EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 16, No. 4, 2023, 2035-2048
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Outer-Convex Hop Domination in Graphs Under Some Binary Operations

Al-Amin Y. Isahac[1], Javier A. Hassan[1,*], Ladznar S. Laja[1], Hounam B. Copel[1]

[1] *Mathematics and Sciences Department, College of Arts and Sciences, MSU Tawi-Tawi College of Technology and Oceanography, Bongao, Tawi-Tawi, Philippines*

Abstract. Let G be a graph with vertex and edge sets $V(G)$ and $E(G)$, respectively. A set $C \subseteq V(G)$ is called an outer-convex hop dominating if for every two vertices $x, y \in V(G) \setminus C$, the vertex set of every $x-y$ geodesic is contained in $V(G) \setminus C$ and for every $a \in V(G) \setminus C$, there exists $b \in C$ such that $d_G(a, b) = 2$. The minimum cardinality of an outer-convex hop dominating set of G, denoted by $\tilde{\gamma}_{conh}(G)$, is called the outer-convex hop domination number of G. In this paper, we generate some formulas for the parameters of some special graphs and graphs under some binary operations by characterizing first the outer-convex hop dominating sets of each of these graphs. Moreover, we establish realization result that identifies and determines the connection of this parameter with the standard hop domination parameter. It shows that given any graph, this new parameter is always greater than or equal to the standard hop domination parameter.

2020 Mathematics Subject Classifications: 05C69

Key Words and Phrases: Outer-convex set, outer-convex hop dominating set, outer-convex hop domination number

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1. Introduction

A subset S of a vertex set of a simple graph G is called a *hop dominating* in G if $N_G^2[S] = V(G)$, that is, for every $v \in V(G) \setminus S$, there exists $u \in S$ such that $d_G(u, v) = 2$. The minimum cardinality among all hop dominating sets in G, denoted by $\gamma_h(G)$, is called the *hop domination number* of G. This concept was introduced and investigated by Natarajan et al. in [11]. They have studied this concept on some types of graphs and generated some interesting results. Some extreme values and properties of the said parameter can be found in [1, 2, 11]. Recently, Canoy et al. [9] investigated hop domination parameter on graphs under some binary operations.

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v16i4.4862

Email addresses: `al-aminisahac@msutawi-tawi.edu.ph` (A. Isahac)
`javierhassan@msutawi-tawi.edu.ph` (J. Hassan), `ladznarlaja@msutawi-tawi.edu.ph` (L. Laja)
`hounamcopel@msutawi-tawi.edu.ph` (H. Copel)

Because of its nice application in networks and different fields, researchers have introduced variants of hop domination and they studied these variants for some special graphs and graphs under some operations. (see [3–7, 10, 12]).

In this paper, we will introduce new variant of hop domination called outer-convex hop domination. This study is motivated by the introduction of convex hop domination in [5]. We will investigate this concept on some classes of graphs, and join and corona of two graphs. We believe that this new parameter and its results can lead to other interesting research directions in the future.

2. Terminology and Notation

Let $G = V(G), E(G))$ be an undirected graph. Given two vertices u and v of G, the distance $d_G(u, v)$ is the length of a shortest path joining u and v. Any u-v path of length $d_G(u, v)$ is called a *u-v geodesic*. The interval $I_G[u, v]$ consists of u, v, and all vertices lying on a u-v geodesic. The interval $I_G(u, v) = I_G[u, v] \setminus \{u, v\}$.

Let G be a graph. A set $C \subseteq V(G)$ is called a *convex* if for every two vertices $x, y \in C$, the vertex set of every $x - y$ geodesic is contained in C.

A set $C' \subseteq V(G)$ is a *clique* if the subgraph $\langle C' \rangle$ induced by C' is complete. The maximum cardinality among all clique sets of G, denoted by $\omega(G)$, is called the *clique number* of G.

Two vertices x, y of G are *adjacent*, or *neighbors*, if xy is an edge of G. The *open neighborhood* of x in G is the set $N_G(x) = \{y \in V(G) : xy \in E(G)\}$. The *closed neighborhood* of x in G is the set $N_G[x] = N_G(x) \cup \{x\}$. If $X \subseteq V(G)$, the *open neighborhood* of X in G is the set $N_G(X) = \bigcup_{x \in X} N_G(x)$. The *closed neighborhood* of X in G is the set $N_G[X] = N_G(X) \cup X$.

A set $P \subseteq V(G)$ is a *pointwise non-dominating set* if for every $v \in V(G) \backslash P$, there exists $u \in P$ such that $v \notin N_G(u)$. The minimum cardinality of a pointwise non-dominating set of G, denoted by $pnd(G)$, is called a *pointwise non-domination number* of G.

