EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 16, No. 4, 2023, 2476-2498
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

On the j-Edge Intersection Graph of Cycle Graph

Jhon Cris C. Bonifacio[1], Clarence Joy M. Andaya[1], Daryl M. Magpantay[1,*]

[1] *College of Arts and Sciences, Batangas State University - The National Engineering University, Batangas City, Batangas, Philippines*

Abstract. This paper defines a new class of graphs using the spanning subgraphs of a cycle graph as vertices. This class of graphs is called j-edge intersection graph of cycle graph, denoted by $E_{C_{(n,j)}}$. The vertex set of $E_{C_{(n,j)}}$ is the set of spanning subgraphs of cycle graph with j edges where $n \geq 3$ and j is a nonnegative integer such that $1 \leq j \leq n$. Two distinct vertices are adjacent if they have exactly one edge in common. $E_{C_{(n,j)}}$ is considered as a simple graph. Furthermore, $E_{C_{(n,j)}}$ is characterized by the value of j that is when $j = 1$ or $\lceil \frac{n}{2} \rceil < j \leq n$ and $2 \leq j \leq \lceil \frac{n}{2} \rceil$. When $j = 1$ or $\lceil \frac{n}{2} \rceil < j \leq n$, the new graph only produced an empty graph. Hence, the proponents only considered the value when $2 \leq j \leq \lceil \frac{n}{2} \rceil$ in determining the order and size of $E_{C_{(n,j)}}$. Moreover, this paper discusses necessary and sufficient conditions where the j-edge intersection graph of C_n is isomorphic to the cycle graph. Furthermore, the researchers determined a lower bound for the independence number, and an upper bound for the domination number of $E_{C_{(n,j)}}$ when $j = 2$.

2020 Mathematics Subject Classifications: 05

Key Words and Phrases: Edge Intersection Graph, Cycle Graph, Special Classes of Graphs, Parameters of Graphs, Spanning Subgraph

1. Introduction

Graph is a very effective tool to model issues that have their origins in almost every aspect of human life. Studying graphs through a framework provides answers to many arrangement, networking, optimization, matching, and operational problems. The development, computation, and maintenance of multi-part electric circuits are central to the field of electrical engineering, and these circuits are frequently represented graphically using graph theory techniques. Since graphs are very helpful in understanding things, many researchers have created their own graphs to become a new field of study.

All newly created graphs can be used to better understand a concept. A graph is an interesting concept, for this motivates the proponents to create their own graphs and explore a related study. In the study entitled “On the edge-intersection graphs of k-bend

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v16i4.4870

Email addresses: daryl.magpantay@g.batstate-u.edu.ph (D. Magpantay)

paths in grids"[2], the motivation for creating this graph is an application in conflict resolutions of paths in grid networks. Moreover, one interesting topic in graph theory is the edge intersection graph. In an edge intersection graph, the vertices of a graph are usually represented by the members of some family of sets; and two vertices are adjacent if the intersection of their corresponding sets satisfies some specified condition. The set of rules used to define the vertex and edge sets is known as a model. In an intersection graph model, the choice of sets to represent the vertices of a graph pre-determines the edges and the specific sets corresponding to each vertex are a representation of the graph. A graph is representable with respect to a given model if there is some representation.

There are several studies in relation to edge intersection. In the paper [4], they investigated The class of edge intersection graphs of a collection of paths in a tree (EPT graphs) where two paths edge intersect if they share an edge. The cliques of an EPT graph are characterized and shown to have strong. Another is the study [1] that presents some other results about edge intersection graphs of paths on a grid and shows several results of the other clsses of graphs.

This paper sought to introduce the j-edge intersection graph of C_n. This study aims to achieve the following objectives:

(i) To define a j-edge intersection graph of C_n;

(ii) To find necessary and sufficient conditions when j-edge intersection graph of C_n is isomorphic to a special class of graph;

(iii) To identify some of the parameters of a j-edge intersection graph of C_n such as:

  (a) order; and

  (b) size.

(iv) To determine bounds for independence number, and domination number of 2-edge intersection graph of cycle graph.

2. Preliminaries

Graph theory and the principle of counting are both covered in this chapter along with several other essential ideas. For the purpose of further understanding concepts, examples, and illustrations are given. Also, some theorems are presented without proof. The concepts in this section can be found in [3], [5], [7], [9], [8].

Definition 1. A **graph**, denoted by G is an ordered pair $G = (V(G), E(G))$ where the vertex set $V(G)$ is a nonempty set of elements called **vertices**, and the edge set $E(G)$ is a set of unordered pairs of distinct vertices called **edges**.

The edges of a graph is written as $[x_1, x_2]$ where $x_1, x_2 \in V(G)$. Two vertices x_1 and x_2 in G are connected by a line segment whenever x_1 is adjacent to x_2 or x_2 is adjacent to x_1. Note that an edge contains unordered pair of vertices, so $[x_1, x_2] = [x_2, x_1]$. If $[x_1, x_2]$ is an element of $E(G)$, then the vertices x_1 and x_2 are said to be *adjacent* in G. Now, if $[x_1, x_2] \notin E(G)$, then x_1 and x_2 are said to be *non-adjacent* in G. Moreover, edges are *incident* if there is a vertex between these edges. The cardinality of $V(G)$ and $E(G)$ are referred to as the *order* and *size* of G, respectively.

Example 1. Let G be a graph such that $V(G) = \{x_1, x_2, x_3, x_4\}$ and $E(G) = \{[x_1, x_2], [x_1, x_4], [x_4, x_3], [x_3, x_2]\}$. Then $|V(G)| = 4$ and $|E(G)| = 4$. The pictorial representation of G is shown in Figure 1.

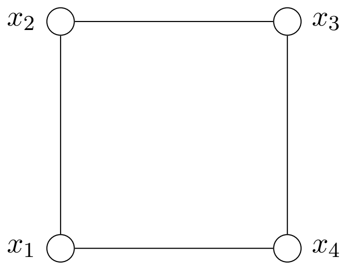

Figure 1: Example of a Graph G

It can be noted that a pictorial representation of a particular graph is not unique. Hence, graph G in Figure 1 can be illustrated differently as shown in Figure 2.

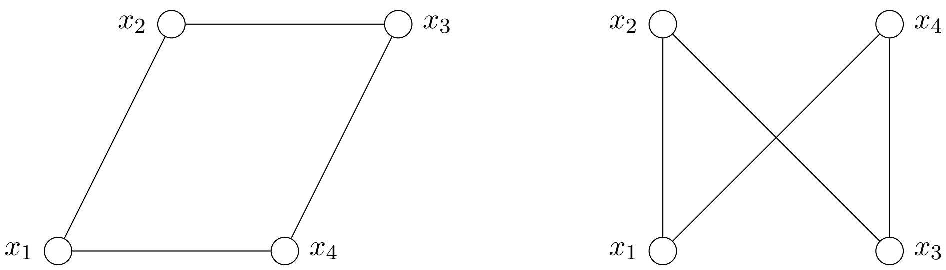

Figure 2: Other Pictorial Representations of G, G_1 and G_2 respectively

Definition 2. An edge of the form $[x, x]$ is called a **loop**. Moreover, **multiple edges** are edges that have the same pair of vertices. A graph having no loops nor multiple edges is called a **simple graph**.

The focus of this paper is on finite graphs, which are finite in both their vertex and edge sets. Also, this paper will be limited to simple finite graphs and we will simply call them graphs.

Example 2. Graph G in Figure 3 is an example of a finite graph because its vertex and edge set is finite and does not contain any loops or multiple edges so it is a simple graph.

However, graph H in Figure 3 contains the edge $[x_5, x_5]$ and there are multiple edges $[x_1, x_2]$ so it is not a simple graph.

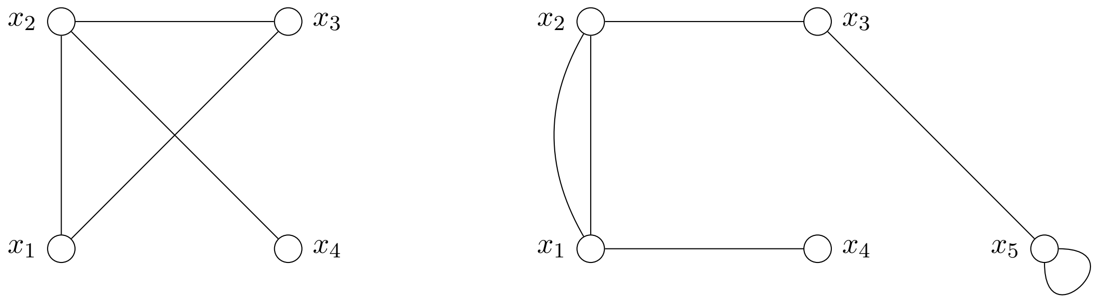

Figure 3: Simple Graph G and Graph H with Loop and Multiple edges

Definition 3. A graph G is **labeled** when each vertex is distinguished from one another by symbols such as $x_1, x_2, ..., x_n$ where n is the order of G. Otherwise, it is called **unlabeled** .

