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Vertex-Weighted (k_1, k_2) E-Torsion Graph of Quasi Self-Dual Codes

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Abstract. In this paper, we have introduced a graph G_{EC} generated by type-(k_1, k_2) E-codes which is (k_1, k_2) E-torsion graph. The binary codewords of the torsion code of C are the set of vertices, and the edges are defined using the construction of E-codes. Moreover, we characterized the graph obtained when $k_1 = 0$ and $k_2 = 0$ and calculated the degrees of every vertex and the number of edges of G_{EC}. Moreover, we presented necessary and sufficient conditions for a vertex to be in the center of a graph given the property of the codeword corresponding to the vertex. Finally, we represent every quasi self-dual codes of short length by defining the vertex-weighted (k_1, k_2) E-torsion graph, where the weight of every vertex is the weight of the codeword corresponding to the vertex.

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1. Introduction

Linear codes, well-studied objects in coding theory, have traditionally been explored over fields or rings with unity. However, recent researches [2–4, 14] have unveiled a fascinating avenue of investigation by extending the study of linear codes to non-unital rings. For instance, Alahmadi, et al [1], introduced the notion of Quasi Self-Dual codes (QSD codes), self-orthogonal linear codes of length n over a non-unital ring E such that the size of the code is 2^n. Moreover, there are some interesting researches in binary codes in the literature, for instance, [15] explored the Z_2-triple cycle codes and their duals, [11] cyclic codes from a sequence over finite fields, and [6] studied self-dual codes over R_k and binary self-dual codes. In continuation to the codes over E, Shi, Minjia, et al. [14] presented a special construction of QSD codes over E, based on combinatorial matrices

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related to two-class association schemes, Strongly Regular Graphs (SRG), and Doubly Regular Tournaments (DRT).

In this article, we delved into the analysis of graphs generated from linear codes over E, called linear E-codes and examine their properties and use these concepts to formulate a definition of graph.

Graph theory provides a powerful framework for visualizing and understanding complex systems, making it an ideal tool for investigating linear codes over non-unital rings. By associating codes with corresponding graphs, we can gain insights into the structure and behavior of these codes, enabling us to extract valuable information related to error correction, network coding, and other areas of interest. For standard notations and concepts in graph theory, the readers are advised to refer to [9].

In this study, we will first establish the foundations of linear codes over E, elucidating the necessary definitions, properties, and construction methods. Next, we will introduce the graph representation of such linear codes, by defining (k_1, k_2) E-torsion graph of an E-code, and will discuss the construction of such graphs and explore the relationship between the code's properties and the resulting graph structure. Moreover, we will study vertex-weighted graph to separate the isomorphic graph generated by two inequivalent E-codes.

The study of coding theory in relation to graph theory is not well-established topic. However, few researchers tried to focus on the subject such as graph theoretic methods in coding theory [13], where it discusses the application of graph theory in coding theory, and codes on graphs [8], where it developed a fundamental theory of realizations of linear and group codes on general graphs using elementary group theory, including basic group duality theory.

Through our comprehensive analysis of graphs produced from linear codes over the non-unital ring E, this article seeks to contribute to the expanding field of coding theory and its applications in diverse domains. By exploring the interplay between graph theory and linear codes over non-unital rings, we strive to unlock new perspectives, insights, and practical solutions that can address challenges in error correction, information transmission, and beyond.

2. Background

2.1. Binary codes

As defined in [14], denoted by $wt(x)$ the Hamming weight of $x \in \mathbb{F}_2^n$. The **dual** of a binary code C is denoted by $C^{\perp}$ and defined as

$$C^{\perp} = \{y \in \mathbb{F}_2^n | \forall x \in C, (x, y) = 0, \}$$

where

$$(x, y) = \sum_{i=1}^{n} x_i y_i,$$

denotes the standard inner product. A code C is **self-orthogonal** if it is included in its dual:

$$C \subseteq C^{\perp}.$$

Two binary codes are **equivalent** if there is a permutation of coordinates that maps one to the other.

2.2. Ring Theory

We describe the main properties of the ring E of order four. The ring E is defined by the relations on two generators a, b and we shall write

$$c = a + b$$

for the given ring.

