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Dokdo filters and deductive systems of Sheffer stroke Hilbert algebras

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Abstract. To investigate the filter and deductive system of the Schaefer stroke Hilbert algebra using the Dokdo structure, the concept of Dokdo filter and Dokdo deductive system is defined, examples are given, and various properties are investigated. The Dokdo filter is formed by attaching appropriate conditions to the given Dokdo structure. Characterization of Dokdo filter is studied. Dokdo filters related to filters are constructed. Dokdo filter and Dokdo deductive system turn out to be the same concept.

2020 Mathematics Subject Classifications: 03B05, 03G25, 06F35, 08A72

Key Words and Phrases: Sheffer stroke Hilbert algebra, filter, deductive system, Dokdo filter, Dokdo deductive system.

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1. Introduction

The shaper stroke represented by the symbol "$|$" is a logical operation for two inputs that produces an invalid result only when both inputs are true, as shown in Table 1. The Sheffer stroke has been applied to several algebraic structures, for example, Boolean algebra, MV-algebra, BL-algebra, BCK-algebra, and ortholattices, etc., and it is also being dealt with in the fuzzy environment (see [1, 4, 5, 10–14]). In 2021, Oner et al. [12] applied the Sheffer stroke to Hilbert algebras. They introduced Sheffer stroke Hilbert algebra and investigated several properties. In [11], Oner et al. introduced the notion of deductive system and filter of Sheffer stroke Hilbert algebras, and dealt with their fuzzification. The Dokdo structure, classified as a hybrid structure, was introduced by Jun [3], and it consists

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Table 1: The truth table for the Sheffer stroke "|"

P	Q	$P\|Q$
F	F	T
F	T	T
T	F	T
T	T	F

of a combination of soft set, bipolar fuzzy set, and interval-value fuzzy set. Here, "Dokdo" is the name of Korea's most beautiful island.

Focusing on examining the filter and deductive system of the Sheffer stroke Hilbert algebra using the Dokdo structure, we define the concept of Dokdo filter and Dokdo deductive system, give examples, and then investigate various properties. We form a Dokdo filter by attaching appropriate conditions to a given Dokdo structure. We study characterizations of Dokdo filters. We construct Dokdo filters that are associated with filters. Ultimately, we show that Dokdo filter and Dokdo deductive system are a matching concept.

2. Preliminaries

2.1. Preliminaries on Sheffer stroke Hilbert algebras

Definition 1 ([9]). *Let $\mathcal{A} := (A, |)$ be a groupoid. Then the operation "|" is said to be Sheffer stroke or Sheffer operation if it satisfies:*

(s1) $(\forall \mathfrak{a}, \mathfrak{b} \in A)\ (\mathfrak{a}|\mathfrak{b} = \mathfrak{b}|\mathfrak{a})$,

(s2) $(\forall \mathfrak{a}, \mathfrak{b} \in A)\ ((\mathfrak{a}|\mathfrak{a})|(\mathfrak{a}|\mathfrak{b}) = \mathfrak{a})$,

(s3) $(\forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in A)\ (\mathfrak{a}|((\mathfrak{b}|\mathfrak{c})|(\mathfrak{b}|\mathfrak{c})) = ((\mathfrak{a}|\mathfrak{b})|(\mathfrak{a}|\mathfrak{b}))|\mathfrak{c})$,

(s4) $(\forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in A)\ ((\mathfrak{a}|((\mathfrak{a}|\mathfrak{a})|(\mathfrak{b}|\mathfrak{b})))|(\mathfrak{a}|((\mathfrak{a}|\mathfrak{a})|(\mathfrak{b}|\mathfrak{b}))) = \mathfrak{a})$.

Definition 2 ([12]). *A Sheffer stroke Hilbert algebra is a groupoid $\mathcal{X} := (X, |)$ with a Sheffer stroke "|" that satisfies:*

(sH1) $(\mathfrak{a}|((A)|(A)))|(((B)|((C)|(C)))|((B)|((C)|(C)))) = \mathfrak{a}|(\mathfrak{a}|\mathfrak{a})$,
where $A := \mathfrak{b}|(\mathfrak{c}|\mathfrak{c})$, $B := \mathfrak{a}|(\mathfrak{b}|\mathfrak{b})$ *and* $C := \mathfrak{a}|(\mathfrak{c}|\mathfrak{c})$,

(sH2) $\mathfrak{a}|(\mathfrak{b}|\mathfrak{b}) = \mathfrak{b}|(\mathfrak{a}|\mathfrak{a}) = \mathfrak{a}|(\mathfrak{a}|\mathfrak{a}) \ \Rightarrow\ \mathfrak{a} = \mathfrak{b}$

for all $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in X$.

Let $\mathcal{X} := (X, |)$ be a Sheffer stroke Hilbert algebra. Then the order relation " $\leq_X$ " on X is defined as follows:

$$(\forall \mathfrak{a}, \mathfrak{b} \in X)(\mathfrak{a} \leq_X \mathfrak{b} \ \Leftrightarrow\ \mathfrak{a}|(\mathfrak{b}|\mathfrak{b}) = 1). \tag{1}$$

We observe that the relation " $\leq_X$ " is a partial order in a Sheffer stroke Hilbert algebra $\mathcal{X} := (X, |)$ (see [12]).