Definition 4. A graph of order $n \geq 1$ having no edges is called an **empty** graph. Furthermore, a graph with only one vertex is referred to as a **trivial graph**.

It can be noted that it cannot form a graph if its vertex set has no elements. Also, it can be observed that all trivial graphs are empty graphs but it is not always true for an empty graph to be a trivial graph.

Example 3. Let G_1 be a graph where $V(G_1) = x_1$ and $E(G_1) = \emptyset$. Since $V(G_1)$ has only an element, it follows that G_1 is a trivial graph. Let G_2 be a graph such that $V(G_2) = \{x_1, x_2, x_3, x_4\}$ and $E(G_2) = \emptyset$. Hence, G_2 is said to be an empty graph. Shown in Figure 4 are pictorial illustrations of G_1 and G_2.

Figure 4: A Trivial Graph G_1 and an Empty Graph G_2

Definition 5. The **degree of a vertex** x, denoted by $deg(x)$, is the number of edges incident with vertex x.

If a vertex x has no degree , it means that it is not adjacent to any other vertices in a graph, then it is called an *isolated vertex*. Thus, $deg(x) = 0$ for every isolated vertex x.

Example 4. Let G be a graph where $V(G) = \{x_1, x_2, x_3, x_4, x_5\}$ and $E(G) = \{[x_1, x_2], [x_2, x_3], [x_2, x_4], [x_3, x_5], [x_4, x_5]\}$. Given in Figure 5 is a pictorial representation of a graph G. Since there are 3 edges incident with vertex x_2, it follows that $deg(x_2) = 3$. Shown in Table 1 is the list for the degree of every vertex in G.

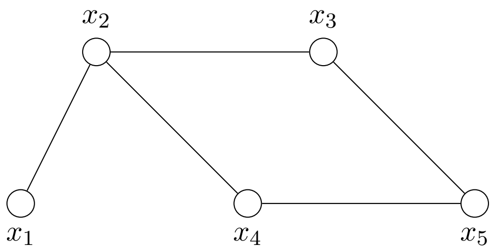

Figure 5: Graph G of order 5 and size 5

Table 1: Degrees of Every Vertex in G in Figure 5

Vertex	$deg(x)$
x_1	1
x_2	3
x_3	2
x_4	2
x_5	2
$\sum deg(x)$	10

By the concept of degrees of a vertex, the next theorem is one of the *fundamental theorems in graph theory.* This theorem represents the equality between the size of a graph G and the totality of the degrees of all vertices of G.

Theorem 1. *If G is a graph with size m, then*

$$\sum_{x \in V(G)} deg(x) = 2m.$$

The idea of the degrees of every vertex is important in finding the order and size of a graph especially if the degrees have the same number. That is, a graph with the same number of degrees of its vertices forms a regular graph.

Definition 6. A graph G is **regular** if every vertex has the same degree. Moreover, G is said to be regular of degree r (or r-**regular**) if $deg(x) = r$ for all vertices x in G.

By using Theorem 1, a formula for the order and size of an r-regular graph can be derived. Given n as its order and m as its size, then

$$\sum_{x \in V(G)} deg(x) = 2m$$

$$nr = 2m. \tag{1}$$

Hence, $n = \frac{2m}{r}$ and $m = \frac{nr}{2}$.

The next concept is the notion of a walk which is a way of traversing a graph by moving from one vertex to another through the edges of the graph.

Definition 7. Let $W : x_1, x_2, \cdots, x_k, x_{k+1}$ be a walk of length $k > 0$. A walk is *closed* if $x_1 = x_{k+1}$. A closed walk is called a **cycle** if the vertices $x_1, x_2, ..., x_k$ are distinct.

Cycles are special kinds of walks in graphs such that these are used to name a special class of graph.

There are numerous notable graphs that have been discovered in graph theory. These graphs have special notations that are used exclusively to denote them. In this section, some of the common classes of graphs such as the cycle graph, and complete graph are discussed.

Definition 8. A graph G of order $n \geq 3$ is called a **cycle graph of order** n, denoted by C_n, if the vertices of G is labeled $x_1, x_2, ..., x_n$ so that the edges are $[x_1, x_2], [x_2, x_3], ..., [x_{n-1}, x_n], [x_n, x_1]$.

A cycle graph is said to be a 2-regular graph. So, by using Equation 1 to determine the size m of a cycle graph of order n we have

$$nr = 2m$$

$$n(2) = 2m$$
$$m = n.$$

Hence, if n is the order of C_n, then the size is also n.

Example 5. A cycle graph C_3 with its pictorial representation shown in Figure 6 is a cycle graph of order 3 since it is a closed walk that contains distinct vertices.

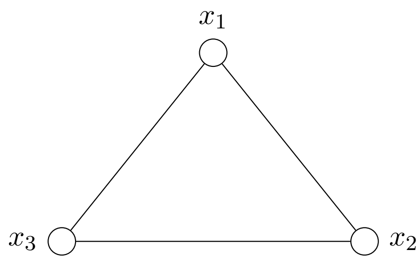

Figure 6: Cycle graph C_3 of order 3

Considering C_n with $V(C_n) = \{x_1, x_2, \cdots, x_n\}$. Throughout the paper, vertices $x_1, x_2, \cdots, x_n$ will be replaced by vertices $1, 2, \cdots, n$, respectively. In addition, an edge $[x_1, x_2]$ will be denoted by 12. Refer to Figure 7.

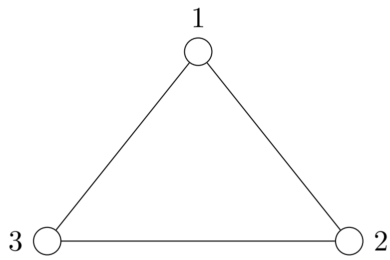

Figure 7: Cycle graph C_3 of order 3

Definition 9. A graph of order n is said to be a **complete graph of order** n, denoted by K_n , if every vertex is adjacent to every other vertex.

A complete graph with one vertex is called a *singleton graph* and is denoted by K_1. Since all the vertices in K_n are adjacent to every other vertex, it can be observed that $deg(x) = n-1$ for all $x \in V(G)$. Hence K_n is an $(n-1)$-regular graph. Using Equation 1, the size of a K_n is given by

$$\begin{aligned} nr &= 2m \\ n(n-1) &= 2m \\ m &= \frac{n(n-1)}{2}. \end{aligned} \tag{2}$$

At this point, the notion of the subgraph of a graph and isomorphism among graphs is discussed.

Definition 10. A graph $H = (V(H), E(H))$ is called a **subgraph** of a graph $G = (V(G), E(G))$, written $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

It can be noted that a graph is a subgraph of itself because when we take the entire set of vertices and edges from a graph, we get the original graph itself.

Example 6. Consider the pictorial representation of graphs G, H_1, and H_2 shown in Figure 8. The graph H_1 is a subgraph of G since $V(H_1) \subseteq V(G)$ and $E(H_1) \subseteq E(G)$. However, graph H_2 is not a subgraph of G since $V(H_2) \subseteq V(G)$ but $E(H_2) \not\subseteq E(G)$.

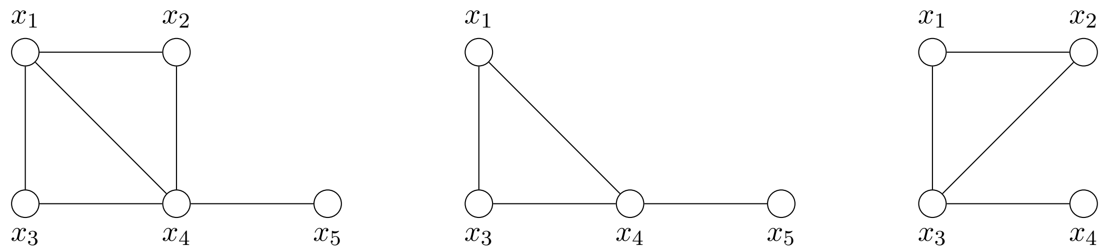

Figure 8: Subgraph H_1 of graph G, and not a subgraph H_2 of graph G

All vertices and edges of the graph H_1 are all subsets of the vertices and edges of graph G. However, edge $[x_2, x_3] \in E(H_2)$ but edge $[x_2, x_3] \notin E(G)$. Hence, H_2 is not a subgraph of G.

Definition 11. A subgraph H of a graph G is called an **induced-subgraph**, written as $\langle H \rangle$, if whenever $x, y \in H$ and $[x, y] \in E(G)$, then $[x, y]$ is an edge of $\langle H \rangle$.

In simple terms, an induced-subgraph $\langle H \rangle$ of a graph G has a vertex set that is a subset of $V(G)$ together with the edges whose vertices are contained in the subset H.

Example 7. Consider the pictorial representation of graphs G and H shown in Figure 9. The Graph H is a subgraph of G since $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Now, consider $E(H) = \{[x_1, x_2], [x_1, x_5], [x_2, x_5], [x_2, x_3]\} \in E(G)$. It follows that H is an induced-subgraph of G.

