(z)}{Q_{\beta,p}^{\alpha-2} f(z)}; z\right) \prec h(z) \quad \left(\alpha > 3,\ \beta > -1; p \in \mathbb{N}; z \in U\right), \tag{32}$$

then

$$\frac{Q_{\beta,p}^{\alpha-1}(z)}{Q_{\beta,p}^{\alpha} f(z)} \prec q(z) \quad (z \in U).$$

Definition 8. *Let Ω be a set in $\mathbb{C}$ and $M > 0$. The class of admissible functions $\Phi_{Q,2}\left[\Omega, M\right]$ consists of those functions $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ such that*

$$\begin{aligned} &\phi\left(Me^{i\theta}, \frac{k-1+(\alpha+\beta+p-1)\,Me^{i\theta}}{\alpha+\beta+p-2}, \frac{1}{\alpha+\beta+p-2}\left\{k-2+(\alpha+\beta+p-1)\,Me^{i\theta}\right.\right. \\ &\left.\left.+\frac{(\alpha+\beta+p-1)\,kM^2e^{i\theta}+kM+Le^{-i\theta}-k^2M}{(k-1)\,M+(\alpha+\beta+p-1)\,M^2e^{i\theta}}\right\}; z\right) \notin \Omega, \end{aligned} \tag{33}$$

whenever $z \in U$, $\theta \in \mathbb{R}$, $\Re\left(Le^{-i\theta}\right) \geq (k-1)\,kM$ for all real θ, $\alpha > 3, \beta > -1, p \in \mathbb{N}$ and $k \geq 1$.

Corollary 8. *Let $\phi \in \Phi_{Q,2}[\Omega, M]$. If $f(z) \in A(p)$ satisfies*

$$\phi\left(\frac{Q_{\beta,p}^{\alpha-1}(z)}{Q_{\beta,p}^{\alpha} f(z)}, \frac{Q_{\beta,p}^{\alpha-2} f(z)}{Q_{\beta,p}^{\alpha-1} f(z)}, \frac{Q_{\beta,p}^{\alpha-3} f(z)}{Q_{\beta,p}^{\alpha-2} f(z)}; z\right) \in \Omega \quad (\alpha > 3; \beta > -1;\ p \in \mathbb{N}; z \in U),$$

then

$$\left|\frac{Q_{\beta,p}^{\alpha-1}(z)}{Q_{\beta,p}^{\alpha} f(z)}\right| < M \quad (z \in U).$$

In the special case $\Omega = q(U) = \{\omega : |\omega| < M\}$, the class $\Phi_{Q,2}[\Omega, M]$ is denoted by $\Phi_{Q,2}[M]$.

Corollary 9. *Let $\phi \in \Phi_{Q,2}[M]$. If $f(z) \in A(p)$ satisfies*

$$\left|\phi\left(\frac{Q_{\beta,p}^{\alpha-1}(z)}{Q_{\beta,p}^{\alpha} f(z)}, \frac{Q_{\beta,p}^{\alpha-2} f(z)}{Q_{\beta,p}^{\alpha-1} f(z)}, \frac{Q_{\beta,p}^{\alpha-3} f(z)}{Q_{\beta,p}^{\alpha-2} f(z)}; z\right)\right| < M \quad (\alpha > 3;\ \beta > -1; p \in \mathbb{N};\ z \in U),$$

then

$$\left|\frac{Q_{\beta,p}^{\alpha-1}(z)}{Q_{\beta,p}^{\alpha} f(z)}\right| < M \quad (z \in U).$$

Remark 3. *The result in the Corollary 9 is extension of the result obtained by Aouf [3, Theorem 4].*

3. Superordination of the Integral Operator

The dual problem of differential subordination, that is, differential superordination of the integral operator $Q_{\beta,p}^{\alpha}$ is investigated in this section. For this purpose the class ofvadmissible functions is given in the following definition.