The ring E is defined by

$$E = \langle a, b | 2a = 2b = 0, a^2 = a, b^2 = b, ab = a, ba = b \rangle.$$

It is a non-unital ring and non-commutative ring with characteristic two. For more details refer to [3, 7, 12]. The ring is local with maximal ideal $\{0, c\}$. Its multiplication table is given in Table 1.

×	0	a	b	c
0	0	0	0	0
a	0	a	0	0
b	0	b	b	0
c	0	c	c	0

Table 1: Multiplication table for the ring E

From Table 1, it is clear E is not commutative, and non-unital. It is local with the maximal ideal

$$J = \{0, c\},$$

and residue field

$$E/J = \mathbb{F} = \{0, 1\},$$

the finite filed of order 2.

If we denote

$$\alpha : E \to E/J = \mathbb{F}_2,$$

the map of reduction modulo J. It follows that

$$\alpha(0) = \alpha(c) = 0,$$

and

$$\alpha(a) = \alpha(b) = 1.$$

This function α is extended in the natural way in a map from E^n to $\mathbb{F}_2^n$. Readers who wanted further details on the properties of ring $\mathcal{R}$, we refer the readers to [1–3, 10].

2.3. Codes over E

A **linear** E-code of length n is a one-sided E-submodule of E^n. Let C be a code of length n over E. With the code, there are two binary codes of length n:

(i) the **residue code** defined by $res(C) = \{\alpha(y) | y \in C\}$,

(ii) the **torsion code** defined by $tor(C) = \{x \in \mathbb{F}_2^n | cx \in C\}$.

The **right dual** $C^{\perp_R}$ of C is the right module defined by

$$C^{\perp_R} = \{y \in E^n | \forall x \in C, (x, y) = 0\}.$$

The **left dual** $C^{\perp_R}$ of C is the left module defined by

$$C^{\perp_L} = \{y \in E^n | \forall x \in C, (y, x) = 0\}.$$

An E-code C is **self-orthogonal** if

$$\forall x, y \in C, (x, y) = 0.$$

It follows that C is **self-orthogonal** if and only if

$$C \subseteq C^{\perp_L}.$$

Similarly, C is **self-orthogonal** if and only if

$$C \subseteq C^{\perp_R}.$$

Hence, for a self-orthogonal code C, it satisfies that

$$C \subseteq C^{\perp_L} \cap C^{\perp_R}.$$

An $E-$code of length n is **Quasi Self-Dual** (QSD for short) [14] if it is self-orthogonal and of size 2^n. A quasi-self dual code is **Type IV** if all its codewords have even weight [5].

3. Some results in linear E-codes

3.1. Linear E-codes

Definition 1. *[3] Let C be a linear E-code. Then C is a* ***type-**(k_1, k_2) *code if*

$$dim(res(C)) = k_1$$

and

$$dim(tor(C)) = k_1 + k_2.$$

Theorem 1. *[3] Let B be a self-orthogonal binary code of length n. The code C defined by the relation*

$$C = aB + cB^{\perp},$$

is a quasi self-dual code. Its residue code is B and its torsion code is $B^{\perp}$.

Corollary 1. *[3] Let B and B' be a binary code of length n such that B is self-orthogonal and $B \subseteq B'$. Then C is a linear E-code defined by the relation*

$$C = aB + cB'.$$

4. Results in (k_1, k_2) E-torsion graph of an E-code

Definition 2. *Let C be a linear E-code and B' be the torsion code of C. Then the simple graph G_{EC} such that the vertex set*

$$V(G_{EC}) = B'$$

and

$$\overline{xy} \in E(G_{EC}),$$

the edge set and $x \neq y$, if

$$ax + cy \in C$$

or

$$ay + cx \in C,$$

*is called the (k_1, k_2) E-**torsion graph** of C.*

To avoid the confusion to whether the binary code is viewed as a codeword in $tor(C)$ or vertex in G_{EC}, we denote the vertex $\widehat{x}$ which corresponds to the codeword x. This means that if

$$x \in tor(C),$$

then

$$\widehat{x} \in V(G_{EC}).$$

Example 1. *Let*

$$C = aB + cB'$$

where

$$B = \langle 1100 \rangle$$

and

$$B' = \langle 1100, 0011 \rangle.$$

This means that

$$V(G_{EC}) = \{\widehat{0000}, \widehat{1100}, \widehat{0011}, \widehat{1111}\}.$$

By computation, we get

$$E(G_{EC}) = \{(\widehat{0000}, \widehat{1100}), (\widehat{0000}, \widehat{0011}), (\widehat{0000}, \widehat{1111}), (\widehat{1100}, \widehat{0011}), (\widehat{1100}, \widehat{1111})\}.$$

Thus, the (k_1, k_2)-torsion graph of C, G_{EC}, is illustrated in Figure 1.