In some cases, a graph H is a subgraph of G and the order of H and G are equal. This idea gives the notion of a spanning subgraph.

Definition 12. A graph H is a **spanning subgraph** of a graph G if H is a subgraph of G and $V(G) = V(H)$.

Note that the edge set of graph H is a subset of the edge set of G.

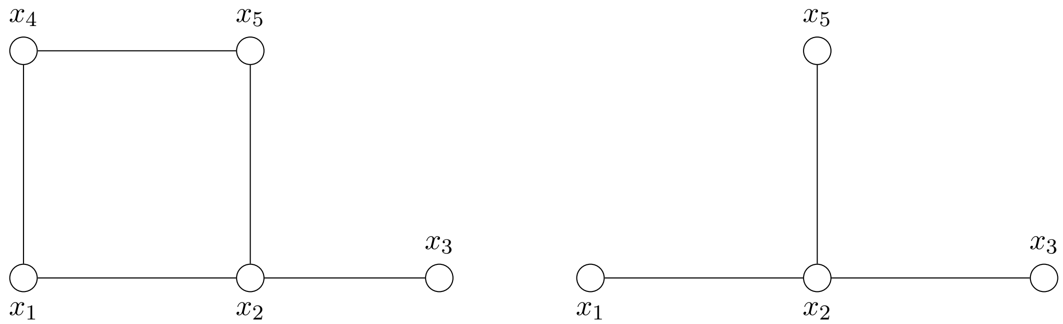

Figure 9: Graph G and its subgraph H

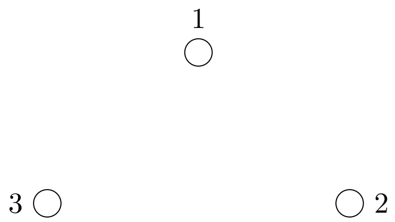

Figure 10: Spanning Subgraphs of C_3 with 0 edge

Example 8. Consider cycle graph C_3 with pictorial representation shown in Figure 7. By Definition 12, the spanning subgraphs of C_3 have 3 vertices, and the edges are subset of C_3. To get the spanning subgraph of C_3, first we consider the spanning subgraph with 0 edge. Refer to Figure 10.

Next, consider the spanning subgraphs with 1 edge.
Consider the first spanning subgraphs with 1 edge. It can be seen that the edge 12 is present hence, this spanning subgraph is denoted as $\{12\}$. Similarly, we get $\{23\}$ and $\{31\}$ are obtained. Now, consider the spanning subgraphs with 2 edges.
Consider the first spanning subgraph with 2 edges, the edges 12 and 23 are present so, this is denoted by $\{12, 23\}$. Similarly, we get $\{12, 31\}$ and $\{23, 31\}$. Lastly, consider the spanning subgraph with 3 edges.
It can be observed that the edges 12, 23, and 31 are present in the spanning subgraph shown in Figure 13. Hence, this is denoted by $\{12, 23, 31\}$. Note that $\{12\}$ and $\{21\}$ are just the same since undirected graphs are considered, which means the edges have no directions.

The following theorem determines the number of spanning subgraphs with j edges of a graph. The proof of Theorem 2 can be seen in [6].

Theorem 2. *Let G be a graph of size m, then there are $\binom{m}{j}$ spanning subgraph with exactly j edges where $0 \leq j \leq m$.*

Illustration 1 shows the spanning subgraphs of C_n when $m = 3$ and $j = 2$.

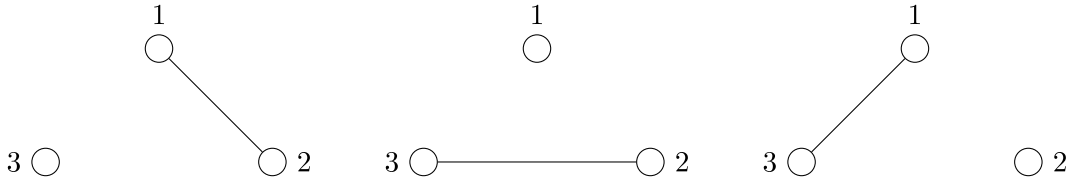

Figure 11: Spanning Subgraphs of C_3 with 1 edge

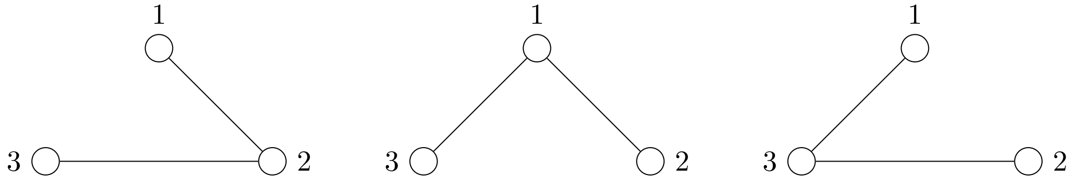

Figure 12: Spanning Subgraphs of C_3 with 2 edges

Illustration 1. Consider the spanning subgraphs of C_3 shown in Example 8. If $j = 2$, then by counting, there are 3 spanning subgraphs with 2 edges. Verifying this by using Theorem 2, we have $\binom{m}{j} = \binom{3}{2} = 3$.

Two graphs G and G' are said to be *equal* if $V(G) = V(G')$ and $E(G) = E(G')$. However, graphs are possible of similar form even if they have unequal vertex and/or edge sets, that is, if there exists an isomorphism between them.

Definition 13. Let $G = (V(G), E(G))$ and $G' = (V(G'), E(G'))$ be graphs. A mapping $\phi : V(G) \longmapsto V(G')$ is called **isomorphism** if the following conditions are satisfied:

(i) ϕ is bijective, that is both one-to-one and onto;

(ii) $[a, b] \in E(G) \Rightarrow [\phi(a), \phi(b)] \in E(G')$;

(iii) $[c, d] \in E(G') \Rightarrow [\phi^{-1}(c), \phi^{-1}(d)]; \in E(G)$.

A function mapping is one-to-one and onto if every element of $V(G)$ is mapped into exactly one element of set $V(G')$.

Example 9. Consider graphs G and G' in Figure 14.

Define the mapping $\phi : V(G) \longmapsto V(G')$ by

$$\begin{aligned} \phi : \quad x_1 &\longmapsto y_1 \\ x_2 &\longmapsto y_2 \\ x_3 &\longmapsto y_4 \\ x_4 &\longmapsto y_3 \end{aligned}$$

By mapping ϕ, it can be observed that ϕ is one-to-one and onto, hence condition (i) of the definition is satisfied. To verify condition (ii), we have:

$$[x_1, x_2] \in E(G) \Rightarrow [\phi(x_1), \phi(x_2)] = [y_1, y_2] \in E(G')$$

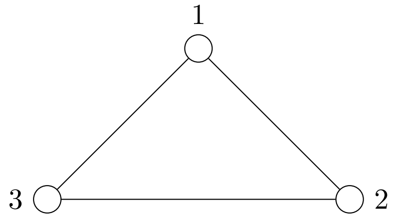

Figure 13: Spanning Subgraphs of C_3 with 3 edge

Figure 14: Graph G and G'

$$[x_3, x_4] \in E(G) \Rightarrow [\phi(x_3), \phi(x_4)] = [y_4, y_3] \in E(G')$$
$$[x_2, x_4] \in E(G) \Rightarrow [\phi(x_2), \phi(x_4)] = [y_2, y_3] \in E(G')$$
$$[x_1, x_3] \in E(G) \Rightarrow [\phi(x_1), \phi(x_3)] = [y_1, y_4] \in E(G')$$

Hence, the condition (ii) of the Definition 13 is satisfied. Lastly, to verify condition (iii) we have:

$$[y_1, y_2] \in E(G') \Rightarrow [\phi^{-1}(x_1), \phi^{-1}(x_2)] = [x_1, x_2] \in E(G)$$
$$[y_4, y_3] \in E(G') \Rightarrow [\phi^{-1}(x_1), \phi^{-1}(x_2)] = [x_3, x_4] \in E(G)$$
$$[y_2, y_3] \in E(G') \Rightarrow [\phi^{-1}(x_1), \phi^{-1}(x_2)] = [x_2, x_4] \in E(G)$$
$$[y_1, y_4] \in E(G') \Rightarrow [\phi^{-1}(x_1), \phi^{-1}(x_2)] = [x_1, x_3] \in E(G)$$

Thus, condition (iii) of the Definition 13 is satisfied. Therefore, ϕ is an isomorphism.

Definition 14. Let G and G' be graphs. A graph G is **isomorphic** to G', denoted by $G \simeq G'$, if there exists an isomorphism $\phi : V(G) \longmapsto V(G')$.

Note that if a graph G has cycle, the isomorphic graph G' should also preserves the cycle. Also, it preserves the degree sequence of the graph which is just the list of degrees of each vertex in a particular graph.

Example 10. In Example 9, since there exists a function mapping $\phi : V(G) \longmapsto V(H)$ which is an isomorphism, it follows that G is isomorphic to H.

