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On Degenerate Laplace-type Integral Transform

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Abstract. This paper is motivated by the work of Taekyun Kim and Dae San Kim on the degenerate Laplace transform and degenerate gamma function, as published in the Russian Journal of Mathematical Physics. We introduce the degenerate Laplace-type integral transform and delve into its properties and interrelations. This paper focuses on the degenerate Laplace-type integral transforms of several fundamental functions, including the degenerate sine, degenerate cosine, degenerate hyperbolic sine, and degenerate hyperbolic cosine functions. Furthermore, we establish crucial connections between the degenerate Laplace-type integral transform and existing degenerate integral transforms. Specifically, we explore its relationships with the degenerate Laplace transform, the degenerate Elzaki transform, and the degenerate Sumudu transforms.

2020 Mathematics Subject Classifications: 44A99

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1. Introduction

Integral transforms have long captivated the mathematical world due to their multifaceted properties and widespread applications across diverse scientific fields. During the 20th and 21st centuries, the Laplace transform has been extensively studied and employed in various scientific disciplines. Among the noteworthy contributions in this domain is the investigation of work the intrinsic structure and properties of Laplace-typed integral transforms by H. Kim [7] and some additional properties of Laplace-type integral transforms by H. Kim et.al[5] . This integral transform is defined as

$$F_\alpha(u) = \mathcal{G}_\alpha\{f(t)\} = u^\alpha \int_0^\infty e^{\frac{-t}{u}} f(t)dt,$$

where $\alpha \in \mathbb{Z}$.

In recent years, there has been growing interest in degenerate versions of existing integral transforms. Pioneering work of T. Kim and D. S. Kim [8] introduced the concept of

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degenerate gamma functions and degenerate Laplace transforms, as well as the derivation of fundamental properties. Subsequently, L. M. Upadhyaya [14–16] further delved into properties of the degenerate Laplace transform, while U. Duran [4] investigated the degenerate Sumudu transform and L. M. Upadhyaya et.al [1] defined the degenerate Elzaki transform and its properties.

In light of the growing significance of degenerate integral transforms, this article seeks to contribute to this research by introducing the degenerate Laplace-type integral transform. The goal is to derive some properties of this transform and explore its relationship with other degenerate integral transforms.

2. Definition and Some Explicit Formulas

Taekyun Kim and Dae San Kim [8] introduced the concept of degenerate Laplace transform, where $f(t)$ be a function defined for $t \geq 0$ and $\lambda \in (0, \infty)$. Then the integral

$$\mathcal{L}_\lambda\{f(t)\} = \int_0^\infty e_\lambda^{-s}(t) f(t) dt = \int_0^\infty (1+\lambda t)^{-\frac{s}{\lambda}} f(t) dt, \tag{1}$$

is said to be the degenerate Laplace transform of f if the integral converges. Letting $s = \dfrac{1}{u}$, the degenerate Laplace transform can be rewritten as

$$\mathcal{L}_\lambda\{f(t)\} = \int_0^\infty e_\lambda^{-\frac{1}{u}}(t) f(t) dt = \int_0^\infty (1+\lambda t)^{-\frac{1}{u\lambda}} f(t) dt$$

Ugur Duran of Iskenderun Technical University [4] introduced the concept of degenerate Sumudu transform of $f(t)$ which is defined by the improper integral

$$\mathcal{S}_\lambda\{f(t)\} = \frac{1}{u}\int_0^\infty e_\lambda^{\frac{-1}{u}}(t) f(t) dt = \frac{1}{u}\int_0^\infty (1+\lambda t)^{-\frac{1}{u\lambda}} f(t) dt,$$

where $\lambda \in (0, \infty)$, and $f(t)$ be a function defined for $t \geq 0$.

On the paper of Lalit Mohan Upadhyaya et.al [1], they defined the degenerate of Elzaki transform and its properties. The degenerate Elzaki transform is defined by the integral

$$\mathcal{E}_\lambda\{f(t)\} = u\int_0^\infty e_\lambda^{-\frac{1}{u}}(t) f(t) dt = u\int_0^\infty (1+\lambda t)^{-\frac{1}{u\lambda}} f(t) dt$$

where $\lambda \in (0, \infty)$, and $f(t)$ be a function defined for $t \geq 0$.

We now have the following definition:

Definition 1. [3, 8–14] For any nonzero real number λ, the **degenerate exponential function** is defined as follows:

$$e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}}, \qquad e_\lambda(t) = e_\lambda^1(t) = (1+\lambda t)^{\frac{1}{\lambda}} \tag{2}$$

That is, the degenerate of the exponential function e^{xt} is equal to $e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}}$, where $\lambda \in \mathbb{R} - \{0\}$.
Here, we note that

$$e_\lambda^x(t) = \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!},$$

where $(x)_{0,\lambda} = 1$, $(x)_{n,\lambda} = x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda)$ for $n \geq 1$.
It is noteworthy to mention that

$$\lim_{\lambda \to 0} e_\lambda^x(t) = \lim_{\lambda \to 0} (1+\lambda t)^{\frac{x}{\lambda}} = e^{xt}.$$

Definition 2. [8] A function $f(t)$ is said to be of **degenerate exponential order** C if there exists $C, M > 0$ and $T > 0$ such that

$$|f(t)| \leq M(1+\lambda t)^{\frac{C}{\lambda}} = Me_\lambda^C(t) \text{ for all } t > T.$$

Definition 3. [2, 4, 6] The **degenerate sine function** is defined by the relation

$$\sin_\lambda^{(x)}(t) = \frac{e_\lambda^{ix}(t) - e_\lambda^{-ix}(t)}{2i} = \sin\left(\frac{x}{\lambda}\log(1+\lambda t)\right), \quad \text{where} \quad i = \sqrt{-1}. \tag{3}$$

It can be noted that,

$$\lim_{\lambda \to 0} \sin_\lambda^{(x)}(t) = \sin xt.$$

Definition 4. [2, 4, 6] The **degenerate cosine function** is defined by the relation

$$\cos_\lambda^{(x)}(t) = \frac{e_\lambda^{ix}(t) + e_\lambda^{-ix}(t)}{2} = \cos\left(\frac{x}{\lambda}\log(1+\lambda t)\right), \quad \text{where} \quad i = \sqrt{-1}. \tag{4}$$

It can be noted that,

$$\lim_{\lambda \to 0} \cos_\lambda^{(x)}(t) = \cos xt.$$

Definition 5. [4, 6, 14] The **degenerate Euler function** is defined by the relation

$$e_\lambda^{ix}(t) = \cos_\lambda^{(x)}(t) + i\sin_\lambda^{(x)}(t) \tag{5}$$

where

$$\cos_\lambda^{(x)}(t) = \cos\left(\frac{x}{\lambda}\log(1+\lambda t)\right) \text{ and } \sin_\lambda^{(x)}(t) = \sin\left(\frac{x}{\lambda}\log(1+\lambda t)\right).$$

It can be noted that,

$$\lim_{\lambda \to 0} e_\lambda^{ix}(t) = \cos xt + i\sin xt.$$

Definition 6. [4, 8, 14] The **degenerate hyperbolic sine function** is defined by the relation

$$\sinh_\lambda^{(x)}(t) = \frac{e_\lambda^x(t) - e_\lambda^{-x}(t)}{2}. \tag{6}$$

It can be noted that,

$$\lim_{\lambda \to 0} \sinh_\lambda^{(x)}(t) = \sinh xt.$$

Definition 7. [4, 8, 14] The **degenerate hyperbolic cosine function** is defined by the relation

$$\cosh_{\lambda}^{(x)}(t) = \frac{e_{\lambda}^{x}(t) + e_{\lambda}^{-x}(t)}{2}. \tag{7}$$

It can be noted that,

$$\lim_{\lambda \to 0} \cosh_{\lambda}^{(x)}(t) = \cosh xt.$$

3. Degenerate Laplace-type Integral Transform

Definition 8. Let $\lambda \in (0, \infty)$, $\alpha \in \mathbb{Z}$ and let $f(t)$ be a function defined for $t \geq 0$. Then the integral

$$F_{\alpha,\lambda}(u) = \mathcal{G}_{\alpha,\lambda}\{f(t)\} = u^{\alpha} \int_0^{\infty} e_{\lambda}^{-\frac{1}{u}}(t) f(t) dt = u^{\alpha} \int_0^{\infty} (1 + \lambda t)^{-\frac{1}{u\lambda}} f(t) dt, \tag{8}$$

is said to be the **degenerate Laplace-type integral transform** of $f(t)$. If the improper integral is convergent, then we say that the function $f(t)$ possesses a degenerate Laplace-type integral transform.