Definition 9. *Let Ω be a set in $\mathbb{C}$ and $q(z) \in H[0, p]$ with $zq'(z) \neq 0$. The class of admissible functions $\Phi_Q'[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times \bar{U} \to \mathbb{C}$ that satisfy the admissibility condition:*

$$\phi(u, v, w; \zeta) \in \Omega$$

whenever

$$u = q(z),\ v = \frac{\zeta q'(z) + m(\alpha + \beta - 1) q(z)}{m(\alpha + \beta + p - 1)},$$

$$\begin{aligned} &\Re\left\{\frac{(\alpha+\beta+p-1)(\alpha+\beta+p-2) w - (\alpha+\beta-1)(\alpha+\beta-2) u}{(\alpha+\beta+p-1) v - (\alpha+\beta-1) u} - 2(\alpha+\beta) + 3\right\} \\ \leq\ & \frac{1}{m} \Re\left\{1 + \frac{zq''(z)}{q'(z)}\right\}, \end{aligned}$$

where $z \in U,\ \zeta \in \partial U,\ \alpha > 2, \beta > -1, p \in \mathbb{N}$ and $m \geq p$.

Theorem 9. *Let* $\phi \in \Phi'_Q[\Omega, q]$. *If* $f(z) \in A(p)$, $Q^{\alpha}_{\beta,p} f(z) \in \mathscr{F}_0$ *and*

$$\phi\left(Q^{\alpha}_{\beta,p} f(z), Q^{\alpha-1}_{\beta,p} f(z), Q^{\alpha-2}_{\beta,p} f(z); z\right)$$

is univalent in U, then

$$\Omega \subset \left\{\phi\left(Q^{\alpha}_{\beta,p} f(z), Q^{\alpha-1}_{\beta,p} f(z), Q^{\alpha-2}_{\beta,p} f(z); z\right) : z \in U\right\} \ \left(\alpha > 2; \beta > -1;\ p \in \mathbb{N}\right), \tag{34}$$

implies

$$q(z) \prec Q^{\alpha}_{\beta,p} f(z) \quad (z \in U).$$

Proof. From (11) and (34), we have

$$\Omega \subset \left\{\psi(g(z), zg'(z), z^2 g''(z); z) : z \in U\right\}.$$

From (9), we see that the admissibility condition for $\phi \in \Phi'_Q[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 2. Hence $\psi \in \Psi'_p[\Omega, q]$, and by Lemma 2,

$$q(z) \prec g(z) \quad or \quad q(z) \prec Q^{\alpha}_{\beta,p} f(z) \quad (z \in U).$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(U)$ for some conformal mapping $h(z)$ of U onto Ω. In this case the class $\Phi'_Q[h(U), q]$ is written as $\Phi'_Q[h, q]$.

Proceeding similarly as in the previous section, the following result is an immediate consequence of Theorem 9.

Theorem 10. *Let* $h(z)$ *is analytic on U and* $\phi \in \Phi'_Q[h, q]$. *If* $f(z) \in A(p)$, $Q^{\alpha}_{\beta,p} f(z) \in \mathscr{F}_0$ *and*

$$\phi\left(Q^{\alpha}_{\beta,p} f(z), Q^{\alpha-1}_{\beta,p} f(z), Q^{\alpha-2}_{\beta,p} f(z); z\right)$$

is univalent in U, then

$$h(z) \prec \phi\left(Q^{\alpha}_{\beta,p} f(z), Q^{\alpha-1}_{\beta,p} f(z), Q^{\alpha-2}_{\beta,p} f(z); z\right) \quad \left(\alpha > 2;\ \beta > -1;\ p \in \mathbb{N};\ z \in U\right), \tag{35}$$

implies

$$q(z) \prec Q^{\alpha}_{\beta,p} f(z) \quad (z \in U).$$

Theorems 9 and 10 can only be used to obtain subordinants of differential superordination of the form (34) or (35). The following theorem proves the existence of the best subordinant of (35) for certain ϕ.

Theorem 11. *Let* $h(z)$ *be analytic in U and* $\phi : \mathbb{C}^3 \times \bar{U} \to \mathbb{C}$. *Suppose that the differential equation*

$$\phi\left(q(z), zq'(z), z^2 q''(z); z\right) = h(z)$$

has a solution $q(z) \in \mathscr{F}_0$. *If* $\phi \in \Phi'_Q[h, q]$, $f(z) \in A(p)$, $Q^{\alpha}_{\beta,p} f(z) \in \mathscr{F}_0$ *and*

$$\phi\left(Q^{\alpha}_{\beta,p} f(z), Q^{\alpha-1}_{\beta,p} f(z), Q^{\alpha-2}_{\beta,p} f(z); z\right)$$

is univalent in U, then

$$h(z) \prec \phi\left(Q^{\alpha}_{\beta,p}f(z), Q^{\alpha-1}_{\beta,p}f(z), Q^{\alpha-2}_{\beta,p}f(z); z\right) \quad \left(\alpha > 2;\ \beta > -1;\ p \in \mathbb{N};\ z \in U\right)$$

implies

$$q(z) \prec Q^{\alpha}_{\beta,p}f(z) \quad (z \in U).$$

and $q(z)$ is the best subordinant.