Recall that a cycle graph of order 3 and size 3 is a 2-regular graph. Moreover, it can be observed that a complete graph of order 3 is also a 2-regular graph which has also a size of 3. Now, since C_3 have the same order, size, and the degree of every vertex as that of K_3, it can be verified that there is an isomorphism between the two graphs. Refer to Figure 15.

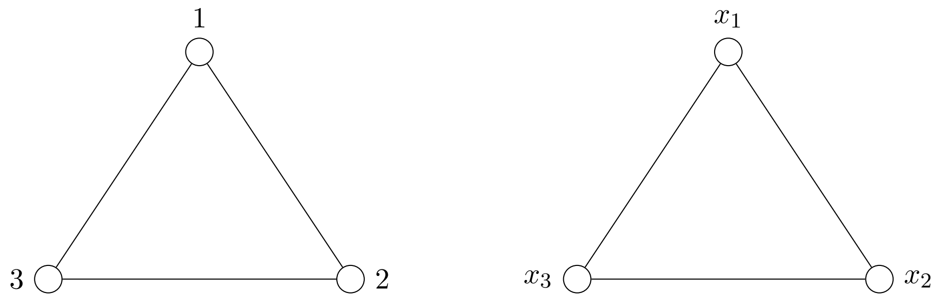

Figure 15: Pictorial Representations of C_3 and K_3

Remark 1. *Let C_3 be a cycle graph of order 3 and let K_3 be a complete graph of order 3. Then $C_3 \simeq K_3$.*

This section discusses some of the parameters that will be helpful in determining a graph.

Definition 15. Let G be a graph. The nonempty set $I \subseteq V(G)$ is called an **independent set** in a graph G if for every $x, y \in I$, then $[x, y] \notin E(G)$. The **independence number** of a graph G, denoted by $\alpha(G)$, is the cardinality of the largest independent set of G.

Note that if there exists a set $I \subseteq V(G)$ that is an independent set of G, then it can be verified that $\alpha(G) \geq |I|$.

Definition 16. Let $G = (V(G), E(G))$ be a graph. A nonempty subset I of $V(G)$ is called **dominating set** of G if every element of $V(G)\backslash I$ is adjacent to some element of I. Moreover, **domination number**, written as $\gamma(G)$, of a graph G is the minimum cardinality among all the dominating set of G.

The statement that if $I = V(G)$ then I is a dominating set is vacuously true since $V(G)\backslash I = \emptyset$ so there are no elements to be considered. In mathematics, a vacuous truth is a universal or conditional statement that is deemed to be true. Also, it can be observed that if $I \subseteq V(G)$ is dominating set in G, thus, $\gamma(G) \leq |I|$.

3. j-edge Intersection Graph of Cycle Graph and its Basic Parameters

Recall that for an arbitrary edge of the cycle graph C_n, say $[1, 2]$ is relabeled as 12. Moreover, the variable j is used as the number of edges of the spanning subgraph of C_n and the researchers are focusing only on the spanning subgraph when $1 \leq j \leq n$.

Definition 17. Let C_n be a cycle graph of order n where $n \geq 3$. For $1 \leq j \leq n$, a ***j*-edge intersection graph of** C_n, denoted by $E_{C_{(n,j)}}$, is the graph whose vertex set is

$$V(E_{C_{(n,j)}}) = \{\{e_1, e_2, \cdots, e_j\} | e_i \text{ is an edge in } E(C_n), 1 \leq i \leq j\}.$$

Moreover, two distinct vertices $A, B \in V(E_{C_{(n,j)}})$ are adjacent whenever $|A \cap B| = 1$.

Note that the spanning subgraphs of C_n can be uniquely determined by the vertices of j-edge intersection graph of C_n. Now, the elements of $V(E_{C_{(n,j)}})$ are the spanning subgraphs of C_n with j edges where $1 \leq j \leq n$. Moreover, distinct pairs of vertices are elements of $E(E_{C_{(n,j)}})$ if they share exactly one edge. To understand this, given in Example 11 is an illustration of $E_{C_{(n,j)}}$ where $n = 4$ and $j = 2$.

Example 11. Consider the cycle graph C_4 where $V(C_4) = \{1, 2, 3, 4\}$, $E(C_4) = \{12, 23, 34, 41\}$ and let $j = 2$. The vertex set of $E_{C_{(4,2)}}$ is given by

$$V(E_{C_{(4,2)}}) = \{\{12, 23\}, \{12, 34\}, \{12, 41\}, \{23, 34\}, \{23, 41\}, \{34, 41\}\}.$$

Now, since $\{12, 23\} \cap \{12, 34\} = \{12\}$, it follows that $[\{12, 23\}, \{12, 34\}] \in E(E_{C_{(4,2)}})$. Similarly, $\{23, 41\} \cap \{34, 41\} = \{41\}$, so $[\{23, 41\}, \{34, 41\}]$ is also in $E(E_{C_{(4,2)}})$. However, vertices $\{12, 23\}$ and $\{34, 41\}$ are not adjacent since $\{12, 23\} \cap \{34, 41\} = \emptyset$. Doing the same process for any two distinct vertices in $V(E_{C_{(4,2)}})$, we have

$$\begin{aligned} E(E_{C_{(4,2)}}) = \{ & [\{12, 23\}, \{12, 34\}], [\{12, 23\}, \{12, 41\}], [\{12, 23\}, \{23, 34\}], [\{12, 23\}, \{23, 41\}], \\ & [\{12, 34\}, \{12, 41\}], [\{12, 34\}, \{23, 34\}], [\{12, 34\}, \{34, 41\}], [\{12, 41\}, \{23, 41\}], \\ & [\{12, 41\}, \{34, 41\}], [\{23, 34\}, \{23, 41\}], [\{23, 34\}, \{34, 41\}], [\{23, 41\}, \{34, 41\}]\}. \end{aligned}$$

It can be noted that the order of $E_{C_{(4,2)}}$ is 6 and its size is 12. A pictorial representation of $E_{C_{(4,2)}}$ is given in Figure 16.

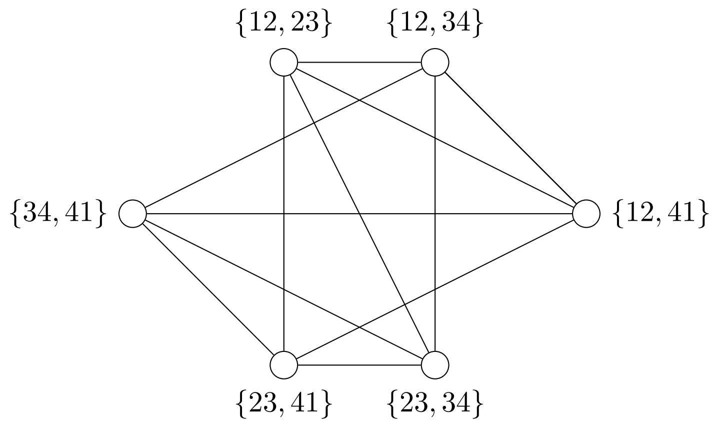

Figure 16: 2-edge Intersection Graph of C_4

It can be observed that $E_{C_{(n,j)}}$ does not contain any loop. Moreover, since $V(E_{C_{(n,j)}})$

is the collection of all distinct spanning subgraphs of C_n with j edges, it follows that $E(E_{C_{(n,j)}})$ does not have the same pair of vertices which means that $E_{C_{(n,j)}}$ has no multiple edges. Equivalently, the following remark is given.

Remark 2. *A j-edge graph of cycle graph $E_{C_{(n,j)}}$ is a simple graph.*

The first theorem determines the order of $E_{C_{(n,j)}}$.

Theorem 3. *Let $n \geq 3$ and $1 \leq j \leq n$. If $E_{C_{(n,j)}}$ is the j-edge intersection graph of C_n. Then the order of $E_{C_{(n,j)}}$ is $\binom{n}{j}$.*

Proof. By Definition 17, $V(E_{C_{(n,j)}})$ contains the spanning subgraphs of C_n with exactly j edges. By Theorem 2, for any graph G of size m, there are $\binom{m}{j}$ spanning subgraphs with exactly j edges. Since the size of C_n is n, it follows that there are $\binom{n}{j}$ spanning subgraph of C_n. Therefore, $|V(E_{C_{(n,j)}})| = \binom{n}{j}$.

Note that $\binom{n}{j}$ is a positive integer. This means that $V(E_{C_{(n,j)}})$ is always nonempty for values of $n \geq 3$ and $1 \leq j \leq n$.

Illustration 2. Consider $E_{C_{(4,2)}}$ with the pictorial illustration in Figure 16. It can be noted that the order of $E_{C_{(4,2)}}$ is 6. Using Theorem 3, with $n = 4$ and $j = 2$, we have $|V(E_{C_{(4,2)}})| = \binom{4}{2} = 6$.