We note that

$$\lim_{\lambda \to 0} \mathcal{G}_{\alpha,\lambda}\{f(t)\} = \mathcal{G}_{\alpha}\{f(t)\}. \tag{9}$$

Theorem 1. *Suppose that $f(t)$ is a piecewise-continuous function on the interval $[0, \infty)$ and has a degenerate exponential order at infinity with $|f(t)| \leq Me_{\lambda}^{C}(t)$ for $t > P$, where $M \geq 0$ and P, C are constants. Then, $\mathcal{G}_{\alpha,\lambda}\{f(t)\}$ exists for $\frac{1 - uC}{\lambda u} > 1$.*

Proof. Suppose that $f(t)$ is a piecewise-continuous function on the interval $[0, \infty)$ and has a degenerate exponential order at infinity with $|f(t)| \leq Me_{\lambda}^{C}(t)$. Then

$$u^{\alpha} \int_0^{\infty} e_{\lambda}^{-\frac{1}{u}}(t) f(t) dt = u^{\alpha} \int_0^{P} e_{\lambda}^{-\frac{1}{u}}(t) f(t) dt + u^{\alpha} \int_P^{\infty} e_{\lambda}^{-\frac{1}{u}}(t) f(t) dt. \tag{10}$$

Since the function $f(t)$ is piecewise-continuous in every finite interval $0 \leq t \leq P$, the first integral on the right-hand side of equation (10) exists. Since

$$\left| e_{\lambda}^{-\frac{1}{u}}(t) f(t) \right| \leq M e_{\lambda}^{-\frac{1}{u}}(t) e_{\lambda}^{C}(t)$$

for $t > P$, we have

$$\begin{aligned} \left| u^{\alpha} \int_P^{\infty} e_{\lambda}^{-\frac{1}{u}}(t) f(t) dt \right| &\leq u^{\alpha} \int_P^{\infty} \left| e_{\lambda}^{-\frac{1}{u}}(t) f(t) \right| dt \\ &\leq u^{\alpha} \int_P^{\infty} e_{\lambda}^{-\frac{1}{u}}(t) M e_{\lambda}^{C}(t) dt \\ &= M u^{\alpha} \int_P^{\infty} (1 + \lambda t)^{\frac{-1+uC}{u\lambda}} dt \end{aligned}$$

$$
\begin{aligned}
&= Mu^{\alpha} \lim_{R\to\infty} \int_P^R (1+\lambda t)^{-\left(\frac{1-uC}{u\lambda}\right)} dt \\
&= \frac{Mu^{\alpha}}{\lambda} \lim_{R\to\infty} \left[\frac{(1+\lambda t)^{1-\left(\frac{1-uC}{u\lambda}\right)}}{\frac{1}{u\lambda}\big(u\lambda - (1-uC)\big)} \right] \Bigg|_P^R \\
&= Mu^{\alpha+1} \lim_{R\to\infty} \left[\frac{(1+\lambda(R))^{1-\left(\frac{1-uC}{u\lambda}\right)}}{u\lambda - 1 + uC} - \frac{(1+\lambda(P))^{1-\left(\frac{1-uC}{u\lambda}\right)}}{u\lambda - 1 + uC} \right] \\
&= \frac{Mu^{\alpha+1}}{1 - u\lambda - uC} (1+P\lambda)^{1-\left(\frac{1-uC}{u\lambda}\right)} < \infty, \\
&\quad \text{for } \frac{1-uC}{u\lambda} > 1.
\end{aligned}
$$

Hence, the second integral converges for $\frac{1-uC}{u\lambda} > 1$.

Since the first integral on the right hand side of equation (10) converges and the second integral on the right hand side of equation (10) also converges for $\frac{1-uC}{u\lambda} > 1$. Thus $f(t)$ has a degenerate Laplace-type integral transform, for $\frac{1-uC}{u\lambda} > 1$.

Theorem 2. *Let $a, b \in \mathbb{R}$ and let $f(t)$ and $h(t)$ be function whose degenerate Laplace-type integral exists. Then*

$$
\mathcal{G}_{\alpha,\lambda}\{af(t) + bh(t)\} = a\mathcal{G}_{\alpha,\lambda}\{f(t)\} + b\mathcal{G}_{\alpha,\lambda}\{h(t)\}.
$$

Proof. Let $a, b \in \mathbb{R}$ and $f(t)$ and $h(t)$ be any function whose degenerate Laplace-type integral exists. Then

$$
\begin{aligned}
\mathcal{G}_{\alpha,\lambda}\{af(t) + bh(t)\} &= u^{\alpha} \int_0^{\infty} e_{\lambda}^{-\frac{1}{u}}(t) \Big[af(t) + bh(t) \Big] dt \\
&= au^{\alpha} \int_0^{\infty} e_{\lambda}^{-\frac{1}{u}}(t) f(t) dt + bu^{\alpha} \int_0^{\infty} e_{\lambda}^{-\frac{1}{u}}(t) h(t) dt \\
&= a\mathcal{G}_{\alpha,\lambda}\{f(t)\} + b\mathcal{G}_{\alpha,\lambda}\{h(t)\}.
\end{aligned}
$$

Thus, linearity property of the degenerate Laplace-type integral transform holds true.

4. Degenerate Laplace-type Integral Transform of Some Elementary Functions

In this section the researcher establish the degenerate Laplace-type integral transform of some elementary functions.

Theorem 3. *The degenerate Laplace-type integral transform of the function $f(t) = 1$ is given by*

$$
\mathcal{G}_{\alpha,\lambda}\{1\} = \frac{u^{\alpha+1}}{1-\lambda u}, \ \text{for } \lambda u < 1. \tag{11}
$$

Proof. By Definition 8, for $f(t) = 1$, we have

$$\begin{aligned}\mathcal{G}_{\alpha,\lambda}\{1\} =& u^{\alpha}\int_0^{\infty} e_{\lambda}^{-\frac{1}{u}}(t)dt = u^{\alpha}\lim_{R\to\infty}\int_0^{R}(1+\lambda t)^{-\frac{1}{u\lambda}}dt\\ =& u^{\alpha+1}\lim_{R\to\infty}\left[\frac{(1+\lambda t)^{1-\frac{1}{u\lambda}}}{(\lambda u-1)}\right]\Bigg|_0^R\\ =& u^{\alpha+1}\lim_{R\to\infty}\left[\frac{(1+\lambda R)^{1-\frac{1}{u\lambda}}}{(\lambda u-1)}-\frac{1}{(\lambda u-1)}\right]=\frac{u^{\alpha+1}}{1-\lambda u},\ \text{for } \lambda u<1.\end{aligned}$$

Remark 1. It is clear from Theorem 3 and equation (9) that

$$\lim_{\lambda\to 0}\mathcal{G}_{\alpha,\lambda}\{1\} = \lim_{\lambda\to 0}\frac{u^{\alpha+1}}{1-\lambda u} = u^{\alpha+1} = \mathcal{G}_{\alpha}\{1\}.$$

Theorem 4. *The degenerate Laplace-type integral transform of the function $f(t) = t$ is given by*

$$\mathcal{G}_{\alpha,\lambda}\{t\} = \frac{u^{\alpha+2}}{(1-u\lambda)(1-2u\lambda)},\ \textit{for } 2u\lambda < 1. \tag{12}$$