Proposition 1 ([12]). *Every Sheffer stroke Hilbert algebra $\mathcal{X} := (X, |)$ satisfies:*

$$(\forall \mathfrak{a} \in X)(\mathfrak{a}|(\mathfrak{a}|\mathfrak{a}) = 1), \tag{2}$$
$$(\forall \mathfrak{a} \in X)(\mathfrak{a}|(1|1) = 1), \tag{3}$$
$$(\forall \mathfrak{a} \in X)(1|(\mathfrak{a}|\mathfrak{a}) = \mathfrak{a}), \tag{4}$$
$$(\forall \mathfrak{a}, \mathfrak{b} \in X)(\mathfrak{a} \leq_X \mathfrak{b}|(\mathfrak{a}|\mathfrak{a})), \tag{5}$$
$$(\forall \mathfrak{a}, \mathfrak{b} \in X)((\mathfrak{a}|(\mathfrak{b}|\mathfrak{b}))|(\mathfrak{b}|\mathfrak{b}) = (\mathfrak{b}|(\mathfrak{a}|\mathfrak{a}))|(\mathfrak{a}|\mathfrak{a})), \tag{6}$$
$$(\forall \mathfrak{a}, \mathfrak{b} \in X)\left(((\mathfrak{a}|(\mathfrak{b}|\mathfrak{b}))|(\mathfrak{b}|\mathfrak{b}))|(\mathfrak{b}|\mathfrak{b}) = \mathfrak{a}|(\mathfrak{b}|\mathfrak{b})\right), \tag{7}$$
$$(\forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in X)\left(\mathfrak{a}|((\mathfrak{b}|(\mathfrak{c}|\mathfrak{c}))|(\mathfrak{b}|(\mathfrak{c}|\mathfrak{c}))) = \mathfrak{b}|((\mathfrak{a}|(\mathfrak{c}|\mathfrak{c}))|(\mathfrak{a}|(\mathfrak{c}|\mathfrak{c})))\right), \tag{8}$$

Definition 3 ([11]). *Let $(X, |)$ be a Sheffer stroke Hilbert algebra. A subset F of X is called*

- *a deductive system of $(X, |)$ if it satisfies:*

$$1 \in F, \tag{9}$$
$$(\forall \mathfrak{a}, \mathfrak{b} \in X)(\mathfrak{a} \in F,\ \mathfrak{a}|(\mathfrak{b}|\mathfrak{b}) \in F \ \Rightarrow\ \mathfrak{b} \in F), \tag{10}$$

- *a filter of $(X, |)$ if it satisfies* (9) *and*

$$(\forall \mathfrak{a}, \mathfrak{b} \in X)(\mathfrak{b} \in F \ \Rightarrow\ \mathfrak{a}|(\mathfrak{b}|\mathfrak{b}) \in F), \tag{11}$$
$$(\forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in X)(\mathfrak{b}, \mathfrak{c} \in F \ \Rightarrow\ (\mathfrak{a}|(\mathfrak{b}|\mathfrak{c}))|(\mathfrak{b}|\mathfrak{c}) \in F). \tag{12}$$

2.2. Basic concepts about Dokdo structure

Let X be a set. A *bipolar fuzzy set* in X (see [6]) is an object of the following type

$$\mathring{f} = \{(\mathfrak{a}, \mathring{f}^-(\mathfrak{a}), \mathring{f}^+(\mathfrak{a})) \mid \mathfrak{a} \in X\} \tag{13}$$

where $\mathring{f}^- : X \to [-1, 0]$ and $\mathring{f}^+ : X \to [0, 1]$ are mappings. The bipolar fuzzy set which is described in (13) is simply denoted by $\mathring{f} := (X; \mathring{f}^-, \mathring{f}^+)$.

A bipolar fuzzy set can be reinterpreted as a function:

$$\mathring{f} : X \to [-1, 0] \times [0, 1],\ \mathfrak{a} \mapsto (\mathring{f}^-(\mathfrak{a}), \mathring{f}^+(\mathfrak{a})).$$

Denote by $BF(X)$ the set of all bipolar fuzzy sets in X. We define a binary relation "$\leq_b$" on $BF(X)$ as follows:

$$(\forall \mathring{f}, \mathring{g} \in BF(X))\left(\ \mathring{f} \leq_b \mathring{g} \ \Leftrightarrow \begin{cases} \mathring{f}^-(\mathfrak{a}) \geq \mathring{g}^-(\mathfrak{a}) \\ \mathring{f}^+(\mathfrak{a}) \leq \mathring{g}^+(\mathfrak{a}) \end{cases} \text{for all } \mathfrak{a} \in X\right). \tag{14}$$

Then $(\mathrm{BF(X)}, \leq_b)$ is a poset.

Let U be an initial universe set and X be a set of parameters. For any subset A of X, a pair (f^s, A) is called a *soft set* over U (see [7, 8]), where f^s is a mapping described as follows:

$$f^s : A \to 2^U$$

where 2^U is the power set of U. If $A = X$, the soft set (f^s, A) over U is simply denoted by f^s only.