There are times that $E_{C_{(n,j)}}$ contains only one vertex. Since $j \geq 1$ and using Theorem 3, $\binom{n}{j} = 1$ if and only if $j = n$. Theorem 4 discusses a property of $E_{C_{(n,j)}}$ when $j = n$.

Theorem 4. *Let $n \geq 3$ and $1 \leq j \leq n$. If $E_{C_{(n,j)}}$ is the j-edge intersection graph of C_n. Then $E_{C_{(n,j)}}$ is a trivial graph if and only if $j = n$.*

Proof. Assume $E_{C_{(n,j)}}$ is a trivial graph. By Definition 4, $|V(E_{C_{(n,j)}})| = 1$. Since the order of $E_{C_{(n,j)}}$ is equal to $\binom{n}{j}$, we have $\binom{n}{j} = 1$. Hence, by Theorem 3, j is equal to n. Conversely, assume that $j = n$. If $j = n$, then $V(E_{C_{(n,n)}})$ only contains $\{e_1, e_2, \cdots, e_n\}$. Since there is only one spanning subgraph of C_n with n edges, it follows that $E(E_{C_{(n,n)}})$ is empty. By Remark 2, $[\{e_1, e_2, \cdots, e_n\}, \{e_1, e_2, \cdots, e_n\}] \notin E(E_{C_{(n,n)}})$ it implies that $E(E_{C_{(n,n)}}) = \emptyset$. Therefore, $E_{C_{(n,n)}}$ is a trivial graph.

The illustration below provides an example given that $n = 4$ and $j = 4$.

Illustration 3. Consider cycle graph C_4 where $E(C_4) = \{12, 23, 34, 41\}$. The number of spanning subgraphs of C_4 with 4 edges is $\binom{4}{4} = 1$, by Theorem 3. Thus, the order of $E_{C_{(4,4)}}$ is 1. Figure 17 shows the pictorial representation of $E_{C_{(4,4)}}$.

$\{12, 23, 34, 41\}$

Figure 17: A Pictorial Representation of $E_{C_{(4,4)}}$

The next proposition discusses when $E_{C_{(n,j)}}$ contains no edge, that is when $j = 1$.

Proposition 1. *Let $n \geq 3$ and $1 \leq j \leq n$ are integers. If $E_{C_{(n,j)}}$ is the j-edge intersection graph of C_n and $j = 1$, then $E_{C_{(n,j)}}$ is an empty graph of order n.*

Proof. Assume that $j = 1$. Then $V(E_{C_{(n,1)}}) = \{\{e_1\}, \{e_2\}, \cdots, \{e_n\}\}$. Now, observe that $e_a \cap e_b = \emptyset$ for all $1 \leq a, b \leq n$. It means that there are no adjacent vertices in $E_{C_{(n,1)}}$. Since $|V(E_{C_{(n,1)}})| = n$, it follows that $E_{C_{(n,1)}}$ is an empty graph of order n.

Illustration 4 shows that $E_{C_{(n,1)}}$ is an empty graph when $j = 1$.

Illustration 4. Consider the cycle graph C_3 where $E(C_3) = \{12, 23, 31\}$ and let $j = 1$. Then we have $V(E_{C_{(3,1)}}) = \{\{12\}, \{23\}, \{31\}\}$. It can be observed that the spanning subgraphs have no common edge. Thus, $E(E_{C_{(3,1)}}) = \emptyset$. It follows that $E_{C_{(3,1)}}$ is an empty graph of order 3 shown in Figure 18.

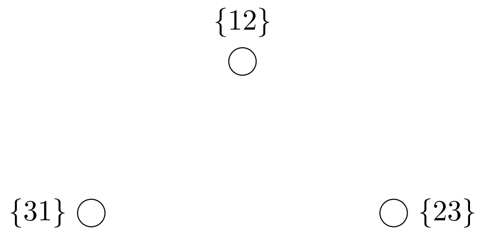

Figure 18: An Empty Graph $E_{C_{(3,1)}}$

In Figure 18, it can be observed that the vertices are not adjacent to each other, thus the degree of every vertex in $E_{C_{(n,j)}}$ when $j = 1$ is equal to zero. Note that the degree of every vertex in $E_{C_{(n,j)}}$ depends on the value of the nonnegative integer j.

Lemma 1 shows that for any two distinct spanning subgraphs of C_n with j edges, their common edge is always greater than 1 that is when $\lceil \frac{n}{2} \rceil < j \leq n$.

Lemma 1. *If $\lceil \frac{n}{2} \rceil < j \leq n$, then for all $A, B \in V(E_{C_{(n,j)}})$, $|A \cap B| > 1$.*

Proof. Suppose $|A \cap B| \not> 1$. Then $|A \cap B|$ is either 0 or 1. Consider two cases:
CASE 1: If $|A \cap B| = 0$, then A and B are disjoint. Now, if n is even, by the definition of ceiling function, let $j = \lceil \frac{n}{2} \rceil + 1 = \frac{n}{2} + 1$ where it is the minimum value of j. Then

$$|A \cup B| = \left(\frac{n}{2} + 1\right) + \left(\frac{n}{2} + 1\right) = n + 2.$$

This is a contradiction to the fact that $j \leq n$. Moreover, if n is odd, let $j = \lceil \frac{n}{2} \rceil + 1 = \frac{n+1}{2} + 1$. Then

$$|A \cup B| = \left(\frac{n+1}{2} + 1\right) + \left(\frac{n+1}{2} + 1\right) = n + 3.$$

This is also a contradiction to the fact that $j \leq n$.

CASE 2: If $|A \cap B| = 1$, then there is exactly one common element. Now, if n is even, by Definition **??**, let $j = \lceil \frac{n}{2} \rceil + 1 = \frac{n}{2} + 1$. Then

$$|A \cup B| = \left(\frac{n}{2} + 1 - 1\right) + \left(\frac{n}{2} + 1 - 1\right) + 1 = n + 1.$$

This is a contradiction to the fact that $j \leq n$. Moreover, if n is odd, let $j = \lceil \frac{n}{2} \rceil + 1 = \frac{n+1}{2} + 1$. Then

$$|A \cup B| = \left(\frac{n+1}{2} + 1 - 1\right) + \left(\frac{n+1}{2} + 1 - 1\right) + 1 = n + 2.$$

This is also a contradiction to the fact that $j \leq n$. Therefore, if $\lceil \frac{n}{2} \rceil < j \leq n$, then $|A \cap B| > 1$.

Illustration 5 shows that for two distinct spanning subgraphs of C_n with j edges, their common edge is always greater than 1 that is when $\lceil \frac{n}{2} \rceil < j \leq n$. Setting $n = 4$ and $j = 3$.

Illustration 5. Consider C_4 with $E(C_4) = \{12, 32, 34, 41\}$ and let $j = 3$. It can be observed that $j = 3 > \lceil \frac{4}{2} \rceil$. The vertex set of $E_{C_{(4,3)}}$ is given by

$$V(E_{C_{(4,3)}}) = \{\{12, 23, 34\}, \{12, 23, 41\}, \{12, 34, 41\}, \{23, 34, 41\}\}.$$

Now, take two arbitrary elements in $V(E_{C_{(4,3)}})$, say $\{12, 23, 34\}$ and $\{12, 23, 41\}$, $\{12, 23, 34\} \cap \{12, 23, 41\} = \{12, 23\}$ with cardinality equal to 2.

It can be noted that when two distinct spanning subgraphs of C_n with j edges share more than 1 edge, the degree of every vertex in $E_{C_{(n,j)}}$ is equal to 0.

Theorem 5. *If $\lceil \frac{n}{2} \rceil < j \leq n$, then for all $A \in V(E_{C_{(n,j)}})$, the $deg(A) = 0$.*

Proof. By Lemma 1, if $A, B \in V(E_{C_{(n,j)}})$, then $|A \cap B| > 1$ when $\lceil \frac{n}{2} \rceil < j \leq n$. By Definition 17, two distinct vertices are adjacent if they share exactly one edge. So, if $|A \cap B| > 1$, then $[A, B] \notin E(E_{C_{(n,j)}})$ for all $A \in V(E_{C_{(n,j)}})$. Thus, when $\lceil \frac{n}{2} \rceil < j \leq n$, $deg(A) = 0$.

Illustration 6 shows that when $\lceil \frac{n}{2} \rceil < j \leq n$ the degree of every vertex in $E_{C_{(n,j)}}$ is equal to 0.

Illustration 6. Let C_4 be a cycle graph of order 4 with $E(C_4) = \{12, 23, 34, 41\}$ and let $j = 3$. Observe that $j = 3 > \lceil \frac{4}{2} \rceil$. The vertex set of $E_{C_{(4,3)}}$ is given by

$$V(E_{C_{(4,3)}}) = \{\{12, 23, 34\}, \{12, 23, 41\}, \{12, 34, 41\}, \{23, 34, 41\}\}$$

Now, observe that $\{12, 23, 34\} \cap \{12, 23, 41\} = \{12, 23\}$ with cardinality equal to 2, by Definition 17, $\{12, 23, 34\}$ and $\{12, 23, 41\}$ in $V(E_{C_{(4,3)}})$ are not adjacent. Similarly, $\{12, 34, 41\}$ and $\{23, 34, 41\}$ in $V(E_{C_{(4,3)}})$ are also not adjacent since $|\{12, 34, 41\} \cap \{23, 34, 41\}| = 2$. Hence, the degree of any vertex in $E_{C_{(4,3)}}$ is equal to 0.