Proof. By Definition 8, for $f(t) = t$, we have

$$\begin{aligned}\mathcal{G}_{\alpha,\lambda}\{t\} =& u^{\alpha}\int_0^{\infty} e_{\lambda}^{-\frac{1}{u}}(t)tdt\\ =& \frac{u^{\alpha}}{\lambda^2}\lim_{R\to\infty}\left[\frac{(1+\lambda t)^{2-\frac{1}{u\lambda}}}{\frac{1}{u\lambda}(2u\lambda-1)}-\frac{(1+\lambda t)^{1-\frac{1}{u\lambda}}}{\frac{1}{u\lambda}(u\lambda-1)}\right]\Bigg|_0^R\\ =& \frac{u^{\alpha+1}}{\lambda}\lim_{R\to\infty}\left[\left(\frac{(1+\lambda R)^{\frac{2u\lambda-1}{u\lambda}}}{2u\lambda-1}-\frac{(1+\lambda R)^{\frac{u\lambda-1}{u\lambda}}}{u\lambda-1}\right)\right.\\ &\left.-\left(\frac{1}{2u\lambda-1}-\frac{1}{u\lambda-1}\right)\right]\\ =& \frac{u^{\alpha+2}}{(1-u\lambda)(1-2u\lambda)},\ \text{for } 2u\lambda<1.\end{aligned}$$

Remark 2. It is clear from Theorem 4 and equation (9) that

$$\lim_{\lambda\to 0}\mathcal{G}_{\alpha,\lambda}\{t\} = \lim_{\lambda\to 0}\left[\frac{u^{\alpha+2}}{(1-u\lambda)(1-2u\lambda)}\right] = u^{\alpha+2} = \mathcal{G}_{\alpha}\{t\}.$$

Theorem 5. *The degenerate Laplace-type integral transform of the function $f(t) = t^n$ is given by*

$$\mathcal{G}_{\alpha,\lambda}\{t^n\} = \frac{n!u^{\alpha+1+n}}{(1-u\lambda)\cdots(1-(n+1)u\lambda)},\quad \textit{for}\quad (n-k+1)u\lambda < 1. \tag{13}$$

Proof. By Definition 8, for $f(t) = t^n$, we obtain

$$\begin{aligned}
\mathcal{G}_{\alpha,\lambda}\{t^n\} =& u^\alpha \int_0^\infty t^n e_\lambda^{-\frac{1}{u}}(t)dt \\
=& \frac{u^\alpha}{\lambda^{n+1}} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \lim_{R\to\infty} \left[\frac{(1+\lambda t)^{n-k-\frac{1}{u\lambda}+1}}{\frac{1}{u\lambda}(nu\lambda - ku\lambda - 1 + u\lambda)} \right] \Bigg|_0^R \\
=& \frac{u^{\alpha+1}}{\lambda^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \left[-\frac{1}{nu\lambda - ku\lambda - 1 + u\lambda} \right], \text{ for } (n-k+1)u\lambda < 1 \\
=& \frac{u^{\alpha+1}}{\lambda^n} \left[\frac{n! u^n \lambda^n}{(1-(1)u\lambda)(1-(2)u\lambda)\cdots(1-(n)u\lambda)(1-(n+1)u\lambda)} \right], \\
& \text{for } (n-k+1)u\lambda < 1 \\
=& \frac{n! u^{\alpha+1+n}}{(1-u\lambda)(1-2u\lambda)\cdots(1-nu\lambda)(1-(n+1)u\lambda)}, \text{ for } (n-k+1)u\lambda < 1.
\end{aligned}$$

Remark 3. It is clear from Theorem 5 and equation (9) that

$$\lim_{\lambda\to 0} \mathcal{G}_{\alpha,\lambda}\{t^n\} = \lim_{\lambda\to 0} \left[\frac{n! u^{\alpha+1+n}}{(1-u\lambda)(1-2u\lambda)\cdots(1-(n+1)u\lambda)} \right] = n! u^{\alpha+1+n} = \mathcal{G}_\alpha\{t^n\}.$$

Theorem 6. *The degenerate Laplace-type integral transform of a function $f(t) = e_\lambda^a(t)$ is given by*

$$\mathcal{G}_{\alpha,\lambda}\{e_\lambda^a(t)\} = \frac{u^{\alpha+1}}{1-u(a+\lambda)}, \text{ for } (a+\lambda)u < 1. \tag{14}$$

Proof. By Definition 8, for $f(t) = e_\lambda^a(t)$, we set

$$\begin{aligned}
\mathcal{G}_{\alpha,\lambda}\{e_\lambda^a(t)\} =& u^\alpha \int_0^\infty e_\lambda^{-\frac{1}{u}}(t) \left[e_\lambda^a(t) \right] dt = u^\alpha \lim_{R\to\infty} \int_0^R (1+\lambda t)^{\frac{ua-1}{u\lambda}} dt. \\
=& u^{\alpha+1} \lim_{R\to\infty} \left[\frac{(1+\lambda t)^{\frac{ua-1+u\lambda}{u\lambda}}}{ua - 1 + u\lambda} \right] \Bigg|_0^R \\
=& u^{\alpha+1} \left[-\frac{1}{u(a+\lambda)-1} \right], \text{ for } (a+\lambda)u < 1 \\
=& \frac{u^{\alpha+1}}{1-u(a+\lambda)}, \text{ for } (a+\lambda)u < 1.
\end{aligned}$$

Remark 4. It is clear from Theorem 6 and equation (9) that

$$\lim_{\lambda\to 0} \mathcal{G}_{\alpha,\lambda}\{e_\lambda^a(t)\} = \lim_{\lambda\to 0} \left[\frac{u^{\alpha+1}}{1-u(a+\lambda)} \right] = \frac{u^{\alpha+1}}{1-ua} = \mathcal{G}_\alpha\{e^{at}\}.$$

Theorem 7. *The degenerate Laplace-type integral transform of a function $f(t) = e_\lambda^{ia}(t)$ is given by*

$$\mathcal{G}_{\alpha,\lambda}\{e_\lambda^{ia}(t)\} = \frac{u^{\alpha+1}}{1-u(ia+\lambda)},\ \textit{for } u\lambda < 1, \tag{15}$$

where a is any positive constant.

Proof. By Definition 8, for $f(t) = e_\lambda^{ia}(t)$, we set

$$\begin{aligned}\mathcal{G}_{\alpha,\lambda}\{e_\lambda^{ia}(t)\} =& u^\alpha \int_0^\infty e_\lambda^{-\frac{1}{u}}(t)\Big[e_\lambda^{ia}(t)\Big]dt = u^\alpha \lim_{R\to\infty}\int_0^R (1+\lambda t)^{\frac{uia-1}{u\lambda}}dt \\ =& u^{\alpha+1}\lim_{R\to\infty}\left[\frac{(1+\lambda t)^{\frac{-1+u\lambda}{u\lambda}}e_\lambda^{ia}(t)}{uia-1+u\lambda}\right]\Bigg|_0^R.\end{aligned}$$

By the definition of degenerate Euler formula in Definition 5,

$$e_\lambda^{ia}(t) = \cos_\lambda^{(a)}(t) + i\sin_\lambda^{(a)}(t) = \cos\left(\frac{a}{\lambda}\log\big(1+\lambda t\big)\right) + i\sin\left(\frac{a}{\lambda}\log\big(1+\lambda t\big)\right).$$

Thus, we have

$$\begin{aligned}\mathcal{G}_{\alpha,\lambda}\{e_\lambda^{ia}(t)\} =& u^{\alpha+1}\lim_{R\to\infty}\left[\frac{(1+\lambda t)^{\frac{-1+u\lambda}{u\lambda}}e_\lambda^{ia}(t)}{uia-1+u\lambda}\right]\Bigg|_0^R \\ =& u^{\alpha+1}\lim_{R\to\infty}\left[\frac{(1+\lambda t)^{\frac{-1+u\lambda}{u\lambda}}\left[\cos\left(\frac{a}{\lambda}\log(1+\lambda t)\right)+i\sin\left(\frac{a}{\lambda}\log(1+\lambda t)\right)\right]}{uia-1+u\lambda}\right]\Bigg|_0^R \\ =& u^{\alpha+1}\left[-\frac{1}{uia-1+u\lambda}\right] = \frac{u^{\alpha+1}}{1-u(ia-\lambda)},\ \text{for } u\lambda<1.\end{aligned}$$

Theorem 8. *The degenerate Laplace-type integral transform of the degenerate sine function $f(t) = \sin_\lambda^{(a)}(t)$ is given by*

$$\mathcal{G}_{\alpha,\lambda}\{\sin_\lambda^{(a)}(t)\} = \frac{au^{\alpha+2}}{(1-\lambda u)^2+u^2a^2}. \tag{16}$$