A mapping $\tilde{f} : X \to [[0, 1]]$ is called an *interval-valued fuzzy set* (briefly, an IVF set) in X (see [2, 15]) where $[[0, 1]]$ is the set of all closed subintervals of $[0, 1]$, and members of $[[0, 1]]$ are called *interval numbers* and are denoted by $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, etc., where $\tilde{a} = [a_l, a_r]$ with $0 \leq a_l \leq a_r \leq 1$.

For every two interval numbers $\tilde{a}$ and $\tilde{b}$, we define

$$\tilde{a} \trianglelefteq \tilde{b} \text{ (or } \tilde{b} \trianglerighteq \tilde{a}) \ \Leftrightarrow \ a_l \leq b_l,\ a_r \leq b_r, \tag{15}$$

$$\tilde{a} = \tilde{b} \ \Leftrightarrow \ \tilde{a} \trianglelefteq \tilde{b},\ \tilde{b} \trianglelefteq \tilde{a}, \tag{16}$$

$$\text{rmin}\{\tilde{a}, \tilde{b}\} = [\min\{a_l, b_l\}, \min\{a_r, b_r\}]. \tag{17}$$

Let U be an initial universe set and X a set of parameters. A triple $Dok_f := (\mathring{f}, f^s, \tilde{f})$ is called a *Dokdo structure* in (U, X) (see [3]) if $\mathring{f} : X \to [-1, 0] \times [0, 1]$ is a bipolar fuzzy set in X, $f^s : X \to 2^U$ is a soft set over U and $\tilde{f} : X \to [[0, 1]]$ is an interval-valued fuzzy set in X.

The Dokdo structure $Dok_f := (\mathring{f}, f^s, \tilde{f})$ in (U, X) can be represented as follows:

$$\begin{aligned} Dok_f := (\mathring{f}, f^s, \tilde{f}) : X \to ([-1, 0] \times [0, 1]) \times 2^U \times [[0, 1]], \\ x \mapsto \left(\mathring{f}(x), f^s(x), \tilde{f}(x)\right) \end{aligned} \tag{18}$$

where $\mathring{f}(x) = (\mathring{f}^-(x), \mathring{f}^+(x))$ and $\tilde{f}(x) = [\tilde{f}_L(x), \tilde{f}_R(x)]$.

Given a Dokdo structure $Dok_f := (\mathring{f}, f^s, \tilde{f})$ in a Dokdo universe (U, X), we consider the following sets:

$$\mathring{f}(M, m) := \left\{ \frac{x}{(y,z)} \in \frac{X}{X \times X} \,\middle|\, \begin{array}{l} \mathring{f}^-(x) \leq \max\{\mathring{f}^-(y), \mathring{f}^-(z)\} \\ \mathring{f}^+(x) \geq \min\{\mathring{f}^+(y), \mathring{f}^+(z)\} \end{array} \right\}$$

and

$$\begin{aligned} &\mathring{f}(t^-) := \{x \in X \mid \mathring{f}^-(x) \leq t^-\}, \\ &\mathring{f}(t^+) := \{x \in X \mid \mathring{f}^+(x) \geq t^+\}, \\ &\mathring{f}(t^-, t^+) := \mathring{f}(t^-) \cap \mathring{f}(t^+), \\ &f^s_\alpha := \{x \in X \mid f^s(x) \supseteq \alpha\}, \\ &\tilde{f}_{\tilde{a}} := \{x \in X \mid \tilde{f}(x) \trianglerighteq \tilde{a}\}, \end{aligned}$$

where $(t^-, t^+) \in [-1, 0] \times [0, 1]$, $\alpha \in 2^U$ and $\tilde{a} = [a_l, a_r]$.

3. Dokdo filters

Let U be an initial universe set and X a set of parameters. We say that the pair $(U, \mathcal{X})$ is the *SSH-Dokdo universe* if $\mathcal{X} := (X, |)$ is a Sheffer stroke Hilbert algebra. In what follows, let $(U, \mathcal{X})$ denote the SSH-Dokdo universe unless otherwise specified.

Definition 4. *A Dokdo structure $Dok_f := (\mathring{f},\ f^s,\ \tilde{f})$ is called a Dokdo filter of $(U, \mathcal{X})$ if it satisfies:*

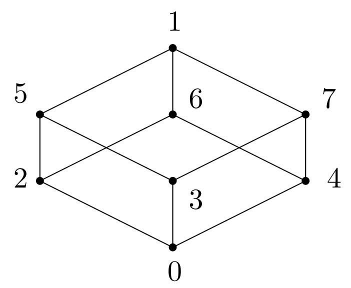
$$(\forall x \in X) \left(\begin{array}{l} \frac{1}{(x,x)} \in \mathring{f}(M, m) \\ f^s(1) \supseteq f^s(x),\ \tilde{f}(1) \trianglerighteq \tilde{f}(x) \end{array} \right), \tag{19}$$

$$(\forall x, y \in X) \left(\begin{array}{l} \frac{x|(y|y)}{(y,y)} \in \mathring{f}(M, m) \\ f^s(x|(y|y)) \supseteq f^s(y) \\ \tilde{f}(x|(y|y)) \trianglerighteq \tilde{f}(y) \end{array} \right), \tag{20}$$

$$(\forall x, y, z \in X) \left(\begin{array}{l} \frac{(x|(y|z))|(y|z)}{(y,z)} \in \mathring{f}(M, m) \\ f^s((x|(y|z))|(y|z)) \supseteq f^s(y) \cap f^s(z) \\ \tilde{f}((x|(y|z))|(y|z)) \trianglerighteq \mathrm{rmin}\{\tilde{f}(y), \tilde{f}(z)\} \end{array} \right). \tag{21}$$