A *path graph* is a non-empty graph with vertex-set $\{x_1, x_2, ..., x_n\}$ and edge-set $\{x_1x_2, x_2x_3, ..., x_{n-1}x_n\}$, where the $x_i^{'}s$ are all distinct. The path of order n is denoted by P_n. If G is a graph and u and v are vertices of G, then a path from vertex u to vertex v is sometimes called a *u-v path*. The *cycle graph* $C_n = [x_1, x_2, \ldots, x_n, x_1]$ is the graph of order $n \geq 3$ with vertex-set $\{x_1, x_2, ..., x_n\}$ and edge-set $\{x_1x_2, x_2x_3, ..., x_{n-1}x_n, x_nx_1\}$.

Let G and H be any two graphs. The *join* $G + H$ is the graph with vertex set $V(G + H) = V(G) \cup V(H)$ and edge set

$$E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}.$$

The *corona* $G \circ H$ is the graph obtained by taking one copy of G and $|V(G)|$ copies of H, and then joining the *ith* vertex of G to every vertex of the *ith* copy of H. We denote by H^v the copy of H in $G \circ H$ corresponding to the vertex $v \in G$ and write $v + H^v$ for $\langle \{v\} \rangle + H^v$.

The *distance* $d_G(u,v)$ in G of two vertices u,v is the length of a shortest *u-v path* in G. The greatest distance between any two vertices in G, denoted by $diam(G)$, is called the *diameter* of G.

A vertex v in G is a *hop neighbor* of vertex u in G if $d_G(u,v) = 2$. The set $N_G^2(u) = \{v \in V(G) : d_G(v,u) = 2\}$ is called the *open hop neighborhood* of u. The *closed hop neighborhood* of u in G is given by $N_G^2[u] = N_G^2(u) \cup \{u\}$. The *open hop neighborhood* of $X \subseteq V(G)$ is the set $N_G^2(X) = \bigcup_{u \in X} N_G^2(u)$. The *closed hop neighborhood* of X in G is the set $N_G^2[X] = N_G^2(X) \cup X$.

A set $S \subseteq V(G)$ is a *hop dominating set* of G if $N_G^2[S] = V(G)$, that is, for every $v \in V(G)\backslash S$, there exists $u \in S$ such that $d_G(u,v) = 2$. The minimum cardinality among all hop dominating sets of G, denoted by $\gamma_h(G)$, is called the *hop domination number* of G. Any hop dominating set with cardinality equal to $\gamma_h(G)$ is called a γ_h-set of G.

3. Results

We begin this section by defining the new concept called outer-convex hop domination in a graph.

Definition 1. Let G be a simple graph with vertex and edge-sets $V(G)$ and $E(G)$, respectively. Then $C \subseteq V(G)$ is called an outer-convex hop dominating set if C is hop dominating and $V(G) \setminus C$ is convex in G. The minimum cardinality among all outer-convex hop dominating sets of G, denoted by $\tilde{\gamma}_{conh}(G)$, is called the *outer-convex hop domination number* of G. Any outer-convex hop dominating set C satisfying $|C| = \tilde{\gamma}_{conh}(G)$, is called a $\tilde{\gamma}_{conh}$-set of G.

Example 1. Consider the graph G given in Figure 1. Let $C = \{a_5, a_6, \ldots, a_{13}\}$. Then $N_G^2[C] = V(G)$ and so C is a hop dominating set of G. Observe that $V(G) \setminus C = \{a_1, a_2, a_3, a_4\}$ is convex in G. Hence, C is an outer-convex hop dominating set of G. Next, consider $C' = \{a_5, a_6, a_7, a_{11}, a_{12}\}$. Then C' is a hop dominating set in G. However, C' is not an outer-convex hop dominating set in G since $V(G) \setminus C'$ is not convex in G. Moreover, it can be verified that $\tilde{\gamma}_{conh}(G) = 9$.

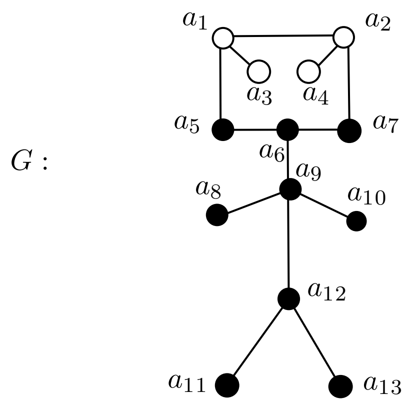

Figure 1: Graph G with $\tilde{\gamma}_{conh}(G)$=9

Remark 1. *(i) Any graph G admits an outer-convex hop domination.*

(ii) If S is an outer-convex hop dominating set of G, then S and $V(G) \setminus C$ are not necessarily convex and hop dominating sets in G, respectively.

Remark 2. *Let G be any graph. Then every outer-convex hop dominating set C in G is hop dominating but the converse is not always true.*

The converse part can be seen by considering C' in the previous example.