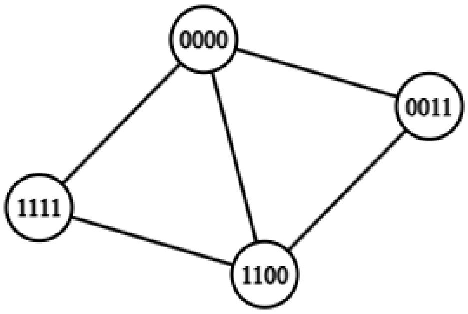

Figure 1: (k_1, k_2) E-torsion graph of C

Theorem 2. *If C is a type-(k_1, k_2) of an E-code, then*

$$|V(G_{EC})| = 2^{k_1+k_2}$$

and

$$|E(G_{EC})| = \sum_{i=1}^{2^{k_1}} 2^{k_1+k_2} - i.$$

Proof. The equation

$$|V(G_{EC})| = 2^{k_1+k_2}$$

follows from the fact that the torsion of a type-(k_1, k_2) E-code has dimension $k_1 + k_2$. On the other hand, from the definition of $E(G_{EC})$,

$$E(G_{EC}) = \{(\widehat{x}, \widehat{y}) : x \in res(C), y \in tor(C)\},$$

that is, each of the 2^{k_1} elements of the residue will be connected by an edge to the

$$2^{k_1+k_2} - 1$$

elements of the torsion. We can enumerate the edges by starting at an element in the residue with $2^{k_1+k_2} - 1$ edges containing that element, then if there is another element of the residue, we will enumerate the $2^{k_1+k_2} - 2$ edges containing the second element, since there is one edge common to the set of edges containing the first element and set of edges containing the second element, hence the second set of edges is 1 less than the previous set of edges. We continue the process by subtracting 1 from the number of the previous set of edges. Using this algorithm, the number of distinct pairs would be

$$\sum_{i=1}^{2^{k_1}} 2^{k_1+k_2} - i.$$

■

Corollary 2. *Let $\widehat{x} \in V(G_{EC})$. If $x \in res(C)$, then*

$$deg(\widehat{x}) = 2^{k_1+k_2} - 1.$$

If $x \notin res(C)$, then

$$deg(\widehat{x}) = 2^{k_1}.$$

Proof. The proof follows from Theorem 2. ■

Corollary 3. *If C is a type-(k_1, k_2) E-code, then*

$$|E(G_{EC})| = 2^{2k_1+k_2} - 2^{2k_1-1} - 2^{k_1-1}.$$

Proof. The proof follows directly from Corollary 2. ■

Lemma 1. $r(G_{EC}) = 1$.

Proof. If $x \in res(C)$, then the eccentricity of $\widehat{x}$ is 1 since $\widehat{x}$ is connected by an edge to every vertex in G_{EC}. If $x \notin res(C)$, then the eccentricity of $\widehat{x}$ is 2 since every vertex in G_{EC} is connected through a vertex in $res(C)$ to all other vertex not in $res(C)$. Therefore,

$$r(G_{EC}) = 1.$$

■

Lemma 2. *Let $G_{EC} \neq P_2$, path of order 2. If there exists $x \notin res(C)$, then there exists $y \neq x$ such that $y \notin res(C)$.*

Proof. Let $x \notin res(C)$. Then

$$|res(C)| < |tor(C)|.$$

This means $k_1 < k_1 + k_2$, that is, $k_2 > 0$. Now,

$$|tor(C)| - |res(C)| = 2^{k_1+k_2} - 2^{k_1} = 2^{k_1}\left(2^{k_2} - 1\right).$$