Example 1. *Consider a set $X = \{0, 1, 2, 3, 4, 5, 6, 7\}$. The Hasse diagram and the Sheffer stroke "$|$" on X are given by Figure 1 and Table 2, respectively.*

Figure 1: Hasse Diagram



Then $\mathcal{X} := (X, |)$ is a Sheffer stroke Hilbert algebra (see [12]). Let $Dok_f := (\mathring{f},\ f^s,\ \tilde{f})$ be a Dokdo structure in $(X, U = \mathbb{Z})$ which is given by Table 3.
It is routine to verify that $Dok_f := (\mathring{f},\ f^s,\ \tilde{f})$ is a Dokdo filter of $(U = \mathbb{Z}, \mathcal{X})$.

Proposition 2. *Every Dokdo filter $Dok_f := (\mathring{f},\ f^s,\ \tilde{f})$ of $(U, \mathcal{X})$ satisfies:*

$$(\forall x, y \in X) \left(\begin{array}{l} \frac{(x|(y|y))|(y|y)}{(x,x)} \in \mathring{f}(M, m) \\ f^s((x|(y|y))|(y|y)) \supseteq f^s(x) \\ \tilde{f}((x|(y|y))|(y|y)) \trianglerighteq \tilde{f}(x) \end{array} \right), \tag{22}$$

$$(\forall x, y \in X) \left(x \leq_X y \ \Rightarrow \left\{ \begin{array}{l} \frac{y}{(x,x)} \in \mathring{f}(M, m) \\ f^s(x) \subseteq f^s(y) \\ \tilde{f}(x) \trianglelefteq \tilde{f}(y) \end{array} \right. \right). \tag{23}$$

Table 2: Cayley table for the Sheffer stroke "|"

$\mid$	0	2	3	4	5	6	7	1
0	1	1	1	1	1	1	1	1
2	1	7	1	1	7	7	1	7
3	1	1	6	1	6	1	6	6
4	1	1	1	5	1	5	5	5
5	1	7	6	1	4	7	6	4
6	1	7	1	5	7	3	5	3
7	1	1	6	5	6	5	2	2
1	1	7	6	5	4	3	2	0

Table 3: Tabular representation of $Dok_f := (\mathring{f},\, f^s,\, \tilde{f})$

X	$\mathring{f}(x)$	$f^s(x)$	$\tilde{f}(x)$
0	$(-0.41, 0.48)$	$16\mathbb{N}$	$[0.28, 0.65]$
2	$(-0.55, 0.67)$	$16\mathbb{N}$	$[0.28, 0.65]$
3	$(-0.41, 0.48)$	$8\mathbb{N}$	$[0.28, 0.65]$
4	$(-0.41, 0.48)$	$16\mathbb{N}$	$[0.32, 0.73]$
5	$(-0.63, 0.78)$	$8\mathbb{N}$	$[0.28, 0.65]$
6	$(-0.55, 0.67)$	$16\mathbb{N}$	$[0.38, 0.76]$
7	$(-0.41, 0.48)$	$4\mathbb{N}$	$[0.32, 0.73]$
1	$(-0.71, 0.82)$	$2\mathbb{N}$	$[0.42, 0.91]$

Proof. Let $Dok_f := (\mathring{f},\, f^s,\, \tilde{f})$ be a Dokdo filter of $(U, \mathcal{X})$. Then

$$\mathring{f}^-((x|(y|y))|(y|y)) = \mathring{f}^-((y|(x|x))|(x|x)) \leq \max\{\mathring{f}^-(x), \mathring{f}^-(x)\} = \mathring{f}^-(x),$$

and

$$\mathring{f}^+((x|(y|y))|(y|y)) = \mathring{f}^+((y|(x|x))|(x|x)) \geq \min\{\mathring{f}^+(x), \mathring{f}^+(x)\} = \mathring{f}^+(x)$$

by (6) and (21), that is, $\frac{(x|(y|y))|(y|y)}{(x,x)} \in \mathring{f}(M, m)$ for all $x, y \in X$. Also, we have

$$f^s((x|(y|y))|(y|y)) = f^s((y|(x|x))|(x|x)) \supseteq f^s(x) \cap f^s(x) = f^s(x)$$

and

$$\tilde{f}((x|(y|y))|(y|y)) = \tilde{f}((y|(x|x))|(x|x)) \trianglerighteq \text{rmin}\{\tilde{f}(x), \tilde{f}(x)\} = \tilde{f}(x)$$

for all $x, y \in X$. Therefore (22) is valid. Let $x, y \in X$ be such that $x \leq_X y$. Then $x|(y|y) = 1$ by (1). Using (4) and (22), we have

$$\mathring{f}^-(y) = \mathring{f}^-(1|(y|y)) = \mathring{f}^-((x|(y|y))|(y|y)) \leq \mathring{f}^-(x),$$

$$\mathring{f}^{+}(y) = \mathring{f}^{+}(1|(y|y)) = \mathring{f}^{+}((x|(y|y))|(y|y)) \geq \mathring{f}^{+}(x),$$

which shows that $\frac{y}{(x,x)} \in \mathring{f}(M,m)$. Also we get

$$f^{s}(x) \subseteq f^{s}((x|(y|y))|(y|y)) = f^{s}(1|(y|y)) = f^{s}(y)$$

and $\tilde{f}(x) \trianglelefteq \tilde{f}((x|(y|y))|(y|y)) = \tilde{f}(1|(y|y)) = \tilde{f}(y)$.