Proof. By the definition of the degenerate sine in Definition 3, we have

$$\mathcal{G}_{\alpha,\lambda}\{\sin_\lambda^{(a)}(t)\} = \mathcal{G}_{\alpha,\lambda}\left\{\frac{e_\lambda^{ia}(t)-e_\lambda^{-ia}(t)}{2i}\right\}.$$

Now using Theorem 2 and Theorem 7, we obtain

$$\mathcal{G}_{\alpha,\lambda}\{\sin_\lambda^{(a)}(t)\} = \frac{1}{2i}\left[\mathcal{G}_{\alpha,\lambda}\{e_\lambda^{ia}(t)\} - \mathcal{G}_{\alpha,\lambda}\{e_\lambda^{-ia}(t)\}\right]$$

$$
\begin{aligned}
&= \frac{1}{2i}\left[\frac{u^{\alpha+1}}{1-u(ia+\lambda)} - \frac{u^{\alpha+1}}{1-u(-ia+\lambda)}\right] \\
&= \frac{au^{\alpha+2}}{(1-\lambda u)^2 + u^2a^2}.
\end{aligned}
$$

Remark 5. It is clear from Theorem 8 and equation (9) that

$$
\lim_{\lambda\to 0} \mathcal{G}_{\alpha,\lambda}\{\sin_\lambda^{(a)}(t)\} = \lim_{\lambda\to 0}\left[\frac{au^{\alpha+2}}{(1-\lambda u)^2 + u^2a^2}\right] = \frac{au^{\alpha+2}}{1+u^2a^2} = \mathcal{G}_\alpha\{\sin at\}.
$$

Theorem 9. *The degenerate Laplace-type integral transform of the degenerate cosine function* $f(t) = \cos_\lambda^{(a)}(t)$ *is given by*

$$
\mathcal{G}_{\alpha,\lambda}\{\cos_\lambda^{(a)}(t)\} = \frac{(1-\lambda u)u^{\alpha+1}}{(1-\lambda u)^2 + u^2a^2}. \tag{17}
$$

Proof. By the definition of the degenerate cosine in Definition 4 and using Theorem 2 and Theorem 7, we obtain

$$
\mathcal{G}_{\alpha,\lambda}\{\cos_\lambda^{(a)}(t)\} = \frac{1}{2}\left[\mathcal{G}_{\alpha,\lambda}\{e_\lambda^{ia}(t)\} + \mathcal{G}_{\alpha,\lambda}\{e_\lambda^{-ia}(t)\}\right] = \frac{(1-\lambda u)u^{\alpha+1}}{(1-\lambda u)^2 + u^2a^2}.
$$

Remark 6. It is clear from Theorem 9 and equation (9) that

$$
\lim_{\lambda\to 0} \mathcal{G}_{\alpha,\lambda}\{\cos_\lambda^{(a)}(t)\} = \lim_{\lambda\to 0}\left[\frac{(1-\lambda u)u^{\alpha+1}}{(1-\lambda u)^2 + u^2a^2}\right] = \frac{u^{\alpha+1}}{1+u^2a^2} = \mathcal{G}_\alpha\{\cos at\}.
$$

Theorem 10. *The degenerate Laplace-type integral transform of the degenerate hyperbolic sine function* $f(t) = \sinh_\lambda^{(a)}(t)$ *is given by*

$$
\mathcal{G}_{\alpha,\lambda}\{\sinh_\lambda^{(a)}(t)\} = \frac{au^{\alpha+2}}{(1-\lambda u)^2 - u^2a^2}. \tag{18}
$$

Proof. By the definition of the degenerate hyperbolic sine in Definition 6, we have

$$
\mathcal{G}_{\alpha,\lambda}\{\sinh_\lambda^{(a)}(t)\} = \mathcal{G}_{\alpha,\lambda}\left\{\frac{e_\lambda^{a}(t) - e_\lambda^{-a}(t)}{2}\right\}.
$$

Now, using Theorem 2 and Theorem 6, we obtain

$$
\begin{aligned}
\mathcal{G}_{\alpha,\lambda}\{\sinh_\lambda^{(a)}(t)\} &= \frac{1}{2}\left[\mathcal{G}_{\alpha,\lambda}\{e_\lambda^{a}(t)\} - \mathcal{G}_{\alpha,\lambda}\{e_\lambda^{-a}(t)\}\right] \\
&= \frac{1}{2}\left[\frac{u^{\alpha+1}}{1-u(a+\lambda)} - \frac{u^{\alpha+1}}{1-u(-a+\lambda)}\right] \\
&= \frac{u^{\alpha+1}}{2}\left[\frac{2au}{(1-\lambda u)^2 - u^2a^2}\right] = \frac{au^{\alpha+2}}{(1-\lambda u)^2 - u^2a^2}.
\end{aligned}
$$

Remark 7. It is clear from Theorem 10 and equation (9) that

$$\lim_{\lambda\to 0}\mathcal{G}_{\alpha,\lambda}\{\sinh_\lambda^{(a)}(t)\} = \lim_{\lambda\to 0}\left[\frac{au^{\alpha+2}}{(1-\lambda u)^2-u^2a^2}\right] = \frac{au^{\alpha+2}}{1-u^2a^2} = \mathcal{G}_\alpha\{\sinh at\}.$$

Theorem 11. *The degenerate Laplace-type integral transform of the degenerate hyperbolic cosine function $f(t)=\cosh_\lambda^{(a)}(t)$ is given by*

$$\mathcal{G}_{\alpha,\lambda}\{\cosh_\lambda^{(a)}(t)\} = \frac{(1-\lambda u)u^{\alpha+1}}{(1-\lambda u)^2-u^2a^2}. \tag{19}$$

Proof. By the definition of the degenerate hyperbolic cosine in [**?**] and using Theorem 2 and Theorem 7, we obtain

$$\begin{aligned}\mathcal{G}_{\alpha,\lambda}\{\cosh_\lambda^{(a)}(t)\} =&\frac{1}{2}\left[\mathcal{G}_{\alpha,\lambda}\{e_\lambda^a(t)\}+\mathcal{G}_{\alpha,\lambda}\{e_\lambda^{-a}(t)\}\right]\\ =&\frac{1}{2}\left[\frac{u^{\alpha+1}}{1-u(a+\lambda)}+\frac{u^{\alpha+1}}{1-u(-a+\lambda)}\right] = \frac{(1-\lambda u)u^{\alpha+1}}{(1-\lambda u)^2-u^2a^2}.\end{aligned}$$

Remark 8. It is clear from Theorem 11 and equation (9) that

$$\lim_{\lambda\to 0}\mathcal{G}_{\alpha,\lambda}\{\cosh_\lambda^{(a)}(t)\} = \lim_{\lambda\to 0}\left[\frac{(1-\lambda u)u^{\alpha+1}}{(1-\lambda u)^2-u^2a^2}\right] = \frac{u^{\alpha+1}}{1-u^2a^2} = \mathcal{G}_\alpha\{\cosh at\}.$$

Theorem 12. *The degenerate Laplace-type integral transform of the function $f(t) = e_\lambda^a(t)\sin_\lambda^{(b)}(t)$ is given by*

$$\mathcal{G}_{\alpha,\lambda}\{e_\lambda^a(t)\sin_\lambda^{(b)}(t)\} = \frac{bu^{\alpha+2}}{(1-au-u\lambda)^2+b^2u^2}. \tag{20}$$

Proof. By the definition of the degenerate sine in in Definition 3, we have

$$e_\lambda^a(t)\sin_\lambda^{(b)}(t) =e_\lambda^a(t)\left[\frac{e_\lambda^{ib}(t)-e_\lambda^{-ib}(t)}{2i}\right] = \frac{e_\lambda^{a+ib}(t)-e_\lambda^{a-ib}(t)}{2i}.$$