It can be noted that when $\lceil \frac{n}{2} \rceil < j \leq n$, $E_{C_{(n,j)}}$ is a empty graph of order $\binom{n}{j}$ since the degree of every vertex in $E_{C_{(n,j)}}$ is equal to 0. It means that there are no adjacent vertices in $E_{C_{(n,j)}}$.

Since we have already explored the case when $j = 1$ in Theorem 1, we will proceed to the case when $2 \leq j \leq \lceil \frac{n}{2} \rceil$. Theorem 6 determines the degree of every vertex in $E_{C_{(n,j)}}$ when $2 \leq j \leq \lceil \frac{n}{2} \rceil$.

Theorem 6. *For any arbitrary vertex $A \in V(E_{C_{(n,j)}})$ where $2 \leq j \leq \lceil \frac{n}{2} \rceil$, $deg(A) = j\binom{n-j}{j-1}$.*

Proof. Let $A \in V(E_{C_{(n,j)}})$. The vertices adjacent to A are the spanning subgraphs of C_n with j edges and having exactly one common edge. Without loss of generality, fix e_1 as the common edge. Hence, there are $j-1$ edges different from $\{e_2, \cdots, e_j\}$. These edges must be chosen from the other $n-j$ edges of C_n. Thus, these are $\binom{n-j}{j-1}$ ways to do this. Since there are j edges contained in each vertex, it follows that there are $j\binom{n-j}{j-1}$ vertices adjacent to A.

In the succeeding discussions, we just focus on $E_{C_{(n,j)}}$ when $2 \leq j \leq \lceil \frac{n}{2} \rceil$ since when $j = 1$ and $\lceil \frac{n}{2} \rceil < j \leq n$ we produce an empty graph. Presented in Illustration 7 is an example for the degree of every vertex of a $E_{C_{(n,j)}}$ when $2 \leq j \leq \lceil \frac{n}{2} \rceil$ where $n = 5$ and $j = 3$.

Illustration 7. Consider the cycle graph C_5 where $E(C_5) = \{12, 23, 34, 45, 51\}$ and $j = 3$. The vertex set of $E_{C_{(5,3)}}$ is given by

$$\begin{aligned} V(E_{C_{(5,3)}}) =& \{\{12, 23, 34\}, \{12, 23, 45\}, \{12, 23, 51\}, \{12, 34, 45\}, \{12, 34, 51\}, \{12, 45, 51\}, \\ & \{23, 34, 45\}, \{23, 34, 51\}, \{23, 45, 51\}, \{34, 45, 51\}\} \end{aligned}$$

Figure 19 is a pictorial representation of $S_{\mathcal{C}_{(5,3)}}$.

Observe that the degree of each vertex of $E_{C_{(5,3)}}$ is 3. Now, to verify this using Theorem 6, the degree of every vertex A of $E_{C_{(5,3)}}$ is given by

$$\begin{aligned} deg(A) =& j\binom{n-j}{j-1} \\ =& 3\binom{5-3}{3-1} \\ =& 3\binom{2}{2} \\ =& 3(1) \\ =& 3. \end{aligned}$$

It can be observed that the vertices of $E_{C_{(n,j)}}$ have the same degree which means that $E_{C_{(n,j)}}$ is a regular graph.

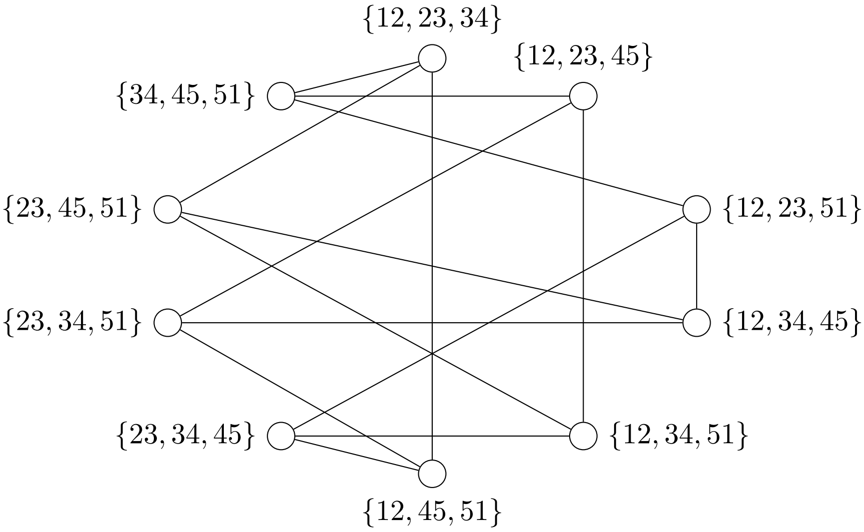

Figure 19: A Pictorial Representation of $E_{C_{(5,3)}}$

Corollary 1. *Let $E_{C_{(n,j)}}$ be a j-edge intersection graph of C_n. Then $E_{C_{(n,j)}}$ is an r-regular graph where $r = j\binom{n-j}{j-1}$ if $2 \leq j \leq \lceil \frac{n}{2} \rceil$.*

Proof. This is the direct consequence of Theorem 6.

Illustration 8. Consider the pictorial representation of $E_{C_{(5,3)}}$ in Figure 19. Since $deg(A) = 3$ for all $A \in E_{C_{(5,3)}}$, it follows that $E_{C_{(5,3)}}$ is a 3-regular graph.

In describing a graph, the size of the graph is one important characteristic to consider. It can be noted that $E_{C_{(n,j)}}$ is a regular graph; thus, Corollary 1 and Equation 1 can be used to determine the size of $E_{C_{(n,j)}}$.

Theorem 7. *Let $E_{C_{(n,j)}}$ be a j-edge intersection graph of C_n. If $2 \leq j \leq \lceil \frac{n}{2} \rceil$, then the size of $E_{C_{(n,j)}}$ is given by $|E(E_{C_{(n,j)}})| = \frac{j\binom{n-j}{j-1}\binom{n}{j}}{2}$.*

Proof. By Theorem 11, the order of $E_{C_{(n,j)}}$ is $\binom{n}{j}$ and by Theorem 1, $E_{C_{(n,j)}}$ is a regular graph. Using Equation 1, the size of $E_{C_{(n,j)}}$ is $\frac{\binom{n}{j}(r)}{2}$ where r is the degree of every vertex in $E_{C_{(n,j)}}$. Now, if $2 \leq j \leq \lceil \frac{n}{2} \rceil$, $S_{(C_n,j)}$ is a $\{j\binom{n-j}{j-1}\}$-regular graph. Hence, $|E(E_{C_{(n,j)}})| = \frac{j\binom{n-j}{j-1}\binom{n}{j}}{2}$.

The next illustration shows the size of $E_{C_{(n,j)}}$ given that $2 \leq j \leq \lceil \frac{n}{2} \rceil$.

Illustration 9. Given $E_{C_{(5,3)}}$ shown in Figure 19. We know that $j = 3$ which means that $\lceil \frac{5}{2} \rceil = 3 = j$. Since $E_{C_{(5,3)}}$ is a graph of order 10 and a 3-regular graph, using Theorem 7, $|E(E_{C_{(5,3)}})| = \frac{10(3)}{2} = 15$.

There are times that a $E_{C_{(n,j)}}$ is isomorphic to some special classes of a graph. The next theorem provides necessary and sufficient conditions when $E_{C_{(n,j)}}$ is a cycle graph of order 3.

Theorem 8. *A j-edge intersection graph of C_n $E_{C_{(n,j)}}$ is a cycle graph of order 3 if and only if $n = 3$ and $j = 2$.*

Proof. Assume that $E_{C_{(n,j)}}$ is a cycle graph of order 3. We know that every cycle graph is a 2-regular graph. Suppose that $n \neq 3$ or $j \neq 2$. Now, if $n > 3$, then $\binom{n}{j} = 1$ or $\binom{n}{j} \geq 4$. This contradicts the fact that the order of $E_{C_{(n,j)}}$ is 3. On the other hand, if $j < 2$, then j is 1. By Proposition 4, $E_{C_{(n,j)}}$ is a trivial graph, this is a contradiction to the assumption that $E_{C_{(n,j)}}$ is a cycle graph of order 3. Furthermore, if $j > 2$, then $\binom{n}{j}$ is equal to 1 or greater than or equal to 4 which is a contradiction that $E_{C_{(n,j)}}$ is a cycle graph of order 3. Therefore, $n = 3$ and $j = 2$. Conversely, assume that $n = 3$ and $j = 2$. Observe that $\lceil \frac{3}{2} \rceil = 2$. By Lemma 6, the degree of every vertex in $E_{C_{(3,2)}}$ is equal to 2. By Theorem 3, $\binom{3}{2} = 3$. Thus, every vertex in $E_{C_{(3,2)}}$ is adjacent to each other. Therefore, $E_{C_{(3,2)}}$ is a cycle graph of order 3.