Proof. The proof is similar to the proof of Theorem 4 and is therefore omitted.

Combining Theorems 2 and 10, we obtain the following sandwich-type theorem.

Corollary 10. *Let $h_1(z)$ and $q_1(z)$ be analytic functions in U, $h_2(z)$ be univalent function in U, $q_2(z) \in \mathscr{F}_0$ with $q_1(0) = q_2(0) = 0$ and $\phi \in \Phi_Q\left[h_2, q_2\right] \cap \Phi'_Q\left[h_1, q_1\right]$. If $f(z) \in A(p)$, $Q^{\alpha}_{\beta,p}f(z) \in H[0,p] \cap \mathscr{F}_0$ and*

$$\phi\left(Q^{\alpha}_{\beta,p}f(z), Q^{\alpha-1}_{\beta,p}f(z), Q^{\alpha-2}_{\beta,p}f(z); z\right) \quad \left(\alpha > 2;\ \beta > -1; p \in \mathbb{N};\ z \in U\right)$$

is univalent in U, then

$$h_1(z) \prec \phi\left(Q^{\alpha}_{\beta,p}f(z), Q^{\alpha-1}_{\beta,p}f(z), Q^{\alpha-2}_{\beta,p}f(z); z\right) \prec h_2(z) \qquad \left(\alpha > 2;\ p \in \mathbb{N};\ z \in U\right),$$

implies

$$q_1(z) \prec Q^{\alpha}_{\beta,p}f(z) \prec q_2(z) \qquad (z \in U).$$

Definition 10. *Let Ω be a set in $\mathbb{C}$ and $q(z) \in H_0$ with $zq'(z) \neq 0$. The class of admissible functions $\Phi'_{Q,1}\left[\Omega, q\right]$ consists of those functions $\phi \mathbb{C}^3 \times \bar{U} \to \mathbb{C}$ that satisfy the admissibility condition:*

$$\phi(u, v, w; \zeta) \in \Omega \tag{36}$$

whenever

$$u = q(z),\ v = \frac{zq'(z) + m\left(\alpha+\beta+p-2\right)q(z)}{m\left(\alpha+\beta+p-1\right)},$$

$$\begin{aligned} &\Re\left\{\frac{\left(\alpha+\beta+p-2\right)\left[\left(\alpha+\beta+p-1\right)w - \left(\alpha+\beta+p-3\right)u\right]}{\left(\alpha+\beta+p-1\right)v - \left(\alpha+\beta+p-2\right)u} - 2\left(\alpha+\beta\right) + 5\right\} \\ \leq\ & \frac{1}{m}\Re\left\{1 + \frac{zq''(z)}{q'(z)}\right\}, \end{aligned}$$

where $z \in U,\ \zeta \in \partial U$ and $m \geq 1$

Now we will give the dual result of Theorem 5 for differential superordination.

Theorem 12. *Let* $\phi \in \Phi'_{Q,1}\left[\Omega, q\right]$. *If* $f(z) \in A\left(p\right)$, $\frac{Q^{\alpha}_{\beta,p}f(z)}{z^{p-1}} \in \mathscr{F}_0$ *and*

$$\phi\left(\frac{Q^{\alpha}_{\beta,p}f(z)}{z^{p-1}}, \frac{Q^{\alpha-1}_{\beta,p}f(z)}{z^{p-1}}, \frac{Q^{\alpha-2}_{\beta,p}f(z)}{z^{p-1}}; z\right)$$

is univalent in U, then

$$\Omega \subset \left\{\phi\left(\frac{Q^{\alpha}_{\beta,p}f(z)}{z^{p-1}}, \frac{Q^{\alpha-1}_{\beta,p}f(z)}{z^{p-1}}, \frac{Q^{\alpha-2}_{\beta,p}f(z)}{z^{p-1}}; z\right) : z \in U\right\} \quad \left(\alpha > 2; \beta > -1;\ p \in \mathbb{N}\right) \tag{37}$$

implies

$$q(z) \prec \frac{Q^{\alpha}_{\beta,p}f(z)}{z^{p-1}} \qquad (z \in U).$$

Proof. From (21) and (37), we have

$$\Omega \subset \left\{\psi\left(g(z), zg'(z), z^2 g''(z); z\right) : z \in U\right\} \quad \left(\alpha > 2; \beta > -1; p \in \mathbb{N}\right).$$