We have a question: If a Dokdo structure $Dok_f := (\mathring{f},\ f^s,\ \tilde{f})$ in $(U, \mathcal{X})$ satisfies the condition (23) then is it a Dokdo filter of $(U, \mathcal{X})$? The example below provides a negative answer to the question.

Example 2. *Consider a set $X = \{0, 1, 2, 3\}$. The Hasse diagram and the Sheffer stroke "$|$" on X are given by Figure 2 and Table 4, respectively.*

Figure 2: Hasse Diagram

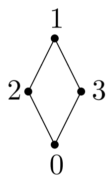


Table 4: Cayley table for the Sheffer stroke "$|$"

$\vert$	1	2	3	0
1	0	3	2	1
2	3	3	1	1
3	2	1	2	1
0	1	1	1	1

Then $\mathcal{X} := (X, |)$ is a Sheffer stroke Hilbert algebra (see [12]). Let $Dok_f := (\mathring{f},\ f^s,\ \tilde{f})$ be a Dokdo structure in $(U = \mathbb{Z}, X)$ which is given by Table 5.

Table 5: Tabular representation of $Dok_f := (\mathring{f},\ f^s,\ \tilde{f})$

X	$\mathring{f}(x)$	$f^s(x)$	$\tilde{f}(x)$
0	$(-0.13, 0.10)$	$8\mathbb{N}$	$[0.29, 0.63]$
2	$(-0.38, 0.17)$	$4\mathbb{N}$	$[0.32, 0.67]$
3	$(-0.55, 0.29)$	$4\mathbb{Z}$	$[0.36, 0.75]$
1	$(-0.82, 0.63)$	$2\mathbb{Z}$	$[0.47, 0.89]$

It is routine to check that $Dok_f := (\mathring{f},\ f^s,\ \tilde{f})$ *in* $(U, \mathcal{X})$ *satisfies the condition* (23)*. But it is not a Dokdo filter of* $(U = \mathbb{Z}, X)$ *since*

$$\tfrac{(0|(2|3))|(2|3)}{(2,3)} = \tfrac{0}{(2,3)} \notin \mathring{f}(M, m),$$

$f^s((0|(2|3))|(2|3)) = f^s(0) = 8\mathbb{N} \subsetneq 4\mathbb{N} = f^s(2) \cap f^s(3)$ *or*

$$\tilde{f}((0|(2|3))|(2|3)) = \tilde{f}(0) = [0.29, 0.63] \not\trianglerighteq [0.32, 0.67] = \text{rmin}\{\tilde{f}(2), \tilde{f}(3)\}.$$

We provide conditions for the Dokdo structure to be the Dokdo filter.

Theorem 1. *Let* $Dok_f := (\mathring{f},\ f^s,\ \tilde{f})$ *be a Dokdo structure in* (U, X)*. Then it is Dokdo filter of* (U, X) *if and only if it satisfies the condition* (23) *and*

$$(\forall x, y \in X) \left(\begin{array}{l} \frac{(x|y)|(x|y)}{(x,y)} \in \mathring{f}(M, m), \\ f^s((x|y)|(x|y)) \supseteq f^s(x) \cap f^s(y), \\ \tilde{f}((x|y)|(x|y)) \trianglerighteq \text{rmin}\{\tilde{f}(x), \tilde{f}(y)\} \end{array} \right). \tag{24}$$

Proof. Let $Dok_f := (\mathring{f},\ f^s,\ \tilde{f})$ be a Dokdo filter of (U, X). The condition (23) is valid by Proposition 2. Since

$$\begin{aligned} ((1|1)|(x|y))|(x|y) &\overset{(s1)}{=} ((x|y)|(1|1))|(x|y) \overset{(3)}{=} 1|(x|y) \\ &\overset{(s2)}{=} 1|(((x|y)|(x|y))|((x|y)|(x|y))) \\ &\overset{(4)}{=} (x|y)|(x|y) \end{aligned}$$

for all $x, y \in X$, it follows from (21) that

$$\begin{aligned} &\tfrac{(x|y)|(x|y)}{(x,y)} = \tfrac{((1|1)|(x|y))|(x|y)}{(x,y)} \in \mathring{f}(M, m), \\ &f^s((x|y)|(x|y)) = f^s(((1|1)|(x|y))|(x|y)) \supseteq f^s(x) \cap f^s(y)\} \end{aligned}$$

and $\tilde{f}((x|y)|(x|y)) = \tilde{f}(((1|1)|(x|y))|(x|y)) \trianglerighteq \text{rmin}\{\tilde{f}(x), \tilde{f}(y)\}$ for all $x, y \in X$.