The next illustration shows that a $E_{C_{(n,j)}}$ is isomorphic to the cycle graph when $n = 3$ and $j = 2$.

Illustration 10. Consider cycle graph C_3 and let $j = 2$. The vertex set of $E_{C_{(3,2)}}$ is given by

$$V(E_{C_{(3,2)}}) = \{\{12, 23\}, \{12, 31\}, \{23, 31\}\}.$$

Now, since $\{12, 23\} \cap \{12, 31\} = \{12\}$, it follows that $[\{12, 23\}, \{12, 31\}] \in E(E_{C_{(3,2)}})$. Similarly, $[\{12, 23\}, \{23, 31\}]$ and $[\{12, 31\}, \{23, 31\}]$ are also elements of $E(E_{C_{(3,2)}})$. Thus, $E_{C_{(3,2)}}$ is a cycle graph of order 3. Figure 20 is a pictorial representation of $E_{C_{(3,2)}}$.

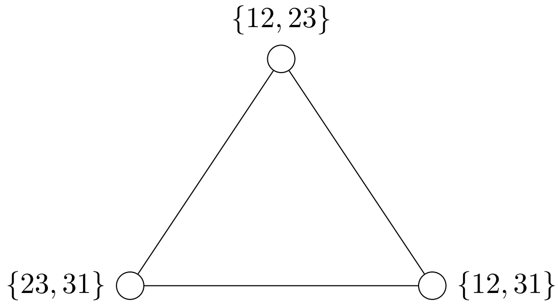

Figure 20: Pictorial Representation of $E_{C_{(3,2)}}$

4. Additional Parameters of $E_{C_{(n,j)}}$

In this section, other parameters of a graph such as independence number, and domination number are discussed.

4.1. Independence Number of $E_{C_{(n,2)}}$

This subsection determines a lower bound for the independence number of $E_{C_{(n,j)}}$ when $j = 2$. Theorem 9 determines the existence of independent set of $E_{C_{(n,2)}}$.

Theorem 9. *Let $E_{C_{(n,2)}}$ be a 2-edge intersection graph of C_n. Then $\alpha(E_{C_{(n,2)}}) \geq \lfloor \frac{n}{2} \rfloor$.*

Proof. Let $I = \{\{e_1, e_2\}, \{e_3, e_4\}, \cdots, \{e_{2\lfloor \frac{n}{2} \rfloor - 1}, e_{2\lfloor \frac{n}{2} \rfloor}\}\}$. For every $A \in I$, the cardinality of A is equal to 2. This means that $I \subseteq V(E_{C_{(n,2)}})$. For all two distinct elements in I, say A and B, $A \cap B = \emptyset$, thus $[A, B] \notin E(E_{C_{(n,2)}})$. Therefore, I is an independent set in $E_{C_{(n,2)}}$ and $\alpha(E_{C_{(n,2)}}) \geq |I|$. To determine $|I|$, two cases are considered. If n is even, then $\lfloor \frac{n}{2} \rfloor = \frac{n}{2}$ which implies that $I = \{\{e_1, e_2\}, \{e_3, e_4\}, \cdots, \{e_{n-1}, e_n\}\}$. In this case, $|I| = \frac{n}{2}$. If n is odd, then $\lfloor \frac{n}{2} \rfloor = \frac{n-1}{2}$ which menas that $I = \{\{e_1, e_2\}, \{e_3, e_4\}, \cdots, \{e_{n-2}, e_{n-1}\}\}$. This indicates that $|I| = \frac{n-1}{2}$. Thus, $\alpha(E_{C_{(n,2)}}) \geq \lfloor \frac{n}{2} \rfloor$.

Illustration 11, shows a lower bound for the independence number of $E_{C_{(n,2)}}$ when $n = 6$.

Illustration 11. Consider C_6, and let $j = 2$. The vertex set of $E_{C_{(6,2)}}$ is given by

$$\begin{aligned} V(E_{C_{(6,2)}}) = &\{\{12, 23\}, \{12, 34\}, \{12, 45\}, \{12, 56\}, \{12, 61\}, \{23, 34\}, \{23, 45\}, \{23, 56\}, \\ &\{23, 61\}, \{34, 45\}, \{34, 56\}, \{34, 61\}, \{45, 56\}, \{45, 61\}, \{56, 61\}\}. \end{aligned}$$

The pictorial representation in Figure 21 shows $E_{C_{(6,2)}}$.

Let $I_a \subseteq V(E_{C_{(6,2)}})$ for $1 \leq a \leq 3$. It can be noticed that $I_1 = \{\{e_i\}\}$ for all i element of the spanning subgraph with 2 edges is an independent set since $E_{C_{(6,2)}}$ is a simple graph. Moreover, the set $I_2 = \{\{12, 23\}, \{34, 45\}\}$ is an independent set since $\{12, 23\} \cap \{34, 45\} = \emptyset$ so there is no edge connecting the vertices $\{12, 23\}$ and $\{34, 45\}$. It means that there exists an independent set with cardinality 2. Furthermore, $I_3 = \{\{12, 23\}, \{34, 45\}, \{56, 61\}\}$ is also an independent set since there is no edge incident to themselves. It can be observed that $\lfloor \frac{6}{2} \rfloor = 3$ and $|I_3| = 3$. Thus, there exists an independent set of $E_{C_{(6,2)}}$ with cardinality 3. Moreover, there are no independent sets in $E_{C_{(6,2)}}$ of cardinality greater than 3. Therefore, $\alpha(E_{C_{(6,2)}}) \geq 3$ To verify this, we will be using Theorem 9, setting $n = 6$ and $j = 2$, we have

$$\begin{aligned} \alpha(E_{C_{(6,2)}}) \geq &\lfloor \frac{6}{2} \rfloor \\ = &3. \end{aligned}$$

4.2. Domination Number of $E_{C_{(n,2)}}$

This subsection determines an upper bound for the domination number of $E_{C_{(n,j)}}$ when $j = 2$. Theorem 10 determines the existence of dominating set of $E_{C_{(n,2)}}$.

Theorem 10. *Let $E_{C_{(n,2)}}$ be a 2-edge intersection graph of C_n. Then $\gamma(E_{C_{(n,2)}}) \leq \lfloor \frac{n}{2} \rfloor$.*