Hence, by Theorem 2 and Theorem 7, we obtain

$$\begin{aligned}\mathcal{G}_{\alpha,\lambda}\{e_\lambda^a(t)\sin_\lambda^{(b)}(t)\} =&\mathcal{G}_{\alpha,\lambda}\left\{\frac{e_\lambda^{a+ib}(t)-e_\lambda^{a-ib}(t)}{2i}\right\}\\ =&\frac{1}{2i}\left[\mathcal{G}_{\alpha,\lambda}\left\{e_\lambda^{a+ib}(t)\right\}-\mathcal{G}_{\alpha,\lambda}\left\{e_\lambda^{a-ib}(t)\right\}\right]\\ =&\frac{1}{2i}\left[\frac{u^{\alpha+1}}{1-u((a+ib)+\lambda)}-\frac{u^{\alpha+1}}{1-u((a-ib)+\lambda)}\right]\\ =&\frac{bu^{\alpha+2}}{(1-au-u\lambda)^2+b^2u^2}.\end{aligned}$$

Remark 9. It is clear from Theorem 12 and equation (9) that

$$\lim_{\lambda\to 0}\mathcal{G}_{\alpha,\lambda}\{e_\lambda^a(t)\sin_\lambda^{(b)}(t)\} = \lim_{\lambda\to 0}\left[\frac{bu^{\alpha+2}}{(1-au-u\lambda)^2+b^2u^2}\right]$$
$$= \frac{bu^{\alpha+2}}{(1-au)^2+b^2u^2} = \mathcal{G}_\alpha\{e^{at}\sin bt\}.$$

Theorem 13. *The degenerate Laplace-type integral transform of the function* $f(t) = e_\lambda^a(t)\cos_\lambda^{(b)}(t)$ *is given by*

$$\mathcal{G}_{\alpha,\lambda}\{e_\lambda^a(t)\cos_\lambda^{(b)}(t)\} = \frac{(1-au-u\lambda)u^{\alpha+1}}{(1-au-u\lambda)^2+b^2u^2}. \tag{21}$$

Proof. By the definition of the degenerate hyperbolic cosine in Definition 7, Theorem 2 and Theorem 7, we obtain

$$\mathcal{G}_{\alpha,\lambda}\{e_\lambda^a(t)\cos_\lambda^{(b)}(t)\} = \mathcal{G}_{\alpha,\lambda}\left\{\frac{e_\lambda^{a+ib}(t)+e_\lambda^{a-ib}(t)}{2}\right\} = \frac{(1-au-u\lambda)u^{\alpha+1}}{(1-au-u\lambda)^2+b^2u^2}.$$

Remark 10. It is clear from Theorem 13 and equation (9) that

$$\lim_{\lambda\to 0}\mathcal{G}_{\alpha,\lambda}\{e_\lambda^a(t)\cos_\lambda^{(b)}(t)\} = \lim_{\lambda\to 0}\left[\frac{(1-au-u\lambda)u^{\alpha+1}}{(1-au-u\lambda)^2+b^2u^2}\right]$$
$$= \frac{(1-au)u^{\alpha+1}}{(1-au)^2+b^2u^2} = \mathcal{G}_\alpha\{e^{at}\cos bt\}.$$

4.1. Degenerate Laplace-type Integral Transform of Derivative

Theorem 14. *If* $f(t), f'(t), ..., f^{(n-1)}(t)$ *are continuous and* $f^{(n)}(t)$ *is a piecewise-continuous function on* $[0,\infty)$ *and has a degenerate exponential order at infinity with* $\left|f^{(n)}(t)\right| \leq Me_\lambda^C(t)$ *for* $t \geq C$*, where* C *is a constant, then the following hold:*

$$(i.)\quad \mathcal{G}_{\alpha,\lambda}\{f'(t)\} = \frac{1}{u}\mathcal{G}_{\alpha,\lambda}\Big\{(1+\lambda t)^{-1}f(t)\Big\} - u^\alpha f(0)$$

$$(ii.)\quad \mathcal{G}_{\alpha,\lambda}\{f''(t)\} = \frac{1}{u^2}(1+\lambda u)\mathcal{G}_{\alpha,\lambda}\Big\{(1+\lambda t)^{-2}f(t)\Big\} - u^{\alpha-1}f(0) - u^\alpha f'(0)$$

$$(iii.)\quad \mathcal{G}_{\alpha,\lambda}\{f^{(n)}(t)\} = \frac{1}{u^n}\mathcal{G}_{\alpha,\lambda}\Big\{(1+\lambda t)^{-n}f(t)\Big\}\prod_{l=1}^{n-1}\Big(1+lu\lambda\Big) - u^{\alpha+1-n}\sum_{i=0}^{n-1}u^i f^{(i)}(0)\left[\prod_{l=1}^{n-i-2}\Big(1+lu\lambda\Big)\right],$$

where $f^{(n)}(t) = \left(\frac{d}{dt}\right)^n f(t)$ *and* $n = 1, 2, 3, 4, \cdots$.

Proof. First we prove $(i.)$. By Definition 8, we have

$$\mathcal{G}_{\alpha,\lambda}\{f'(t)\} = u^{\alpha}\int_0^{\infty}(1+\lambda t)^{-\frac{1}{u\lambda}}f'(t)dt = u^{\alpha}\lim_{R\to\infty}\int_0^{R}(1+\lambda t)^{-\frac{1}{u\lambda}}f'(t)dt.$$

Using integration by parts, we get

$$\begin{aligned}\mathcal{G}_{\alpha,\lambda}\{f'(t)\} =& u^{\alpha}\lim_{R\to\infty}\left[(1+\lambda t)^{-\frac{1}{u\lambda}}f(t)\Big|_0^R\right]+\frac{u^{\alpha}}{u}\lim_{R\to\infty}\left[\int_0^R(1+\lambda t)^{-1-\frac{1}{u\lambda}}f(t)dt\right]\\ =& u^{\alpha}\lim_{R\to\infty}\left[(1+\lambda R)^{-\frac{1}{u\lambda}}f(R)-(1+\lambda 0)^{-\frac{1}{u\lambda}}f(0)\right]+\frac{1}{u}\mathcal{G}_{\alpha,\lambda}\Big\{(1+\lambda t)^{-1}f(t)\Big\}\\ =& \frac{1}{u}\mathcal{G}_{\alpha,\lambda}\Big\{(1+\lambda t)^{-1}f(t)\Big\}-u^{\alpha}f(0).\end{aligned}$$

For $(ii.)$, using Definition 8, we have

$$\mathcal{G}_{\alpha,\lambda}\{f''(t)\} = u^{\alpha}\int_0^{\infty}(1+\lambda t)^{-\frac{1}{u\lambda}}f''(t)dt = u^{\alpha}\lim_{R\to\infty}\int_0^{R}(1+\lambda t)^{-\frac{1}{u\lambda}}f''(t)dt.$$

Using integration by parts, we have

$$\begin{aligned}\mathcal{G}_{\alpha,\lambda}\{f''(t)\} =& u^{\alpha}\lim_{R\to\infty}\left[(1+\lambda t)^{-\frac{1}{u\lambda}}f'(t)\Big|_0^R-\int_0^R\Big(-\frac{1}{u}\Big)(1+\lambda t)^{-1-\frac{1}{u\lambda}}f'(t)dt\right]\\ =& -u^{\alpha}f'(0)+\frac{u^{\alpha}}{u}\lim_{R\to\infty}\left[\int_0^R(1+\lambda t)^{-1-\frac{1}{u\lambda}}f'(t)dt\right]\\ =& -u^{\alpha}f'(0)+\frac{u^{\alpha}}{u}\lim_{R\to\infty}\Bigg[(1+\lambda t)^{-1-\frac{1}{u\lambda}}f(t)\Big|_0^R\\ &-\int_0^R\Big(-\lambda-\frac{1}{u}\Big)(1+\lambda t)^{-2-\frac{1}{u\lambda}}f(t)dt\Bigg]\\ =& -u^{\alpha}f'(0)-u^{\alpha-1}f(0)+\frac{1}{u^2}(1+\lambda u)u^{\alpha}\lim_{R\to\infty}\left[\int_0^R(1+\lambda t)^{-2-\frac{1}{u\lambda}}f(t)dt\right]\\ =& \frac{1}{u^2}(1+\lambda u)\mathcal{G}_{\alpha,\lambda}\Big\{(1+\lambda t)^{-2}f(t)\Big\}-u^{\alpha-1}f(0)-u^{\alpha}f'(0).\end{aligned}$$