Proposition 1. *Let G be any graph. Then $\gamma_h(G) \leq \tilde{\gamma}_{conh}(G)$.*

Proof. Let G be any graph and let S be a minimum outer-convex hop dominating set of G. Then $\tilde{\gamma}_{conh}(G) = |S|$. By Remark 2, S is a hop dominating set of G. It follows that $\gamma_h(G) \leq |S| = \tilde{\gamma}_{conh}(G)$. □

Remark 3. *The bound given in Proposition 1 is sharp. Moreover, strict inequality can also be attained.*

To see this, consider the graph G_1 in Figure 2. Let $C = \{f, g\}$. Then C is both a γ_h-set and a $\tilde{\gamma}_{conh}$-set of G_1. Hence, $\gamma_h(G_1) = 2 = \tilde{\gamma}_{conh}(G_1)$.

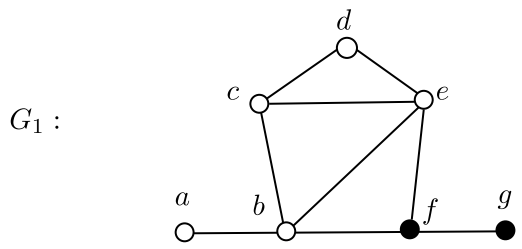

Figure 2: Graph G_1 with $\gamma_h(G_1) = \tilde{\gamma}_{conh}(G_1)$

For strict inequality, consider the graph G_2 in Figure 3. Let $C' = \{e, f\}$ and $C'' = \{e, f, g, h, i\}$. Then C' and C'' are γ_h-set and $\tilde{\gamma}_{conh}$-set of G_2, respectively. Thus, $\gamma_h(G_2) = 2 < 5 = \tilde{\gamma}_{conh}(G_2)$.

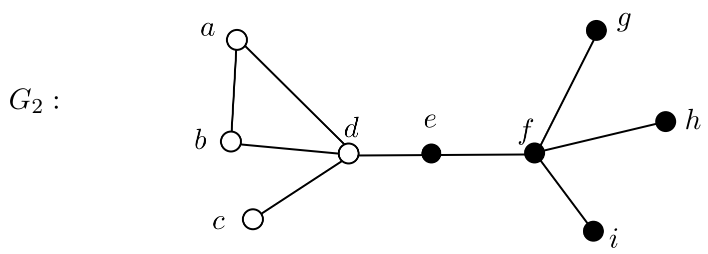

Figure 3: Graph G_2 with $\gamma_h(G_2) < \tilde{\gamma}_{conh}(G_2)$

Theorem 1. *Let G be any graph. Then $1 \leq \tilde{\gamma}_{conh}(G) \leq |V(G)|$. Moreover,*

(i) $\tilde{\gamma}_{conh}(G) = 1$ *if and only if G is trivial.*

(ii) $\tilde{\gamma}_{conh}(G) = 2$ *if and only if G has γ_h-set $C = \{x, y\}$ such that $V(G) \setminus C$ is convex set of G.*

(iii) $\tilde{\gamma}_{conh}(G) = |V(G)|$ *if and only if every component of G is complete.*

Proof. Clearly, $1 \leq \tilde{\gamma}_{conh}(G) \leq |V(G)|$.

(i) Suppose that $\tilde{\gamma}_{conh}(G) = 1$. Then $\gamma_h(G) = 1$ by Proposition 1. It follows that $G = K_1$ which is a trivial graph.

The converse is clear.

(ii) Suppose that $\tilde{\gamma}_{conh}(G) = 2$, say $C = \{x, y\}$ is a $\tilde{\gamma}_{conh}$-set of G. Then G is non-trivial by (i) and so $\gamma_h(G) \geq 2$. By assumption and Proposition 1, $\gamma_h(G) \leq 2$. Thus, $\gamma_h(G) = 2$. In particular, C is a γ_h-set of G. Moreover, $V(G) \setminus C$ is a convex set of G by assumption.

Conversely, suppose that G has γ_h-set $C = \{x, y\}$ of G such that $V(G) \setminus C$ is convex set of G. Then C is an outer-convex hop dominating set of G. Thus, $\tilde{\gamma}_{conh}(G) \leq 2$. Since $\gamma_h(G) = 2$, it follows that $\tilde{\gamma}_{conh}(G) = 2$ by Proposition 1.

(iii) Assume that $\tilde{\gamma}_{conh}(G) = |V(G)|$. Suppose that there is a component C of G which is non-complete. Then there exist $u, v \in V(C) \subseteq V(G)$ such that $d_C(u, v) = 2 = d_G(u, v)$. Let $S^* = V(G) \setminus \{u\}$. Then S^* is an outer-convex hop dominating set in G. Hence, $\tilde{\gamma}_{conh}(G) \leq |V(G)| - 1$, a contradiction. Therefore, every component of G is complete.