Conversely, suppose that a Dokdo structure $Dok_f := (\mathring{f},\ f^s,\ \tilde{f})$ satisfies the conditions (23) and (24). Since $x \leq_X 1$ for all $x \in X$, we have $\frac{1}{(x,x)} \in \mathring{f}(M, m)$, $f^s(x) \subseteq f^s(1)$, and $\tilde{f}(x) \trianglelefteq \tilde{f}(1)$ by (23). Since $y \leq_X x|(y|y)$ for all $x, y \in X$, we have $\frac{x|(y|y)}{(y,y)} \in \mathring{f}(M, m)$, $f^s(y) \subseteq f^s(x|(y|y))$, and $\tilde{f}(y) \trianglelefteq \tilde{f}(x|(y|y))$ by (23). In (5), if we replace $\mathfrak{a}$ and $\mathfrak{b}$ with $(y|z)|(y|z)$ and $x|(y|z)$, respectively, and use (s2), then

$$(y|z)|(y|z) \leq_X (x|(y|z))|(((y|z)|(y|z))|((y|z)|(y|z))) = (x|(y|z))|(y|z)$$

for all $x, y, z \in X$. Using (23) and (24), we have

$$\tfrac{(x|(y|z))|(y|z)}{((y|z)|(y|z),(y|z)|(y|z))} \in \mathring{f}(M, m),$$

and so $\max\{f^-(y), f^-(z)\} \geq f^-((y|z)|(y|z)) \geq f^-((x|(y|z))|(y|z))$ and

$$\min\{f^+(y), f^+(z)\} \leq f^+((y|z)|(y|z)) \leq f^+((x|(y|z))|(y|z)).$$

Hence $\frac{(x|(y|z))|(y|z)}{(y,z)} \in \mathring{f}(M, m)$. Also, we have

$$f^s(y) \cap f^s(z) \subseteq f^s((y|z)|(y|z)) \subseteq f^s((x|(y|z))|(y|z))$$

and $\operatorname{rmin}\{\tilde{f}(y), \tilde{f}(z)\} \trianglelefteq \tilde{f}((y|z)|(y|z)) \trianglelefteq \tilde{f}((x|(y|z))|(y|z))$. Therefore, $Dok_f := (\mathring{f},\ f^s,\ \tilde{f})$ is a Dokdo filter of (U, X).

Theorem 2. *If $Dok_f := (\mathring{f},\ f^s,\ \tilde{f})$ is a Dokdo filter of (U, X), then the sets $\mathring{f}(t^-, t^+)$, f^s_α and $\tilde{f}_{\tilde{a}}$ are filters of $\mathcal{X} := (X, |)$ whenever they are nonempty for all $(t^-, t^+) \in [-1, 0] \times [0, 1]$, $\alpha \in 2^U$ and $\tilde{a} = [a_l, a_r]$.*

Proof. Assume that $Dok_f := (\mathring{f},\ f^s,\ \tilde{f})$ is a Dokdo filter of (U, X) and let $(t^-, t^+) \in [-1, 0] \times [0, 1]$, $\alpha \in 2^U$ and $\tilde{a} = [a_l, a_r]$ be such that $\mathring{f}(t^-, t^+)$, f^s_α and $\tilde{f}_{\tilde{a}}$ are nonempty. It is clear that $1 \in \mathring{f}(t^-, t^+) \cap f^s_\alpha \cap \tilde{f}_{\tilde{a}}$ by (19). Let $x \in X$ and $y \in \mathring{f}(t^-, t^+) \cap f^s_\alpha \cap \tilde{f}_{\tilde{a}}$. Then $\mathring{f}^-(y) \leq t^-$, $\mathring{f}^+(y) \geq t^+$, $f^s(y) \supseteq \alpha$, and $\tilde{f}(y) \trianglerighteq \tilde{a}$. Using (20), we have $\mathring{f}^-(x|(y|y)) \leq \mathring{f}^-(y) \leq t^-$ and $\mathring{f}^+(x|(y|y)) \geq \mathring{f}^+(y) \geq t^+$, that is, $x|(y|y) \in \mathring{f}(t^-, t^+)$. Also, we obtain $f^s(x|(y|y)) \supseteq f^s(y) \supseteq \alpha$ and $\tilde{f}(x|(y|y)) \trianglerighteq \tilde{f}(y) \trianglerighteq \tilde{a}$. Hence $x|(y|y) \in f^s_\alpha \cap \tilde{f}_{\tilde{a}}$. Let $x \in X$ and $y, z \in \mathring{f}(t^-, t^+) \cap f^s_\alpha \cap \tilde{f}_{\tilde{a}}$. Then $\mathring{f}^-(y) \leq t^-$, $\mathring{f}^+(y) \geq t^+$, $f^s(y) \supseteq \alpha$, $\tilde{f}(y) \trianglerighteq \tilde{a}$, $\mathring{f}^-(z) \leq t^-$, $\mathring{f}^+(z) \geq t^+$, $f^s(z) \supseteq \alpha$, and $\tilde{f}(z) \trianglerighteq \tilde{a}$. It follows from (21) that

$$\mathring{f}^-((x|(y|z))|(y|z)) \leq \max\{\mathring{f}^-(y), \mathring{f}^-(z)\} \leq t^-,$$
$$\mathring{f}^+((x|(y|z))|(y|z)) \geq \min\{\mathring{f}^+(y), \mathring{f}^+(z)\} \geq t^+,$$

i.e., $(x|(y|z))|(y|z) \in \mathring{f}(t^-, t^+)$. Also, $f^s((x|(y|z))|(y|z)) \supseteq f^s(y) \cap f^s(z) \supseteq \alpha$ and

$$\tilde{f}((x|(y|z))|(y|z)) \trianglerighteq \operatorname{rmin}\{\tilde{f}(y), \tilde{f}(z)\} \trianglerighteq \tilde{a}.$$

Thus $(x|(y|z))|(y|z) \in f^s_\alpha \cap \tilde{f}_{\tilde{a}}$. Therefore, $\mathring{f}(t^-, t^+)$, f^s_α and $\tilde{f}_{\tilde{a}}$ are filters of $\mathcal{X} := (X, |)$.