From (19), we see that the admissibility condition for $\phi \in \Phi'_{Q,1}\left[\Omega, q\right]$ is equivalent to the admissibility condition for ψ as given in Definition 2. Hence $\psi \in \Psi'\left[\Omega, q\right]$, and by Lemma 2

$$q(z) \prec p(z) \quad or \quad q(z) \prec \frac{Q^{\alpha}_{\beta,p}f(z)}{z^{p-1}} \qquad \left(\alpha > 2; \beta > -1; p \in \mathbb{N}; z \in U\right).$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, and $\Omega = h(U)$ for some conformal mapping $h(z)$ of U onto Ω and the class $\Phi'_{Q,1}\left[h(U), q\right]$ is written as $\Phi'_{Q,1}\left[h, q\right]$.

Proceeding similarly as in the previous section, the following result is an immediate consequence of Theorem 12.

Theorem 13. *Let* $q(z) \in H_0$, $h(z)$ *is analytic on U and* $\phi \in \Phi'_{Q,1}\left[h, q\right]$. *If* $f(z) \in A\left(p\right)$, $\frac{Q^{\alpha}_{\beta,p}f(z)}{z^{p-1}} \in \mathscr{F}_0$ *and*

$$\phi\left(\frac{Q^{\alpha}_{\beta,p}f(z)}{z^{p-1}}, \frac{Q^{\alpha-1}_{\beta,p}f(z)}{z^{p-1}}, \frac{Q^{\alpha-2}_{\beta,p}f(z)}{z^{p-1}}; z\right)$$

is univalent in U, then

$$h(z) \prec \phi\left(\frac{Q^{\alpha}_{\beta,p}f(z)}{z^{p-1}}, \frac{Q^{\alpha-1}_{\beta,p}f(z)}{z^{p-1}}, \frac{Q^{\alpha-2}_{\beta,p}f(z)}{z^{p-1}}; z\right) \quad \left(\alpha > 2;\ \beta > -1; p \in \mathbb{N};\ z \in U\right) \tag{38}$$

implies

$$q(z) \prec \frac{Q^{\alpha}_{\beta,p}f(z)}{z^{p-1}} \qquad (z \in U).$$

Combining Theorems 6 and 13, we obtain the following sandwich-type theorem.

Corollary 11. *Let $h_1(z)$ and $q_1(z)$ be analytic functions in U, $h_2(z)$ be univalent function in U, $q_2(z) \in \mathscr{F}_0$ with $q_1(0) = q_2(0) = 0$ and $\phi \in \Phi_{Q,1}[h_2, q_2] \cap \Phi'_{Q,1}[h_1, q_1]$. If $f(z) \in A(p)$, $\frac{Q^{\alpha}_{\beta,p}f(z)}{z^{p-1}} \in H_0 \cap \mathscr{F}_0$ and*

$$\phi\left(\frac{Q^{\alpha}_{\beta,p}f(z)}{z^{p-1}}, \frac{Q^{\alpha-1}_{\beta,p}f(z)}{z^{p-1}}, \frac{Q^{\alpha-2}_{\beta,p}f(z)}{z^{p-1}}; z\right)$$

is univalent in U, then

$$h_1(z) \prec \phi\left(\frac{Q^{\alpha}_{\beta,p}f(z)}{z^{p-1}}, \frac{Q^{\alpha-1}_{\beta,p}f(z)}{z^{p-1}}, \frac{Q^{\alpha-2}_{\beta,p}f(z)}{z^{p-1}}; z\right) \prec h_2(z) \quad (\alpha > 2; \beta > -1;\ p \in \mathbb{N};\ z \in U),$$

implies

$$q_1(z) \prec \frac{Q^{\alpha}_{\beta,p}f(z)}{z^{p-1}} \prec q_2(z) \quad (z \in U).$$