Note that if $k_1 = 0$ and $k_2 = 1$, $G_{EC} \neq P_2$, which is a contradiction. Thus,

$$2^{k_1}\left(2^{k_2} - 1\right) \geq 2.$$

■

Theorem 3. *Let C be an E-code and G_{EC} be the (k_1, k_2) E-torsion graph of C which is not P_2. Then vertex $\widehat{x} \in C(G_{EC})$ if and only if $x \in res(C)$.*

Proof. Let $\widehat{x} \in C(G_{EC})$. Suppose $x \notin res(C)$. Then, by Lemma 2 there exists $y \in tor(C)$ such that both

$$ax + cy$$

and

$$ay + cx$$

not in C. It follows that eccentricity of $\widehat{x}$ is greater than 1, a contradiction that $\widehat{x} \in C(G_{EC})$ by Lemma 1.
Conversely, suppose $x \in res(C)$. Then $\widehat{x}$ is connected by an edge to every vertex in G_{EC}. Thus, the eccentricity of vertex $\widehat{x}$ is 1, that is, $\widehat{x} \in C(G_{EC})$. ■

4.1. (k_1, k_2) E-torsion graph of QSD codes

Quasi self-dual codes are classified in [3] using their residue codes. But since every residue code corresponds to a unique torsion code, the study of the structure of G_{EC} of a QSD code will be concentrated in this section.

Example 2. *Let*

$$C = aB + cB^{\perp},$$

where

$$B = \langle 1100, 0011 \rangle .$$

Then

$$B^{\perp} = \langle 1100, 0011 \rangle$$

By Theorem 1*, C is a QSD code.*

$$V(G_{EC}) = \{\widehat{0000}, \widehat{1100}, \widehat{0011}, \widehat{1111}\}.$$

By Corollary 3*,*

$$|E(G_{EC})| = 16 - 8 - 2 = 6,$$

that is, G_{EC} is a complete graph.

Theorem 4. *Let G_{EC} be the (k_1, k_2) E-torsion graph of a QSD code*

$$C = aB + cB^{\perp}$$

where B is a binary code. Then B is self-dual if and only if G_{EC} is a complete graph.

Proof. Let B be self-dual. Then

$$res(C) = tor(C).$$

By Corollary 2, the degree of every vertex of G_{EC} is

$$2^{k_1+k_2} - 1,$$

that is, G_{EC} is a complete graph.
Conversely, suppose that G_{EC} is a complete graph. Let $x \in tor(C)$. Then

$$(\widehat{x}, \widehat{y}) \in E(G_{EC})$$

since G_{EC} is complete. It follows that

$$ax + cy \in C$$

for all

$$y \in tor(C).$$

Applying α, we have $x \in res(C)$, that is,

$$tor(C) \subseteq res(C).$$

■

Corollary 4. *If C is a QSD code of type-$(k_1, 0)$, then G_{EC} is a complete graph.*

Theorem 5. *If C is a QSD code of type-$(0, k_2)$, then G_{EC} is a star graph.*

Proof. If $k_1 = 0$, then $res(C)$ is the trivial code which contains only the zero vector. It follows that

$$tor(C) = \mathbb{F}_2^n.$$

Hence,

$$E(G_{EC}) = \{(\widehat{0_v}, \widehat{x}) : x \in \mathbb{F}_2^n\}.$$

■

Remark 1. *Let $k_1, k_2 \in \mathbb{Z}^+$ and C_1, C_2 be type-(k_1, k_2) linear E-codes. Then*

$$G_{EC_1} \cong G_{EC_2}.$$

Looking at Remark 1, (k_1, k_2) E-torsion graph alone cannot be used to classify QSD codes since two inequivalent codes under the same type-(k_1, k_2) code have the same (k_1, k_2) E-torsion graph. So to separate these two inequivalent QSD codes, we use the concept of vertex-weighted graph which is defined in the following.