For $(iii.)$, we prove

$$\begin{aligned}\mathcal{G}_{\alpha,\lambda}\{f^{(n)}(t)\} =& \frac{1}{u^n}\mathcal{G}_{\alpha,\lambda}\Big\{(1+\lambda t)^{-n}f(t)\Big\}\prod_{l=1}^{n-1}\Big(1+lu\lambda\Big)\\ &-u^{\alpha+1-n}\sum_{i=0}^{n-1}u^i f^{(i)}(0)\left[\prod_{l=1}^{n-i-2}\Big(1+lu\lambda\Big)\right]\end{aligned}\tag{22}$$

by induction. From results ($i.$) and ($ii.$), equation (22) holds for $n = 1$ and $n = 2$. Assume that equation (22) is true for $n = k$. Let $g(t) = f^{(k)}(t)$, then by the result of ($i.$),

$$\begin{aligned}\mathcal{G}_{\alpha,\lambda}\{f^{(k+1)}(t)\} =&\mathcal{G}_{\alpha,\lambda}\{g'(t)\} = \frac{1}{u}\mathcal{G}_{\alpha,\lambda}\Big\{(1+\lambda t)^{-1}g(t)\Big\} - u^{\alpha}g(0)\\ =&\frac{1}{u}\mathcal{G}_{\alpha,\lambda}\Big\{(1+\lambda t)^{-1}f^{(k)}(t)\Big\} - u^{\alpha}f^{(k)}(0). \end{aligned} \tag{23}$$

Now,

$$\begin{aligned}\mathcal{G}_{\alpha,\lambda}\Big\{(1+\lambda t)^{-1}f^{(k)}(t)\Big\} =&u^{\alpha}\int_0^{\infty} e_{\lambda}^{-\frac{1}{u}}(t)(1+\lambda t)^{-1}f^{(k)}(t)dt\\ =&u^{\alpha}\int_0^{\infty}(1+\lambda t)^{-(\frac{1+u\lambda}{u\lambda})}f^{(k)}(t)dt\\ =&(1+u\lambda)^{\alpha}\left(\frac{u}{1+u\lambda}\right)^{\alpha}\int_0^{\infty} e_{\lambda}^{-\frac{1}{\frac{u}{1+u\lambda}}}(t)f^{(k)}(t)dt. \end{aligned} \tag{24}$$

By inductive hypothesis, we have

$$\begin{aligned}\left(\frac{u}{1+u\lambda}\right)^{\alpha}\int_0^{\infty} e_{\lambda}^{-\frac{1}{\frac{u}{1+u\lambda}}}(t)f^{(k)}(t)dt =&\frac{1}{(\frac{u}{1+u\lambda})^k}\mathcal{G}_{\alpha,\lambda}\Big\{(1+\lambda t)^{-k}f(t)\Big\}\prod_{l=1}^{k-1}\left(1+\frac{lu\lambda}{1+u\lambda}\right)\\ &-\left(\frac{u}{1+u\lambda}\right)^{\alpha+1-k}\sum_{i=0}^{k-1}\left(\frac{u}{1+u\lambda}\right)^{i}f^{(i)}(0)\\ &\left[\prod_{l=1}^{k-i-2}\left(1+\frac{lu\lambda}{1+u\lambda}\right)\right].\end{aligned}$$

Observe that

$$\begin{aligned}\mathcal{G}_{\alpha,\lambda}\Big\{(1+\lambda t)^{-k}f(t)\Big\} =&\left(\frac{u}{1+u\lambda}\right)^{\alpha}\int_0^{\infty} e_{\lambda}^{-\frac{1}{\frac{u}{1+u\lambda}}}(t)(1+\lambda t)^{-k}f(t)dt\\ =&\frac{1}{(1+u\lambda)^{\alpha}}\left[\mathcal{G}_{\alpha,\lambda}\Big\{(1+\lambda t)^{-(k+1)}f(t)\Big\}\right].\end{aligned}$$

Thus, the RHS of equation (23) becomes

$$\begin{aligned}RHS =&\frac{(1+u\lambda)^k}{u^k}\left[\frac{1}{(1+u\lambda)^{\alpha}}\mathcal{G}_{\alpha,\lambda}\Big\{(1+\lambda t)^{-(k+1)}f(t)\Big\}\right]\prod_{l=1}^{k-1}\frac{(1+(l+1)u\lambda)}{1+u\lambda}\\ &-\frac{u^{\alpha+1-k}}{(1+u\lambda)^{\alpha+1-k}}\sum_{i=0}^{k-1}\frac{u^i}{(1+u\lambda)^i}f^{(i)}(0)\prod_{l=1}^{k-i-2}\frac{(1+(l+1)u\lambda)}{1+u\lambda}\\ =&\frac{1}{u^k(1+u\lambda)^{\alpha}}\left[\mathcal{G}_{\alpha,\lambda}\Big\{(1+\lambda t)^{-(k+1)}f(t)\Big\}\right]\left[(1+u\lambda)\prod_{l=1}^{k-1}\Big(1+(l+1)u\lambda\Big)\right]\\ &-\left(\frac{u^{\alpha-k}}{(1+u\lambda)^{\alpha-k}}\right)\sum_{i=0}^{k-1}\left(\frac{u}{1+u\lambda}\right)^{i+1}f^{(i)}(0)\left[\frac{1}{(1+u\lambda)^{k-i-2}}\prod_{l=1}^{k-i-2}\Big(1+(l+1)u\lambda\Big)\right]\end{aligned}$$

$$
\begin{aligned}
=&\frac{1}{u^k(1+u\lambda)^\alpha}\left[\mathcal{G}_{\alpha,\lambda}\Big\{(1+\lambda t)^{-(k+1)}f(t)\Big\}\right]\left[\prod_{l=1}^{k}\Big(1+lu\lambda\Big)\right]\\
&-\left(\frac{u^{\alpha-k}}{(1+u\lambda)^\alpha}\right)\sum_{i=0}^{k-1}u^{i+1}f^{(i)}(0)\left[\prod_{l=1}^{k-i-1}\Big(1+lu\lambda\Big)\right].
\end{aligned}
$$

Hence, the RHS of equation (22) becomes

$$
\begin{aligned}
RHS =&(1+u\lambda)^\alpha\left[\frac{1}{u^k(1+u\lambda)^\alpha}\mathcal{G}_{\alpha,\lambda}\Big\{(1+\lambda t)^{-(k+1)}f(t)\Big\}\prod_{l=1}^{k}\Big(1+lu\lambda\Big)\right.\\
&\left.-\frac{u^{\alpha-k}}{(1+u\lambda)^\alpha}\sum_{i=0}^{k-1}u^{i+1}f^{(i)}(0)\prod_{l=1}^{k-i-1}\Big(1+lu\lambda\Big)\right]\\
=&\frac{1}{u^k}\mathcal{G}_{\alpha,\lambda}\Big\{(1+\lambda t)^{-(k+1)}f(t)\Big\}\prod_{l=1}^{k}\Big(1+lu\lambda\Big)\\
&-u^{\alpha-k}\sum_{i=0}^{k-1}u^{i+1}f^{(i)}(0)\prod_{l=1}^{k-i-1}\Big(1+lu\lambda\Big).
\end{aligned}
$$

So, the RHS of equation (24) becomes

$$
\begin{aligned}
RHS =&\frac{1}{u}\left[\frac{1}{u^k}\mathcal{G}_{\alpha,\lambda}\Big\{(1+\lambda t)^{-(k+1)}f(t)\Big\}\prod_{l=1}^{k}\Big(1+lu\lambda\Big)\right.\\
&\left.-u^{\alpha-k}\sum_{i=0}^{k-1}u^{i+1}f^{(i)}(0)\prod_{l=1}^{k-i-1}(1+lu\lambda)\right]-u^\alpha f^k(0)\\
=&\frac{1}{u^{k+1}}\mathcal{G}_{\alpha,\lambda}\Big\{(1+\lambda t)^{-(k+1)}f(t)\Big\}\prod_{l=1}^{k}(1+lu\lambda)\\
&-u^{\alpha-k}\sum_{i=0}^{k}u^{i}f^{(i)}(0)\prod_{l=1}^{k-i-1}\Big(1+lu\lambda\Big).
\end{aligned}
$$

Hence, $\mathcal{G}_{\alpha,\lambda}\{f^{(k+1)}(t)\}$ holds. Therefore, by induction equation (22) holds for all $n \geq 1$.