Conversely, suppose every component of G is complete. Then $V(G)$ is the minimum outer-convex hop dominating set of G. Therefore, $\tilde{\gamma}_{conh}(G) = |V(G)|$. □

The next result is a direct consequence of Theorem 1.

Corollary 1. *Let G be any graph of order $k \geq 1$. Then each of the following statements holds.*

- (i) $\tilde{\gamma}_{conh}(G) = k$ *if G is complete.*
- (ii) $\tilde{\gamma}_{conh}(G) \leq k - 1$ *if G is non-complete.*

The following result is a realization result involving outer-convex hop domination and hop domination.

Theorem 2. *Let a and b be positive integers such that $2 \leq a \leq b$. Then there exists a connected graph G such that $\gamma_h(G) = a$ and $\tilde{\gamma}_{conh}(G) = b$.*

Proof. For $a = b$, consider the graph G in Figure 4. Let $C = \{v_1, v_2, \ldots, v_a\}$. Then C is both γ_h-set and $\tilde{\gamma}_{conh}$-set of G, respectively. Hence, $\gamma_h(G) = a = \tilde{\gamma}_{conh}(G)$.

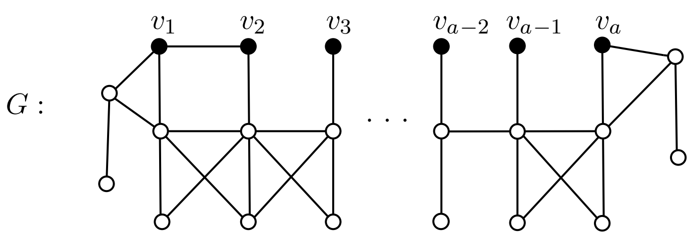

Figure 4: Graph G with $\gamma_h(G) = \tilde{\gamma}_{conh}(G)$

Suppose that $a < b$. For $a = 2$, let $s = b - 2$ and consider the graph H given in Figure 5. Let $C = \{v_1, v_2\}$ and $C' = \{v_1, v_2, y_1, y_2, \ldots, y_s\}$. Then C and C' are γ_h-set and $\tilde{\gamma}_{conh}$-set of H, respectively. Therefore, $\gamma_h(H) = 2$ and $\tilde{\gamma}_{conh}(H) = s + 2 = b$.

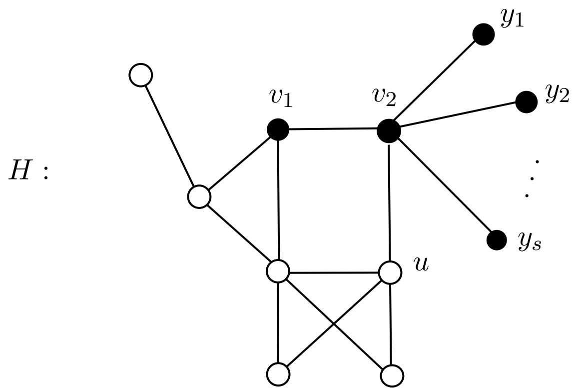

Figure 5: Graph G with $\gamma_h(G) < \tilde{\gamma}_{conh}(G)$

For $a \geq 3$, let $s = b - a$ and consider the graph G' in Figure 6. Let $C' = \{v_1, v_2, \ldots, v_{a-2}, u, v_a\}$ and $C'' = \{v_1, v_2, \ldots, v_a, y_1, y_2, \ldots, y_s\}$. Then C' and C'' are γ_h-set and $\tilde{\gamma}_{conh}$-set of G', respectively. Therefore, $\gamma_h(G') = a$ and $\tilde{\gamma}_{conh}(G') = s + a = b$. $\square$

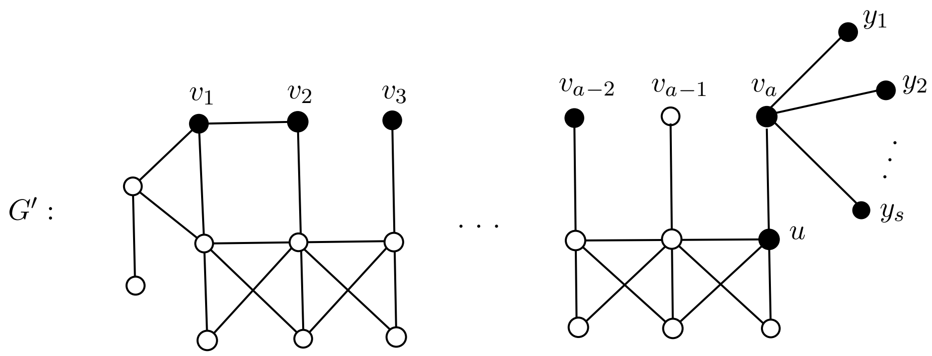

Figure 6: Graph G' with $\gamma_h(G') < \tilde{\gamma}_{conh}(G')$

Corollary 2. *Let n be a positive integer. Then there exists a connected graph G such that $\tilde{\gamma}_{conh}(G) - \gamma_h(G) = n$. In other words, $\tilde{\gamma}_{conh}(G) - \gamma_h(G)$ can be made arbitrarily large.*