Definition 3. *The **vertex-weighted** (k_1, k_2) **E-torsion graph** of a QSD code is the vertex-weighted graph where the weight of a vertex $x \in G_{EC}$ is the weight of the codeword $wt(x)$ of $x \in tor(C)$.*

Example 3. *Let*

$$C_1 = aB_1 + cB_1^{\perp}$$

and

$$C_2 = aB_2 + cB_2^{\perp}$$

where

$$B_1 = \langle 1100 \rangle$$

and

$$B_2 = \langle 1111 \rangle .$$

Note that C_1 and C_2 are two nonequivalents E-codes. Now,

$$V(G_{EC_1}) = \{\widehat{0000}, \widehat{1100}, \widehat{0010}, \widehat{1110}, \widehat{0001}, \widehat{1101}, \widehat{0011}, \widehat{1111}\}$$

and

$$V(G_{EC_2}) = \{\widehat{0000}, \widehat{1111}, \widehat{1100}, \widehat{0011}, \widehat{0110}, \widehat{1001}, \widehat{1010}, \widehat{0101}\}.$$

Figure 2 shows the graph representation of G_{EC_1}:

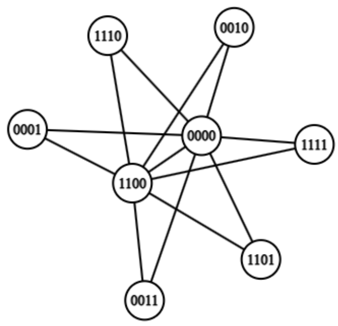

Figure 2: (k_1, k_2) E-torsion graph of G_{EC_1}

Furthermore, Figure 3 is the graph representation of graph G_{EC_2}.

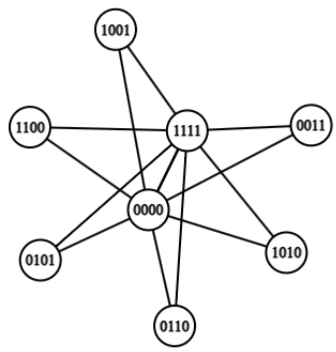

Figure 3: (k_1, k_2) E-torsion graph of G_{EC_2}

Note that the two graphs are isomorphic. However, if we look at the vertex-weighted graph of G_{EC_1} and G_{EC_2}, respectively, (see Figure 4 and 5) using the weights of every codeword, we see the difference between these two vertex-weighted $(1, 2)$ E-torsion graphs. Hence, two codes can have isomorphic graphs but different vertex-weighted (k_1, k_2) E-torsion graphs.

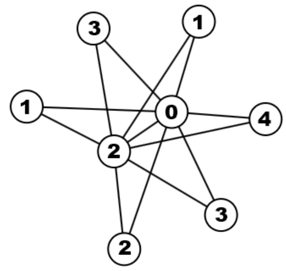

Figure 4: (k_1, k_2) E-torsion graph of G_{EC_1}

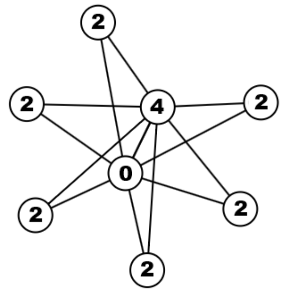

Figure 5: (k_1, k_2) E-torsion graph of G_{EC_2}

5. Vertex-weighted (k_1, k_2) E-torsion graph of QSD codes with $n \leq 4$

Quasi self-dual E-codes of short length were classified in [3]. In this section, we will illustrate those QSD codes using their vertex-weighted (k_1, k_2) E-torsion graphs up to $n = 4$.

5.1. (k_1, k_2) E-torsion graph of QSD codes for n=2.

For

$$C_1 = a \langle 00 \rangle + c \langle 10, 01 \rangle ,$$

we have a $(0, 2)$ E-torsion graph which is illustrated in Figure 6.

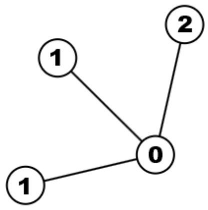

Figure 6: Vertex-weighted (k_1, k_2) E-torsion graph of C_1

For

$$C_2 = a\langle 11\rangle + c\langle 11\rangle,$$

we have a $(1,0)$ E-torsion graph which is illustrated in Figure 7.

Figure 7: Vertex-weighted (k_1, k_2) E-torsion graph of C_2

5.2. (k_1, k_2) E-torsion graph of QSD codes for n=3.

For

$$C_3 = a\langle 000\rangle + c\langle 100, 010, 001\rangle,$$

we have a $(0,3)$ E-torsion graph which is illustrated in Figure 8.