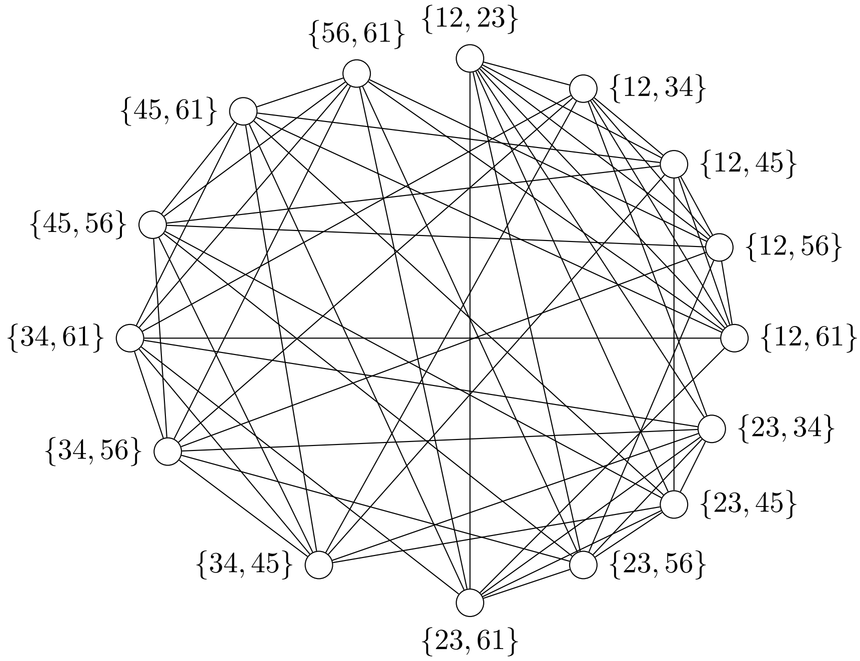

Figure 21: A Pictorial Representation of $E_{C_{(6,2)}}$

Proof. Let $I = \{\{e_1, e_2\}, \{e_3, e_4\}, \cdots, \{e_{2\lfloor \frac{n}{2} \rfloor - 1}, e_{2\lfloor \frac{n}{2} \rfloor}\}\}$. If n is even, then $\lfloor \frac{n}{2} \rfloor = \frac{n}{2}$. In this case, $I = \{\{e_1, e_2\}, \{e_3, e_4\}, \cdots, \{e_{n-1}, e_n\}\}$ and $|I| = \frac{n}{2}$. Since each e_i, $1 < i < n$, is incident with two other edges, it follows that if $\{e_{i-1}, e_i\} \in V(E_{C_{(n,2)}}) \backslash I$ then we have $\{e_i, e_{i+1}\} \in I$ such that $\{e_{i-1}, e_i\}$ is adjacent to $\{e_i, e_{i+1}\}$ since $|\{e_{i-1}, e_i\} \cap \{e_i, e_{i+1}\}| = 1$. Moreover, since e_1 is incident to e_2 and e_n, if $\{e_n, e_1\} \in V(E_{C_{(n,2)}}) \backslash I$ then we have $\{e_1, e_2\} \in I$ such that $\{e_n, e_1\}$ and $\{e_1, e_2\}$ are adjacent. Similarly, since e_n is incident to e_{n-1} and e_1, if $\{e_n, e_1\} \in V(E_{C_{(n,2)}}) \backslash I$ then we have $\{e_{n-1}, e_n\} \in I$ such that $\{e_n, e_1\}$ and $\{e_{n-1}, e_n\}$ are adjacent. Thus, for every $A \in V(E_{C_{(n,2)}}) \backslash I$, there exist $B \in I$ such that $|A \cap B| = 1$. Hence, $[A, B] \in E(E_{C_{(n,2)}})$. Therefore, I is a dominating set in $E_{C_{(n,2)}}$. On the other hand, if n is odd, then $\lfloor \frac{n}{2} \rfloor = \frac{n-1}{2}$. Hence, $I = \{\{e_1, e_2\}, \{e_3, e_4\}, \cdots, \{e_{n-2}, e_{n-1}\}\}$ and $I = \frac{n-1}{2}$. Similarly, for each e_i where $1 < i \leq n$, if $\{e_{i-1}, e_i\} \in V(E_{C_{(n,2)}}) \backslash I$ then we have $\{e_i, e_{i+1}\} \in I$ such that $\{e_{i-1}, e_i\}$ and $\{e_i, e_{i+1}\}$ are adjacent. Moreover, if $\{e_n, e_1\} \in V(E_{C_{(n,2)}}) \backslash I$ then we have $\{e_1, e_2\} \in I$ such that $\{e_n, e_1\}$ and $\{e_1, e_2\}$ are adjacent. Thus, for all $A \in V(E_{C_{(n,2)}}) \backslash I$, there exist $B \in I$ where $|A \cap B| = 1$. This implies that $[A, B] \in E(E_{C_{(n,2)}})$. Thus, I is a dominating set in $E_{C_{(n,2)}}$ and $\gamma(E_{C_{(n,2)}}) \leq |I|$. Therefore, $\gamma(E_{C_{(n,2)}}) \leq \lfloor \frac{n}{2} \rfloor$.

Illustration 12 shows an upper bound for domination number of $E_{C_{(6,2)}}$.

Illustration 12. Consider $E_{C_{(6,2)}}$ with pictorial representation shown in Figure 21. Let $I = \{\{12, 23\}, \{34, 45\}, \{56, 61\}\}$ where $I \subseteq V(E_{C_{(6,2)}})$. Now, the set $V(E_{C_{(6,2)}}) \backslash I$ is given by

$$V(E_{C_{(6,2)}})\backslash I =\{\{12,34\},\{12,45\},\{12,56\},\{12,61\},\{23,34\},\{23,45\},\{23,56\},\{23,61\},\\ \{34,56\},\{34,61\},\{45,56\},\{45,61\}\}.$$

It can be noted that the vertices $\{12,34\},\{12,45\},\{12,56\},\{12,61\},\{23,34\},\{23,45\}$, $\{23,56\},\{23,61\}$ of $V(E_{C_{(6,2)}})\backslash I$ are adjacent to $\{12,23\}$. Also, $\{34,56\},\{34,61\},\{45,56\}$, $\{45,61\}$ are vertices adjacent to $\{34,45\}$. Now, all of the elements of $V(E_{C_{(6,2)}})\backslash I$ are adjacent to either $\{12,23\}$ or $\{34,45\}$. Hence, I is a dominating set with cardinality equal to 3 which is also equal to $\lfloor\frac{6}{2}\rfloor$. Therefore, there exists a dominating set in $E_{C_{(6,2)}}$ with cardinality equal to $\lfloor\frac{n}{2}\rfloor$. Moreover, there is no dominating set in $E_{C_{(6,2)}}$ with cardinality less than 3. Therefore, $\gamma(E_{C_{(6,2)}}) \leq 3$. To verify this, using Theorem 10, setting $n = 6$ and $j = 2$, we have

$$\begin{aligned}\gamma(E_{C_{(6,2)}}) \leq& \left\lfloor\frac{6}{2}\right\rfloor\\ =&3.\end{aligned}$$

5. Conclusion and Recommendations

This study explores and defines a new graph which is called a j-edge graph of C_n, as well as some of its parameters and characteristics. A j-edge graph of C_n, denoted by $E_{C_{(n,j)}}$, is a graph whose vertex set contains the spanning subgraphs of C_n with j edges. Moreover, two distinct vertices are adjacent whenever they share exactly one common edge.

A j-edge graph of C_n does not contain any loop. Since $V(E_{C_{(n,j)}})$ is the collection of all distinct spanning subgraphs of C_n with j edges, it follows that $E(E_{C_{(n,j)}})$ does not have the same pair of vertices which means that $E_{C_{(n,j)}}$ has no multiple edges and it is a simple graph.

The researchers discovered that the order of $E_{C_{(n,j)}}$ is equal to $\binom{n}{j}$. From this, it was determined that $E_{C_{(n,j)}}$ is a trivial graph if $j = n$. Moreover, the degree of every vertex in $E_{C_{(n,j)}}$ when $j = 1$ and $\lceil\frac{n}{2}\rceil < j \leq n$ is both equal to 0 which will both produce an empty graph of order $\binom{n}{2}$. With these, the proponents focused only on $E_{C_{(n,j)}}$ when $2 \leq j \leq \lceil\frac{n}{2}\rceil$. For all $A \in V(E_{C_{(n,j)}})$ where $2 \leq j \leq \lceil\frac{n}{2}\rceil$, $deg(A) = j\binom{n-j}{j-1}$. The size of $E_{C_{(n,j)}}$ is equal to $|E(E_{C_{(n,j)}})| = \frac{j\binom{n-j}{j-1}\binom{n}{j}}{2}$ if $2 \leq j \leq \lceil\frac{n}{2}\rceil$.

Furthermore, this study showed that $E_{C_{(n,j)}}$ is a cycle graph of order 3 if and only if $n = 3$ and $j = 2$.

Finally, this study specified other parameters of $E_{C_{(n,j)}}$ such as independence number, and domination number. The proponents focused on $E_{C_{(n,j)}}$ when $j = 2$ on getting independence number, and domination number since $E_{C_{(n,2)}}$ is defined for all values of n. The researchers found a lower bound of the independence number of $E_{C_{(n,2)}}$ which is greater than or equal to $\lfloor\frac{n}{2}\rfloor$. In addition, it was discovered that an upper bound of

$\gamma(E_{C_{(n,2)}}) \leq \lfloor \frac{n}{2} \rfloor$.

The researcher believed that by focusing on the $E_{C_{(n,j)}}$, the parallel research study could be accomplished. The researchers have made the following recommendations in particular:

(i) It is recommended that future studies explore more in finding the independence number, and domination number for all values of j. Also, finding other parameters of $E_{C_{(n,j)}}$ such as its distance, adjacency matrix, complement, chromatic number, and isolate domination number can help in determining the graph;

(ii) The researchers suggest exploring the j-edge intersection graph of other special classes of a graph such as a path and complete graph. Also, the proponents suggest the notion of an edge-induced subgraph instead of a spanning subgraph.

(iii) The researchers recommend exploring the use of $E_{C_{(n,j)}}$ in solving real-life problems since many of the results in this study are based on combination formula which has much application in solving real-life problems.

References

[1] A. Asinowski, A. Suk. 'Edge intersection graphs of systems of paths on a grid with a bounded number of bends. *Discrete Applied Mathematics*. 2009

[2] T. Biedl, M. Stern. On edge-intersection graphs of k-bends paths grids. *Discrete Mathematics and Theoretical Cmputer Science*.2010.

[3] D. Guichard. Graph Coloring and Independence Number of a Complete Graph. *Withman College*.

[4] M.C. Golumbic, R. Jamison. The k-edge intersection graphs of paths in a tree. *Journal of Combinatorial Theory,*. 1985

[5] J.M. Harris et. al. Combinatorics and Graph Theory. *Springer New York, NY*.2008.

[6] B.E. Sagan, Combinatorics: The Art of Counting , *American Mathematical Society*. 2020.

[7] I. Sahul Hamid, S. Balamurgan. Isolate domination in graphs. *Arab Journal of Mathematical Sciences*. 2015.

[8] S. Selkow, The independence number of graphs in terms of degrees. *Discrete Mathematics*. 1993.

[9] A. Sugumaran, E. Jayachadran. Domination of number of some graph. *International Journal of Scientific Development and Research*. 2018.