Proposition 2. *Let n be any positive integer. Then each of the following holds.*

(i) $\tilde{\gamma}_{conh}(P_n) = \begin{cases} 2 & if\ n = 2, 3 \\ n - 2 & if\ n \geq 4. \end{cases}$

$$(ii)\ \tilde{\gamma}_{conh}(C_n) = \begin{cases} 2 & if\ n = 4,5 \\ 3 & if\ n = 3,6 \\ n-4 & if\ n \geq 7. \end{cases}$$

Proof. (i) Clearly, $\tilde{\gamma}_{conh}(P_n) = 2$ for $n = 2,3$. Suppose that $n \geq 4$. Let $P_n = [v_1, v_2, \ldots, v_n]$ and consider $S = \{v_1, v_2, \ldots, v_{n-3}, v_{n-2}\}$. Clearly, S is a hop dominating set of P_n. Observe that $V(P_n) \setminus S = \{v_{n-1}, v_n\}$ is convex set of P_n. It follows that S is an outer-convex hop dominating set of P_n. Since the induced subgraph of a convex set is always connected, it follows that S is a minimum outer-convex hop dominating set of P_n. Thus, $\tilde{\gamma}_{conh}(P_n) = n - 2$ for all $n \geq 4$.

(ii) Clearly, $\tilde{\gamma}_{conh}(C_4) = 2 = \tilde{\gamma}_{conh}(C_5)$ and $\tilde{\gamma}_{conh}(C_3) = 3 = \tilde{\gamma}_{conh}(C_6)$. Suppose that $n \geq 7$. Let $C_n = [v_1, v_2, \ldots, v_n, v_1]$ and consider $S^* = \{v_1, v_2, \ldots, v_{n-5}, v_{n-4}\}$. Clearly, S^* is hop dominating set of C_n. Notice that $V(C_n) \setminus S^* = \{v_{n-3}, v_{n-2}, v_{n-1}, v_n\}$ is a convex set of C_n. It follows that S^* is an outer-convex hop dominating set of C_n. Since the induced subgraph of any convex set is connected, it follows that S^* is a minimum outer-convex hop dominating set of C_n. Therefore, $\tilde{\gamma}_{conh}(C_n) = n - 4$ for all $n \geq 7$. □

Proposition 3. *Let G be any graph on $n \geq 2$ vertices. If $\tilde{\gamma}_{conh}(G) = 2$, then $\gamma_h(G) = 2$. However, the converse is not always true.*

Proof. Suppose $\tilde{\gamma}_{conh}(G) = 2$. Then $\gamma_h(G) \leq 2$ by Remark 1. Since $\gamma_h(G) \geq 2$ for any graph of order $n \geq 2$, it follows that $\gamma_h(G) = 2$.

To see the converse is not necessarily true, consider $P_5 = [v_1, v_2, v_3, v_4, v_5]$. Let $C = \{v_2, v_3\}$. Then C is a γ_h-set of P_5. Hence, $\gamma_h(P_5) = 2$. However, $\tilde{\gamma}_{conh}(P_5) = 3$ by Proposition 2. □

The following concept can be found in [6] and it will be used to characterize outer-convex hop dominating sets in the join of two graphs.

Definition 2. Let G be a non-complete graph. Then $D \subseteq V(G)$ is called an *outer-clique pointwise non-dominating set* in G if D is pointwise non-dominating set and $V(G) \setminus D$ is clique set in G. The smallest cardinality of an outer-clique pointwise non-dominating set of G, denoted by $ocpnd(G)$, is called the *outer-clique pointwise non-domination number* of G. Any outer-clique pointwise non-dominating set D of G with $|D| = ocpnd(G)$, is called an *ocpnd-set* of G.

The following results will be used to calculate the exact values of the parameter on the join of two graphs.

Theorem 3. *Let G be a non-complete graph of order n. Then $1 \leq ocpnd(G) \leq n - 1$. Moreover,*

(i) *$pnd(G) \leq ocpnd(G)$.*

(*ii*) *ocpnd*$(G) = 1$ *if and only if* G *has an isolated vertex* v *such that* $\langle V(G) \setminus \{v\}\rangle$ *is complete.*

Proof. Since $\varnothing$ is not an outer-clique pointwise non-dominating set of G, it follows that $ocpnd(G) \geq 1$. Also, since $V(G) \setminus \{v\}$ is an outer-clique pointwise non-dominating set of G for every $v \in V(G)$, by definition we have $ocpnd(G) \leq n-1$. Consequently, $1 \leq ocpnd(G) \leq n-1$.