Definition 11. *Let Ω be a set in $\mathbb{C}$, $q(z) \neq 0$, $zq'(z) \neq 0$ and $q(z) \in H$. The class of admissible functions $\phi \in \Phi'_{Q,2}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times \bar{U} \to \mathbb{C}$ that satisfy the admissibility condition:*

$$\phi(u, v, w; \zeta) \in \Omega$$

whenever

$$u = q(z),\ v = \frac{1}{\alpha+\beta+p+2}\left\{-1 + (\alpha+\beta+p-1)\, g(z) + \frac{zg'(z)}{mg(z)}\right\},$$

$$\begin{aligned} &\Re\left\{\frac{[(\alpha+\beta+p-3)\, w - (\alpha+\beta+p-2)\, v + 1]\, v}{(\alpha+\beta+p-2)\, v - (\alpha+\beta+p-1)\, u + 1}\right.\\ + &\left.\quad (\alpha+\beta+p-2)\, v - 2(\alpha+\beta+p-1)\, u + 1\right\}\\ \leq &\quad \frac{1}{m}\Re\left\{1 + \frac{zq''(z)}{q'(z)}\right\}, \end{aligned}$$

where $z \in U$, $\zeta \in \partial U, p \in \mathbb{N}$ and $m \geq 1$.

Now we will give the dual result of Theorem 7 for the differential superordination.

Theorem 14. *Let $\phi \in \Phi'_{I,2}[\Omega, q]$. If $f(z) \in A(p)$, $\frac{Q^{\alpha-1}_{\beta,p}f(z)}{Q^{\alpha}_{\beta,p}f(z)} \in \mathscr{F}_1$ and*

$$\phi\left(\frac{Q^{\alpha-1}_{\beta,p}f(z)}{Q^{\alpha}_{\beta,p}f(z)}, \frac{Q^{\alpha-2}_{\beta,p}f(z)}{Q^{\alpha-1}_{\beta,p}f(z)}, \frac{Q^{\alpha-3}_{\beta,p}f(z)}{Q^{\alpha-2}_{\beta,p}f(z)}; z\right)$$

is univalent in U, then

$$\Omega \subset \left\{ \phi \left(\frac{Q_{\beta,p}^{\alpha-1} f(z)}{Q_{\beta,p}^{\alpha} f(z)}, \frac{Q_{\beta,p}^{\alpha-2} f(z)}{Q_{\beta,p}^{\alpha-1} f(z)}, \frac{Q_{\beta,p}^{\alpha-3} f(z)}{Q_{\beta,p}^{\alpha-2} f(z)}; z \right) : z \in U \right\} \quad (\alpha > 3;\ \beta > -1; p \in \mathbb{N}) \tag{39}$$

implies

$$q(z) \prec \frac{Q_{\beta,p}^{\alpha-1} f(z)}{Q_{\beta,p}^{\alpha} f(z)} \quad (z \in U).$$

Proof. From (31) and (39), we have

$$\Omega \subset \left\{ \psi \left(g(z), zg'(z), z^2 g''(z); z \right) : z \in U \right\}.$$

In view of (29), the admissibility condition for $\phi \in \Phi'_{Q,2}[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 2. Hence $\psi \in \Psi'[\Omega, q]$, and by Lemma 2

$$q(z) \prec g(z) \quad or \quad q(z) \prec \frac{Q_{\beta,p}^{\alpha-1} f(z)}{Q_{\beta,p}^{\alpha} f(z)} \quad (z \in U).$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(U)$ for some conformal mapping $h(z)$ of U onto Ω. In this case the class $\Phi'_{Q,2}[h(U), q]$ is written as $\Phi'_{Q,2}[h, q]$.

Proceeding similarly as in the previous section, The following result is an immediate consequence of Theorem 14.

Theorem 15. *Let $q(z) \in H$, $h(z)$ be analytic in U and $\phi \in \Phi'_{Q,2}[h, q]$. If $f(z) \in A(p)$, $\frac{Q_{\beta,p}^{\alpha-1} f(z)}{Q_{\beta,p}^{\alpha} f(z)} \in \mathscr{F}_1$ and*

$$\phi \left(\frac{Q_{\beta,p}^{\alpha-1} f(z)}{Q_{\beta,p}^{\alpha} f(z)}, \frac{Q_{\beta,p}^{\alpha-2} f(z)}{Q_{\beta,p}^{\alpha-1} f(z)}, \frac{Q_{\beta,p}^{\alpha-3} f(z)}{Q_{\beta,p}^{\alpha-2} f(z)}; z \right)$$