The example below shows that the converse of Theorem 2 may not be true.

Example 3. *Consider the Sheffer stroke Hilbert algebra $\mathcal{X} := (X, |)$ in Example 1 and let $Dok_f := (\mathring{f},\ f^s,\ \tilde{f})$ be a Dokdo structure in $(X, U = \mathbb{Z})$ which is given by Table 6. It is routine to verify that the nonempty sets $\mathring{f}(t^-, t^+)$, f^s_α and $\tilde{f}_{\tilde{a}}$ are filters of $\mathcal{X} := (X, |)$ for all $(t^-, t^+) \in [-1, 0] \times [0, 1]$, $\alpha \in 2^U$ and $\tilde{a} = [a_l, a_r]$. But $Dok_f := (\mathring{f},\ f^s,\ \tilde{f})$ is not a Dokdo filter of (U, X) since $\frac{2|(4|4)}{(4,4)} = \frac{7}{(4,4)} \notin \mathring{f}(M, m)$.*

We provide conditions for a Dokdo structure to be a Dokdo filter.

Theorem 3. *Given a Dokdo stucture $Dok_f := (\mathring{f},\ f^s,\ \tilde{f})$ in (U, X), If the nonempty sets $\mathring{f}(t^-)$, $\mathring{f}(t^+)$, f^s_α and $\tilde{f}_{\tilde{a}}$ are filters of $\mathcal{X} := (X, |)$ for all $(t^-, t^+) \in [-1, 0] \times [0, 1]$, $\alpha \in 2^U$ and $\tilde{a} = [a_l, a_r]$, then $Dok_f := (\mathring{f},\ f^s,\ \tilde{f})$ is a Dokdo filter of (U, X).*

Table 6: Tabular representation of $Dok_f := (\mathring{f},\, f^s,\, \tilde{f})$

X	$\mathring{f}(x)$	$f^s(x)$	$\tilde{f}(x)$
0	$(-0.37, 0.48)$	$8\mathbb{N}$	$[0.26, 0.62]$
2	$(-0.37, 0.67)$	$8\mathbb{N}$	$[0.26, 0.62]$
3	$(-0.37, 0.48)$	$8\mathbb{Z}$	$[0.26, 0.62]$
4	$(-0.55, 0.48)$	$8\mathbb{N}$	$[0.31, 0.70]$
5	$(-0.63, 0.78)$	$8\mathbb{Z}$	$[0.26, 0.62]$
6	$(-0.55, 0.48)$	$8\mathbb{N}$	$[0.36, 0.73]$
7	$(-0.37, 0.48)$	$4\mathbb{Z}$	$[0.32, 0.70]$
1	$(-0.71, 0.82)$	$2\mathbb{Z}$	$[0.41, 0.88]$