(i) Since every outer-clique pointwise non-dominating set is a pointwise non-dominating set, it follows that $pnd(G) \leq ocpnd(G)$.

(ii) Suppose that $ocpnd(G) = 1$, say $\{v\}$ is an *ocpnd*-set of G. Then $pnd(G) = 1$ by (i). It follows that G is either a trivial or G has an isolated vertex v. Since outer-clique pointwise non-domination is not defined on any complete graph, it follows that G has an isolated vertex v. Moreover, $\langle V(G) \setminus \{v\}\rangle$ is complete by assumption.

Conversely, suppose G has an isolated vertex v such that $\langle V(G)\setminus\{v\}\rangle$ is complete. Then $\{v\}$ is an outer-clique pointwise non-dominating set of G. It follows that $ocpnd(G) = 1$. □

Proposition 4. *Let* $n \geq 2$ *be any positive integer. Then each of the following holds:*

(*i*) $ocpnd(P_n) = \begin{cases} 2 & if\ n = 3 \\ n-2 & if\ n \geq 4. \end{cases}$

(*ii*) $ocpnd(C_n) = n - 2$ *for all* $n \geq 4$.

(*iii*) $ocpnd(\overline{K}_n) = n - 1$ *for all* $n \geq 2$.

Proof. (*i*) Clearly, $ocpnd(P_3) = 2$. Suppose that $n \geq 4$. Let $P_n = [v_1, v_2, \ldots, v_n]$ and consider $C = \{v_1, v_2, \ldots, v_{n-3}, v_{n-2}\}$. Observe that C is a pointwise non-dominating set of P_n. Since $\langle V(P_n) \setminus C\rangle \cong K_2$, it follows that $V(P_n) \setminus C$ is clique in P_n. Hence, C is an outer-clique pointwise non-dominating set in P_n. Since K_1 and K_2 are the only complete subgraphs of P_n for all $n \geq 4$, it follows that C is a minimum outer-clique pointwise non-dominating set of P_n. Therefore, $ocpnd(G) = n - 2$ for all $n \geq 4$.

(*ii*) Suppose that $n \geq 4$. Then applying the same argument as in the proof of (*i*), we have $ocpnd(C_n) = n - 2$ for all $n \geq 4$.

(*iii*) Let $V(\overline{K}_n) = \{v_1, v_2, \ldots, v_n\}$, where $n \geq 2$ and consider $C^* = \{v_1, v_2, \ldots, v_{n-1}\}$. Then C^* is a minimum outer-clique pointwise non-dominating set of $\overline{K}_n$ for all $n \geq 2$. Hence, $ocpnd(\overline{K}_n) = n - 1$ for all $n \geq 2$. □

Theorem 4. *[9] Let* G *and* H *be any two graphs. A set* $S \subseteq V(G+H)$ *is hop dominating set of* $G+H$ *if and only if* $S = S_G \cup S_H$*, where* S_G *and* S_H *are pointwise non-dominating sets of* G *and* H*, respectively.*

Theorem 5. *[8] Let G and H be two connected graphs. Then a proper subset $C = S_1 \cup S_2$ of $V(G+H)$, where $S_1 \subseteq V(G)$ and $S_2 \subseteq V(H)$, is a convex set in $G+H$ if and only if S_1 and S_2 induce complete subgraphs of G and H, respectively, where it may occur that $S_1 = \varnothing$ or $S_2 = \varnothing$.*

Theorem 6. *Let G and H be two non-complete graphs. A set $C \subseteq V(G+H)$ is an outer-convex hop dominating set of $G+H$ if and only if $C = C_G \cup C_H$, where C_G and C_H are outer-clique pointwise non-dominating sets of G and H, respectively.*

Proof. Suppose that C is an outer-convex hop dominating set of $G+H$. If $C_G = \varnothing$, then $V(G) \nsubseteq N_G^2[C]$. However, this is a contradiction to the fact that C is a hop dominating set of $G+H$. Thus, $C_G \neq \varnothing$. Similarly, $S_H \neq \varnothing$. Now, since C is a hop dominating set of $G+H$, it follows that C_G and C_H are pointwise non-dominating sets of G and H, respectively by Theorem 4. Moreover, since $V(G+H) \setminus C$ is convex set in $G+H$, $V(G) \setminus C_G$ and $V(H) \setminus C_H$ are cliques in G and H, respectively by Theorem 5. Therefore, C_G and C_H are outer-clique pointwise non-dominating sets of G and H, respectively.