is univalent in U, then

$$h(z) \prec \phi \left(\frac{Q_{\beta,p}^{\alpha-1} f(z)}{Q_{\beta,p}^{\alpha} f(z)}, \frac{Q_{\beta,p}^{\alpha-2} f(z)}{Q_{\beta,p}^{\alpha-1} f(z)}, \frac{Q_{\beta,p}^{\alpha-3} f(z)}{Q_{\beta,p}^{\alpha-2} f(z)}; z \right) \quad (\alpha > 3; \beta > -1; p \in \mathbb{N};\ z \in U), \tag{40}$$

implies

$$q(z) \prec \frac{Q_{\beta,p}^{\alpha-1} f(z)}{Q_{\beta,p}^{\alpha} f(z)} \quad (z \in U).$$

Combining Theorems 8 and 15, we obtain the following sandwich-type theorem.

Corollary 12. *Let $h_1(z)$ and $q_1(z)$ be analytic functions in U, $h_2(z)$ be univalent function in U, $q_2(z) \in \mathscr{F}_1$ with $q_1(0) = q_20) = 1$ and $\phi \in \Phi_{Q,2}\left[h_2, q_2\right] \cap \Phi'_{Q,2}\left[h_1, q_1\right]$. If $f(z) \in A(p)$, $\frac{Q^{\alpha-1}_{\beta,p} f(z)}{Q^{\alpha}_{\beta,p} f(z)} \in H \cap \mathscr{F}_1$ and*

$$\phi\left(\frac{Q^{\alpha-1}_{\beta,p} f(z)}{Q^{\alpha}_{\beta,p} f(z)}, \frac{Q^{\alpha-2}_{\beta,p} f(z)}{Q^{\alpha-1}_{\beta,p} f(z)}, \frac{Q^{\alpha-3}_{\beta,p} f(z)}{Q^{\alpha-2}_{\beta,p} f(z)}; z\right)$$

is univalent in U, then

$$h_1(z) \prec \phi\left(\frac{Q^{\alpha-1}_{\beta,p} f(z)}{Q^{\alpha}_{\beta,p} f(z)}, \frac{Q^{\alpha-2}_{\beta,p} f(z)}{Q^{\alpha-1}_{\beta,p} f(z)}, \frac{Q^{\alpha-3}_{\beta,p} f(z)}{Q^{\alpha-2}_{\beta,p} f(z)}; z\right) \prec h_2(z) \quad \left(\alpha > 3; \beta > -1;\ p \in \mathbb{N};\ z \in U\right),$$

implies

$$q_1(z) \prec \frac{Q^{\alpha-1}_{\beta,p} f(z)}{Q^{\alpha}_{\beta,p} f(z)} \prec q_2(z) \quad (z \in U).$$

References

[1] R. Aghalary, R. M. Ali, S. B. Joshi and V. Ravichandran, Inequalities for analytic functions defined by certain linear operator, Internat. J. Math. Sci, 4(2005), no.2, 267–274.

[2] R. M. Ali, V. Ravichandran and N. Seenivasagan, Differential subordination and superodination of analytic functions defined by the multiplier transformation, Math. Inequal. Appl.12(2009), no.1,123-139.

[3] M. K. Aouf, Inequalities involving certain integral operator, J. Math. Inequal. 2(2008), no.2, 537-547.

[4] M. K. Aouf, H.M. Hossen and A. Y. Lashin, An application of certain integral operators, J. Math. Anal. Appl. 248(2000), no. 2, 475–481.

[5] T. B. Jung, Y. C. Kim and H. M. Srivastava,The Hardy space of analytic functions associated with certain one-parameter families of integral operators, J. Math. Anal. Appl., 176(1993), 138-147.

[6] Y. C. Kim and H. M. Srivastava, Inequalities involving certain families of integral and convolution operators, Math. Inequal. Appl. 7(2004), no. 2, 227–234.

[7] J.-L. Liu and S. Owa, Properties of certain integral operators, Internat. J. Math. Math. Sci., 3(2004), no. 1, 69-75.

[8] S. S. Miller and P. T. Mocanu, Differential Subordinations: Theory and Applications, Series on Monographs and Textbooks in Pure and Applied Mathematics, Vol. 225, Marcel Dekker, New York and Basel, 2000.

[9] S. Miller and P. T. Mocanu, Subordinants of differential superordinations, Complex Variables Theory Appl. 48(2003), no.10, 815–826.