4.2. Degenerate Laplace-type Integral Transform of an Integral

Theorem 15. *Let $\mathcal{G}_{\alpha,\lambda}\{f(t)\} = F_{\alpha\lambda}(u)$. Then*

$$
\mathcal{G}_{\alpha,\lambda}\{f(t)\} = \frac{1}{u}\mathcal{G}_{\alpha,\lambda}\Big\{(1+\lambda t)^{-1}\int_0^t f(s)ds\Big\}.
$$

Proof.

$$\text{Let } g(t) = \int_0^t f(s)ds. \text{ Then } g'(t) = \frac{d}{dt}\left\{\int_0^t f(s)ds\right\} = f(t) \text{ and } g(0) = 0.$$

Note that by Theorem 14 ($i.$) we have

$$\begin{aligned}\mathcal{G}_{\alpha,\lambda}\{g'(t)\} =& \frac{1}{u}\mathcal{G}_{\alpha,\lambda}\Big\{(1+\lambda t)^{-1}g(t)\Big\} - u^{\alpha}g(0)\\ =& \frac{1}{u}\mathcal{G}_{\alpha,\lambda}\Big\{(1+\lambda t)^{-1}\int_0^t f(s)ds\Big\}.\end{aligned}$$

4.3. The First Translation Theorem for the Degenerate Laplace-type Integral Transform

Theorem 16. *If* $\mathcal{G}_{\alpha,\lambda}\{f(t)\} = F_{\alpha\lambda}(u)$ *then*

$$\mathcal{G}_{\alpha,\lambda}\{e_\lambda^a(t)f(t)\} = (1-au)^{\alpha}F_{\alpha,\lambda}\left(\frac{u}{1-ua}\right), \text{ for } a \neq \frac{1}{u}.$$

Proof. From Definition 8, it follows that

$$\begin{aligned}\mathcal{G}_{\alpha,\lambda}\{e_\lambda^a(t)f(t)\} =& u^{\alpha}\int_0^{\infty} e_\lambda^a(t)e_\lambda^{-\frac{1}{u}}(t)f(t)dt\\ =& \frac{(1-au)^{\alpha}}{(1-au)^{\alpha}}u^{\alpha}\int_0^{\infty}(1+\lambda t)^{-\frac{1}{(\frac{u}{1-au})\lambda}}f(t)dt\\ =& (1-au)^{\alpha}\left(\frac{u}{1-au}\right)^{\alpha}\int_0^{\infty}(1+\lambda t)^{-\frac{1}{(\frac{u}{1-au})\lambda}}f(t)dt\\ =& (1-au)^{\alpha}F_{\alpha,\lambda}\left(\frac{u}{1-ua}\right), \text{ for } a \neq \frac{1}{u}.\end{aligned}$$

4.4. The Change of Scale Property for the Degenerate Laplace-type Integral Transform

Theorem 17. *If* $\mathcal{G}_{\alpha,\lambda}\{f(t)\} = F_{\alpha,\lambda}(u)$ *then*

$$\mathcal{G}_{\alpha,\lambda}\{f(at)\} = \frac{1}{a^{\alpha+1}}F_{\alpha,\frac{\lambda}{a}}(au), \text{ for } a > 0.$$

Proof. From Definition 8, it follows that

$$\mathcal{G}_{\alpha,\lambda}\{f(at)\} = u^{\alpha}\int_0^{\infty} e_\lambda^{-\frac{1}{u}}(t)f(at)dt = u^{\alpha}\int_0^{\infty}(1+\lambda t)^{-\frac{1}{u\lambda}}f(at)dt.$$

Let $w = at$, then $dw = adt$ and $t = \dfrac{w}{a}$. Hence

$$\begin{aligned}\mathcal{G}_{\alpha,\lambda}\{f(at)\} =& u^{\alpha}\int_0^{\infty}(1+\lambda t)^{-\frac{1}{u\lambda}}f(at)dt\\ =&\frac{1}{a^{\alpha+1}}\left[(ua)^{\alpha}\int_0^{\infty}\left(1+\left(\frac{\lambda}{a}\right)w\right)^{-\frac{1}{(\frac{\lambda}{a})au}}f(w)dw\right]\\ =&\frac{1}{a^{\alpha+1}}F_{\alpha,\frac{\lambda}{a}}(au),\ \text{for } a>0.\end{aligned}$$

5. Generalization of the Degenerate Laplace, Degenerate Sumudu and Degenerate Elzaki Transform

Definition 9. The degenerate Laplace-type integral transform

$$\mathcal{G}_{\alpha,\lambda}\{f(t)\} = F_{\alpha,\lambda}(u) = u^{\alpha}\int_0^{\infty} e_{\lambda}^{-\frac{1}{u}}(t)f(t)dt = u^{\alpha}\int_0^{\infty}(1+\lambda t)^{-\frac{1}{u\lambda}}f(t)dt,$$

is the generalization of the degenerate Laplace, degenerate Sumudu and degenerate Elzaki transform, where $\alpha = 0, \alpha = -1$ and $\alpha = 1$, respectively.
That is, when $\alpha = 0$, we can have the degenerate Laplace transform, that is

$$\begin{aligned}\mathcal{G}_{0,\lambda}\{f(t)\} = F_{0,\lambda}(u) =& u^{0}\int_0^{\infty} e_{\lambda}^{-\frac{1}{u}}(t)f(t)dt = \int_0^{\infty}(1+\lambda t)^{-\frac{1}{u\lambda}}f(t)dt\\ =&\int_0^{\infty} e_{\lambda}^{-\frac{1}{u}}(t)f(t)dt = \mathcal{L}_{\lambda}\{f(t)\}.\end{aligned}$$

When $\alpha = -1$, we can have the degenerate Sumudu transform, that is

$$\begin{aligned}\mathcal{G}_{-1,\lambda}\{f(t)\} = F_{-1,\lambda}(u) =& u^{-1}\int_0^{\infty} e_{\lambda}^{-\frac{1}{u}}(t)f(t)dt = \frac{1}{u}\int_0^{\infty}(1+\lambda t)^{-\frac{1}{u\lambda}}f(t)dt\\ =&\frac{1}{u}\int_0^{\infty} e_{\lambda}^{-\frac{1}{u}}(t)f(t)dt = \mathcal{S}_{\lambda}\{f(t)\}.\end{aligned}$$

Lastly, when $\alpha = 1$, we can have the degenerate Elzaki transform, given by

$$\begin{aligned}\mathcal{G}_{1,\lambda}\{f(t)\} = F_{1,\lambda}(u) =& u^{1}\int_0^{\infty} e_{\lambda}^{-\frac{1}{u}}(t)f(t)dt = u\int_0^{\infty}(1+\lambda t)^{-\frac{1}{u\lambda}}f(t)dt\\ =&u\int_0^{\infty} e_{\lambda}^{-\frac{1}{u}}(t)f(t)dt = \mathcal{E}_{\lambda}\{f(t)\}.\end{aligned}$$

.

Table 1 gives the summary of some elementary functions of the degenerate Laplace-type and degenerate Laplace **[8, 14–16]**.