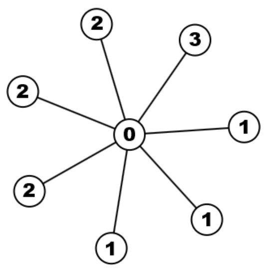

Figure 8: Vertex-weighted (k_1, k_2) E-torsion graph of C_3

5.3. (k_1, k_2) E-torsion graph of QSD codes for n=4.

For

$$C_5 = a\langle 0000\rangle + c\langle 1000, 0100, 0010, 0001\rangle,$$

we have a $(0,4)$ E-torsion graph which is illustrated in Figure 10.

For

$$C_4 = a\langle 101\rangle + c\langle 101, 010\rangle\,,$$

we have a $(1,1)$ E-torsion graph which is illustrated in Figure 9.

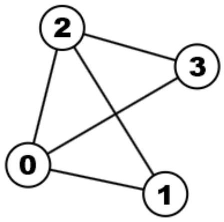

Figure 9: Vertex-weighted (k_1, k_2) E-torsion graph of C_4

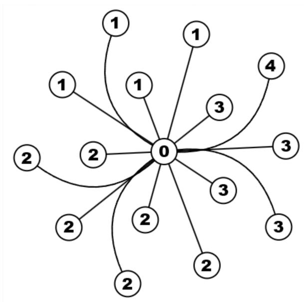

Figure 10: Vertex-weighted (k_1, k_2) E-torsion graph of C_5

For

$$C_6 = a\langle 1100\rangle + c\langle 1100, 0010, 0001\rangle\,,$$

we have a $(1,2)$ E-torsion graph which is illustrated in Figure 11.

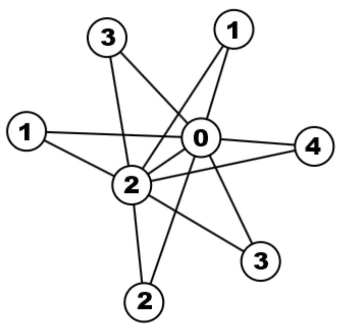

Figure 11: Vertex-weighted (k_1, k_2) E-torsion graph of C_6

For

$$C_7 = a\left\langle 1111 \right\rangle + c\left\langle 1111, 1100, 0110 \right\rangle,$$

we have a $(1,2)$ E-torsion graph which is illustrated in Figure 12.

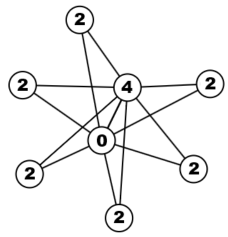

Figure 12: Vertex-weighted (k_1, k_2) E-torsion graph of C_7

For

$$C_8 = a\left\langle 1100, 0011 \right\rangle + c\left\langle 1100, 0011 \right\rangle,$$

we have a $(2,0)$ E-torsion graph which is illustrated in Figure 13.

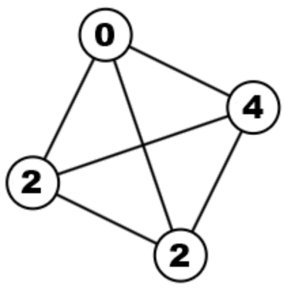

Figure 13: Vertex-weighted (k_1, k_2) E-torsion graph of C_8

6. Conclusion

In this paper, we studied the (k_1, k_2) E-torsion graph of a type-(k_1, k_2) E-codes. In particular, the size of the set of vertices and set of edges. We also characterized (k_1, k_2) E-torsion graph when $k_1 = 0$ and $k_2 = 0$ and introduced the notion of vertex-weighted (k_1, k_2) E-torsion graph to differentiate inequivalent QSD codes of the same type. Finally, we were able to represent QSD codes which were classified in [3] up to $n = 4$ using the vertex-weighted (k_1, k_2) E-torsion graph. By defining a (k_1, k_2) E-torsion graph G such that the $V(G) = 2^{k_1+k_2}$, there are 2^{k_1} vertices that have degree $2^{k_1+k_2} - 1$ with the rest vertices, if there exist, have degree 2^{k_1}. For future study, after graph operations of two (k_1, k_2) E-torsion graphs is a (k_1, k_2) E-torsion graph? Also, one can explore center of (k_1, k_2) E-torsion graphs and the dominating sets of (k_1, k_2) E-torsion graphs.

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