Conversely, suppose that $C = C_G \cup C_H$, where C_G and C_H are outer-clique pointwise non-dominating sets of G and H, respectively. Then $\langle V(G) \setminus C_G \rangle$ and $\langle V(H) \setminus C_H \rangle$ are complete in G and H, respectively. Hence, by Theorem 5, $V(G+H) \setminus C$ is convex set in $G+H$. Since C_G and C_H are pointwise non-dominating sets, $C = C_G \cup C_H$ is a hop dominating set of $G+H$ by Theorem 4. Therefore, $C = C_G \cup C_H$ is an outer-convex hop dominating set of $G+H$. □

The next result follows from Proposition 4 and Theorem 6.

Corollary 3. *Let G and H be two non-complete graphs. Then*

$$\tilde{\gamma}_{conh}(G+H) = ocpnd(G) + ocpnd(H).$$

In particular, given positive integers n and m, we have

$$(i)\ \ \tilde{\gamma}_{conh}(P_n + P_m) = \begin{cases} 4 & if\ n,m = 3 \\ m & if\ n = 3, m \geq 4 \\ n & if\ n \geq 4, m = 3 \\ n+m-4 & if\ n,m \geq 4; \end{cases}$$

(ii) $\tilde{\gamma}_{conh}(C_n + C_m) = n + m - 4$ *for all* $n, m \geq 4$*; and*

$$(iii)\ \ \tilde{\gamma}_{conh}(P_n + C_m) = \begin{cases} m & if\ n = 3, m \geq 4 \\ n+m-4 & if\ n,m \geq 4. \end{cases}$$

Theorem 7. *Let G be any non-complete graph and H be any complete graph. Then $C \subseteq V(G+H)$ is an outer-convex hop dominating set of $G+H$ if and only if $C = C_G \cup V(H)$, where C_G is an outer-clique pointwise non-dominating set of G.*

Proof. Suppose that C is an outer-convex hop dominating set in $G+H$. Since H is complete, it follows that $C = C_G \cup V(H)$, where $C_G \neq \varnothing$. By Theorem 6, C_G is an outer-clique pointwise non-dominating set of G.

Conversely, assume that $C = C_G \cup V(H)$, where C_G is an outer-clique pointwise non-dominating set of G. Then $N^2_{G+H}[C] = V(G+H)$, that is, C is a hop dominating set in $G+H$. Since C_G is an outer-clique set in G, it follows that $V(G+H) \setminus C$ is clique in $G+H$, that is, $V(G+H)\setminus C$ is convex in $G+H$. Therefore, C is an outer-convex hop dominating set of $G+H$. □

The next result follows from Corollary 1, Theorem 3, Proposition 4, and Theorem 7.

Corollary 4. *Let G be any non-complete graph and H be any complete graph. Then*

$$\tilde{\gamma}_{conh}(G+H) = ocpnd(G) + |V(H)|.$$

In particular, given positive integers n and m, we have

(i) $\tilde{\gamma}_{conh}(P_n + K_m) = \begin{cases} m+2 & if\ n=3, m\geq 1 \\ n+m-2 & if\ n\geq 4,\ m\geq 1; \end{cases}$

(ii) $\tilde{\gamma}_{conh}(C_n + K_m) = n+m-2$ *for all* $n \geq 4, m \geq 1$.

Theorem 8. *Let G be a connected non-trivial graph and H be any graph. If $C = \bigcup_{x\in V(G)} V(H^x)$, then C is an outer-convex hop dominating set of $G\circ H$. Moreover, $\tilde{\gamma}_{conh}(G\circ H) \leq |V(G)||V(H)|$.*

Proof. Suppose that $C = \bigcup_{x\in V(G)} V(H^x)$. Let $a \in V(G\circ H)\setminus C$. Then $a \in V(G)$. Since G is connected non-trivial graph, there exists $b \in V(H^x)$ for some $x \in V(G)$ such that $d_{G\circ H}(a,b) = 2$. Thus, C is a hop dominating set in $G\circ H$. Clearly, $V(G\circ H)\setminus C$ is convex in $G\circ H$. Therefore, C is an outer-convex hop dominating set in $G\circ H$. Moreover, since $|C| = |V(G)||V(H)|$, it follows that $\tilde{\gamma}_{conh}(G\circ H) \leq |V(G)||V(H)|$. □

Remark 4. *The sharpness of Theorem 8 is attainable. Moreover, strict inequality can be attained.*

For the sharpness, consider the graph $P_4 \circ P_2$ in Figure 7. Let $C = \{u_1, u_2, \ldots, u_8\}$. Then C is the minimum outer-convex hop dominating set of $P_4\circ P_2$. Thus,

$$\tilde{\gamma}_{conh}(P_4\circ P_2) = 4(2) = |V(G)||V(H)|.$$

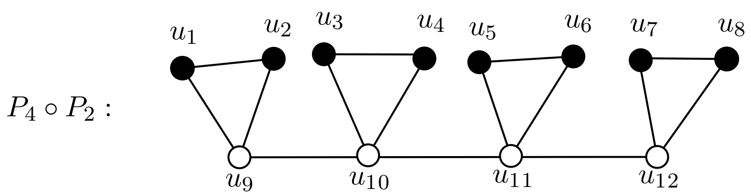

Figure 7: Graph $P_4 \circ P_2$ with $\tilde{\gamma}_{conh}(P_4 \circ P_2) = |V(P_4)||V(P_2)|$