$f(t)$	$\mathcal{G}_{\alpha,\lambda}\{f(t)\}$	$\mathcal{G}_{0,\lambda}\{f(t)\} = \mathcal{L}_\lambda\{f(t)\}, \alpha = 0$
1	$\frac{u^{\alpha+1}}{1-\lambda u}$	$\frac{u}{1-\lambda u}$
t	$\frac{u^{\alpha+2}}{(1-u\lambda)(1-2u\lambda)}$	$\frac{u^2}{(1-u\lambda)(1-2u\lambda)}$
$t^n (n = 0, 1, 2, \cdots)$	$\frac{n!u^{\alpha+1+n}}{(1-u\lambda)\cdots(1-(n+1)u\lambda)}$	$\frac{n!u^{1+n}}{(1-u\lambda)\cdots(1-(n+1)u\lambda)}$
$e^a_\lambda(t)$	$\frac{u^{\alpha+1}}{1-u(a+\lambda)}$	$\frac{u}{1-u(a+\lambda)}$
$\sin^{(a)}_\lambda(t)$	$\frac{au^{\alpha+2}}{(1-\lambda u)^2+u^2a^2}$	$\frac{au^2}{(1-\lambda u)^2+u^2a^2}$
$\cos^{(a)}_\lambda(t)$	$\frac{(1-\lambda u)u^{\alpha+1}}{(1-\lambda u)^2+u^2a^2}$	$\frac{(1-\lambda u)u}{(1-\lambda u)^2+u^2a^2}$
$\sinh^{(a)}_\lambda(t)$	$\frac{au^{\alpha+2}}{(1-\lambda u)^2-u^2a^2}$	$\frac{au^2}{(1-\lambda u)^2-u^2a^2}$
$\cosh^{(a)}_\lambda(t)$	$\frac{(1-\lambda u)u^{\alpha+1}}{(1-\lambda u)^2-u^2a^2}$	$\frac{(1-\lambda u)u}{(1-\lambda u)^2-u^2a^2}$
$e^a_\lambda(t)\sin^{(b)}_\lambda(t)$	$\frac{bu^{\alpha+2}}{(1-au-u\lambda)^2+b^2u^2}$	$\frac{bu^2}{(1-au-u\lambda)^2+b^2u^2}$
$e^a_\lambda(t)\cos^{(b)}_\lambda(t)$	$\frac{(1-au-u\lambda)u^{\alpha+1}}{(1-au-u\lambda)^2+b^2u^2}$	$\frac{(1-au-u\lambda)u}{(1-au-u\lambda)^2+b^2u^2}$

Table 1: The degenerate Laplace-type and degenerate Laplace transform.

Table 2 gives the summary of some elementary functions degenerate Laplace-type and degenerate Sumudu transforms[**4**].

$f(t)$	$\mathcal{G}_{\alpha,\lambda}\{f(t)\}$	$\mathcal{G}_{-1,\lambda}\{f(t)\} = \mathcal{S}_{\lambda}\{f(t)\}, \alpha = -1$
1	$\frac{u^{\alpha+1}}{1-\lambda u}$	$\frac{1}{1-\lambda u}$
t	$\frac{u^{\alpha+2}}{(1-u\lambda)(1-2u\lambda)}$	$\frac{u}{(1-u\lambda)(1-2u\lambda)}$
$t^n (n = 0, 1, 2, \cdots)$	$\frac{n! u^{\alpha+1+n}}{(1-u\lambda)\cdots(1-(n+1)u\lambda)}$	$\frac{n! u^{n}}{(1-u\lambda)\cdots(1-(n+1)u\lambda)}$
$e_{\lambda}^{a}(t)$	$\frac{u^{\alpha+1}}{1-u(a+\lambda)}$	$\frac{1}{1-u(a+\lambda)}$
$\sin_{\lambda}^{(a)}(t)$	$\frac{a u^{\alpha+2}}{(1-\lambda u)^2+u^2a^2}$	$\frac{a u}{(1-\lambda u)^2+u^2a^2}$
$\cos_{\lambda}^{(a)}(t)$	$\frac{(1-\lambda u)u^{\alpha+1}}{(1-\lambda u)^2+u^2a^2}$	$\frac{1-\lambda u}{(1-\lambda u)^2+u^2a^2}$
$\sinh_{\lambda}^{(a)}(t)$	$\frac{a u^{\alpha+2}}{(1-\lambda u)^2-u^2a^2}$	$\frac{a u}{(1-\lambda u)^2-u^2a^2}$
$\cosh_{\lambda}^{(a)}(t)$	$\frac{(1-\lambda u)u^{\alpha+1}}{(1-\lambda u)^2-u^2a^2}$	$\frac{1-\lambda u}{(1-\lambda u)^2-u^2a^2}$
$e_{\lambda}^{a}(t)\sin_{\lambda}^{(b)}(t)$	$\frac{b u^{\alpha+2}}{(1-au-u\lambda)^2+b^2u^2}$	$\frac{b u}{(1-au-u\lambda)^2+b^2u^2}$
$e_{\lambda}^{a}(t)\cos_{\lambda}^{(b)}(t)$	$\frac{(1-au-u\lambda)u^{\alpha+1}}{(1-au-u\lambda)^2+b^2u^2}$	$\frac{1-au-u\lambda}{(1-au-u\lambda)^2+b^2u^2}$

Table 2: The degenerate Laplace-type and degenerate Sumudu transform.

Table 3 gives the summary of some elementary functions degenerate Laplace-type and degenerate Elzaki transforms [**1**].

$f(t)$	$\mathcal{G}_{\alpha,\lambda}\{f(t)\}$	$\mathcal{G}_{1,\lambda}\{f(t)\} = \mathcal{E}_{\lambda}\{f(t)\}, \alpha = 1$
1	$\frac{u^{\alpha+1}}{1-\lambda u}$	$\frac{u^2}{1-\lambda u}$
t	$\frac{u^{\alpha+2}}{(1-u\lambda)(1-2u\lambda)}$	$\frac{u^3}{(1-u\lambda)(1-2u\lambda)}$
$t^n (n = 0, 1, 2, \cdots)$	$\frac{n! u^{\alpha+1+n}}{(1-u\lambda)\cdots(1-(n+1)u\lambda)}$	$\frac{n! u^{2+n}}{(1-u\lambda)\cdots(1-(n+1)u\lambda)}$
$e_{\lambda}^{a}(t)$	$\frac{u^{\alpha+1}}{1-u(a+\lambda)}$	$\frac{u^2}{1-u(a+\lambda)}$
$\sin_{\lambda}^{(a)}(t)$	$\frac{a u^{\alpha+2}}{(1-\lambda u)^2 + u^2 a^2}$	$\frac{a u^3}{(1-\lambda u)^2 + u^2 a^2}$
$\cos_{\lambda}^{(a)}(t)$	$\frac{(1-\lambda u) u^{\alpha+1}}{(1-\lambda u)^2 + u^2 a^2}$	$\frac{(1-\lambda u) u^2}{(1-\lambda u)^2 + u^2 a^2}$
$\sinh_{\lambda}^{(a)}(t)$	$\frac{a u^{\alpha+2}}{(1-\lambda u)^2 - u^2 a^2}$	$\frac{a u^3}{(1-\lambda u)^2 - u^2 a^2}$
$\cosh_{\lambda}^{(a)}(t)$	$\frac{(1-\lambda u) u^{\alpha+1}}{(1-\lambda u)^2 - u^2 a^2}$	$\frac{(1-\lambda u) u^2}{(1-\lambda u)^2 - u^2 a^2}$
$e_{\lambda}^{a}(t) \sin_{\lambda}^{(b)}(t)$	$\frac{b u^{\alpha+2}}{(1-au-u\lambda)^2 + b^2 u^2}$	$\frac{b u^3}{(1-au-u\lambda)^2 + b^2 u^2}$
$e_{\lambda}^{a}(t) \cos_{\lambda}^{(b)}(t)$	$\frac{(1-au-u\lambda) u^{\alpha+1}}{(1-au-u\lambda)^2 + b^2 u^2}$	$\frac{(1-au-u\lambda) u^2}{(1-au-u\lambda)^2 + b^2 u^2}$

Table 3: The degenerate Laplace-type and degenerate Elzaki transform.

6. Conclusion and Recommendations

The concept of degenerate Laplace-type Integral Transform is introduced in this work, and it includes three essential degenerate integral transforms: the degenerate Laplace Integral transform, the degenerate Sumudu Integral transform, and the degenerate Elzaki Integral transform. These transformations offer potentially powerful mathematical tools for addressing a wide range of problems in engineering, physics, and other scientific fields. The degenerate Laplace-type Integral Transform is a unifying framework from which the degenerates of several current Integral Transforms may be derived. This degenerate Laplace-type Integral Transform has a lot of promise and is still being researched and developed. As a result, more study may uncover new applications, features, and generalizations of this groundbreaking concept.

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