For strict inequality, consider the graph $C_4 \circ K_3$ given in Figure 8. Let

$$C' = \{a, b, c, d, e, f, g, h\}.$$

Then C' is a minimum outer-convex hop dominating of $C_4 \circ K_3$. Hence,

$$\tilde{\gamma}_{conh}(C_4 \circ K_3) = 8 < 12 = |V(C_4)||V(K_3)|.$$

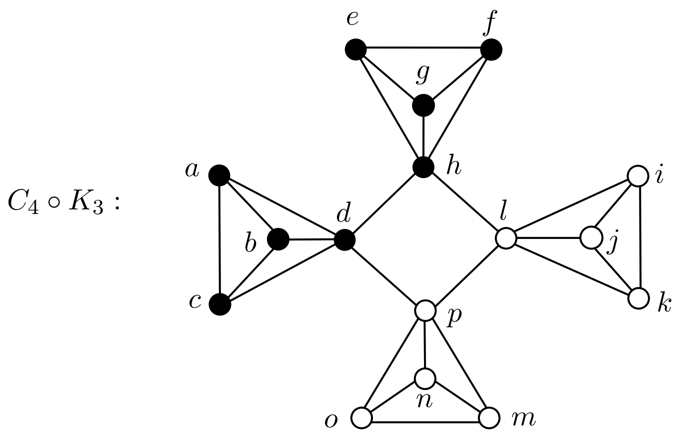

Figure 8: Graph $C_4 \circ K_3$ with $\tilde{\gamma}_{conh}(C_4 \circ K_3) < |V(C_4)||V(K_3)|$

4. Conclusion

The concept of an outer-convex hop domination has been introduced and investigated in this study. Necessary and sufficient conditions for sets in some special graphs and the join of two graphs have been formulated. These results have been used to derive bound or exact value of outer-convex hop domination number of each of these graphs. Moreover, we have established upper bound for the parameter on the corona of two graphs. Other interested researchers may investigate the concept for other graphs that were not considered in this study. Providing a real-life application of the concept will be an interesting topic to consider.

Acknowledgements

The authors would like to thank the referees for their invaluable comments and suggestions that led to the improvement of the paper. Also, the authors are grateful to Mindanao State University - Tawi-Tawi College of Technology and Oceanography for funding this research.

References

[1] S. Ayyaswamy, B. Krishnakumari, B. Natarjan, and Y. Venkatakrishnan. Bounds on the hop domination number of a tree. *Proceedings-Mathematical Sciences.*, 125(4):449–455, 2015.

[2] S. Ayyaswamy, C. Natarajan, and G. Sathiamoorphy. A note on hop domination number of some special families of graphs. *International Journal of Pure and Applied Mathematics.*, 119(12):11465–14171, 2018.

[3] J. Hassan and S. Canoy Jr. Hop independent hop domination in graphs. *Eur. J. Pure Appl. Math.*, 15(4):1783–1796, 2022.

[4] J. Hassan and S. Canoy Jr. Connected grundy hop dominating sequences in graphs. *Eur. J. Pure Appl. Math.*, 16(2):1212–1227, 2023.

[5] J. Hassan, S. Canoy Jr., and Chrisley Jade Saromines. Convex hop domination in graphs. *Eur. J. Pure Appl. Math.*, 16(1):319–335, 2023.

[6] J. Hassan, A. Lintasan, and N.H. Mohammad. Some properties and realization problems involving connected outer-hop independent hop domination in graphs. *Eur. J. Pure Appl. Math.*, 16(3):1848–1861, 2023.

[7] S. Canoy Jr and I.J.L. Garces. Convex sets under some graph operations. *Graphs and Combinatorics*, 18:787–793, 2002.

[8] S. Canoy Jr and J. Hassan. Weakly convex hop dominating sets in graphs. *Eur. J. Pure Appl. Math.*, 16(2):1196–1211, 2023.

[9] S. Canoy Jr., R. Mollejon, and J. G. Canoy. Hop dominating sets in graphs under binary operations. *Eur. J. Pure Appl. Math.*, 12(4):1455–1463, 2019.

[10] S. Nanding and H. Rara. Connected outer-independent hop domination in graphs. *Eur. J. Pure Appl. Math.*, 14(4):1226–1236, 2021.

[11] C. Natarajan and S. Ayyaswamy. Hop domination in graphs ii. *Versita*, 23(2):187–199, 2015.

[12] Y. Pabilona and H. Rara. Connected hop domination in graphs under some binary operations. *Asian-Eur. J. Math.*, 11(5):1850075–1–1850075–11, 2018.