Proof. Assume that $\mathring{f}(t^-)$, $\mathring{f}(t^+)$, f^s_α and $\tilde{f}_{\tilde{a}}$ are nonempty filters of $\mathcal{X} := (X, |)$ for all $(t^-, t^+) \in [-1, 0] \times [0, 1]$, $\alpha \in 2^U$ and $\tilde{a} = [a_l, a_r]$. If there is $\mathfrak{a} \in X$ such that $\frac{1}{(\mathfrak{a},\mathfrak{a})} \notin \mathring{f}(M, m)$, then $\mathring{f}^-(1) > \mathring{f}^-(\mathfrak{a})$ or $\mathring{f}^+(1) < \mathring{f}^+(\mathfrak{a})$. Hence $\mathfrak{a} \in \mathring{f}(\mathring{f}^-(\mathfrak{a})) \cap \mathring{f}(\mathring{f}^+(\mathfrak{a}))$ and $1 \notin \mathring{f}(\mathring{f}^-(\mathfrak{a})) \cap \mathring{f}(\mathring{f}^+(\mathfrak{a}))$, a contradiction. Thus $\frac{1}{(x,x)} \in \mathring{f}(M, m)$ for all $x \in X$. Let $x, \mathfrak{a} \in X$ be such that $f^s(x) = \alpha$ and $\tilde{f}(\mathfrak{a}) = \tilde{a}$. Then $(x, \mathfrak{a}) \in f^s_\alpha \times \tilde{f}_{\tilde{a}}$, i.e., $f^s_\alpha \neq \emptyset \neq \tilde{f}_{\tilde{a}}$, and so $1 \in f^s_\alpha \cap \tilde{f}_{\tilde{a}}$. Hence $f^s(1) \supseteq \alpha = f^s(x)$ and $\tilde{f}(1) \trianglerighteq \tilde{a} = \tilde{f}(\mathfrak{a})$. If there are $\mathfrak{a}, \mathfrak{b} \in X$ such that $\frac{\mathfrak{a}|(\mathfrak{b}|\mathfrak{b})}{(\mathfrak{b},\mathfrak{b})} \notin \mathring{f}(M, m)$, then $\mathring{f}^-(\mathfrak{a}|(\mathfrak{b}|\mathfrak{b})) > \mathring{f}^-(\mathfrak{b})$ or $\mathring{f}^+(\mathfrak{a}|(\mathfrak{b}|\mathfrak{b})) < \mathring{f}^+(\mathfrak{b})$. It follows that $\mathfrak{b} \in \mathring{f}(\mathring{f}^-(\mathfrak{b})) \cap \mathring{f}(\mathring{f}^+(\mathfrak{b}))$ and $\mathfrak{a}|(\mathfrak{b}|\mathfrak{b}) \notin \mathring{f}(\mathring{f}^-(\mathfrak{b})) \cap \mathring{f}(\mathring{f}^+(\mathfrak{b}))$, a contradiction. Hence $\frac{x|(y|y)}{(y,y)} \in \mathring{f}(M, m)$ for all $x, y \in X$. Let $y, \mathfrak{b} \in X$ be such that $f^s(y) = \alpha$ and $\tilde{f}(\mathfrak{b}) = \tilde{a}$. Then $(y, \mathfrak{b}) \in f^s_\alpha \times \tilde{f}_{\tilde{a}}$, which implies that $(x|(y|y), \mathfrak{a}|(\mathfrak{b}|\mathfrak{b})) \in f^s_\alpha \times \tilde{f}_{\tilde{a}}$ for all $x, \mathfrak{a} \in X$. Hence $f^s(x|(y|y)) \supseteq \alpha = f^s(y)$ and $\tilde{f}(\mathfrak{a}|(\mathfrak{b}|\mathfrak{b})) \trianglerighteq \tilde{a} = \tilde{f}(\mathfrak{b})$. If there are $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in X$ such that $\frac{(\mathfrak{a}|(\mathfrak{b}|\mathfrak{c}))|(\mathfrak{b}|\mathfrak{c})}{(\mathfrak{b},\mathfrak{c})} \notin \mathring{f}(M, m)$, then

$$\mathring{f}^-((\mathfrak{a}|(\mathfrak{b}|\mathfrak{c}))|(\mathfrak{b}|\mathfrak{c})) > \max\{\mathring{f}^-(\mathfrak{b}), \mathring{f}^-(\mathfrak{c})\}$$

or $\mathring{f}^+((\mathfrak{a}|(\mathfrak{b}|\mathfrak{c}))|(\mathfrak{b}|\mathfrak{c})) < \min\{\mathring{f}^+(\mathfrak{b}), \mathring{f}^+(\mathfrak{c})\}$. If we take $t^- := \max\{\mathring{f}^-(\mathfrak{b}), \mathring{f}^-(\mathfrak{c})\}$ and $t^+ := \min\{\mathring{f}^+(\mathfrak{b}), \mathring{f}^+(\mathfrak{c})\}$, then $\mathfrak{b}, \mathfrak{c} \in \mathring{f}(t^-) \cap \mathring{f}(t^+)$ and $(\mathfrak{a}|(\mathfrak{b}|\mathfrak{c}))|(\mathfrak{b}|\mathfrak{c}) \notin \mathring{f}(t^-) \cap \mathring{f}(t^+)$. This is a contradiction, and thus $\frac{(x|(y|z))|(y|z)}{(y,z)} \in \mathring{f}(M, m)$ for all $x, y, z \in X$. Let $(x, \mathfrak{a}), (y, \mathfrak{b}), (z, \mathfrak{c}) \in X \times X$ be such that $f^s(y) \cap f^s(z) = \alpha$ and $\text{rmin}\{\tilde{f}(\mathfrak{b}), \tilde{f}(\mathfrak{c})\} = \tilde{a}$. Then $y, z \in f^s_\alpha$ and $\mathfrak{b}, \mathfrak{c} \in \tilde{f}_{\tilde{a}}$. It follows that $(x|(y|z))|(y|z) \in f^s_\alpha$ and $(\mathfrak{a}|(\mathfrak{b}|\mathfrak{c}))|(\mathfrak{b}|\mathfrak{c}) \in \tilde{f}_{\tilde{a}}$. Therefore $f^s((x|(y|z))|(y|z)) \supseteq \alpha = f^s(y) \cap f^s(z)$ and $\tilde{f}((\mathfrak{a}|(\mathfrak{b}|\mathfrak{c}))|(\mathfrak{b}|\mathfrak{c})) \trianglerighteq \tilde{a} = \text{rmin}\{\tilde{f}(\mathfrak{b}), \tilde{f}(\mathfrak{c})\}$. Consequently, $Dok_f := (\mathring{f},\, f^s,\, \tilde{f})$ is a Dokdo filter of (U, X).

Theorem 4. *Given a nonempty subset F of X, let $Dok_{f_F} := (\mathring{f}_F,\, f^s_F,\, \tilde{f}_F)$ be a Dokdo structure in (U, X) defined by*

$$\begin{aligned} Dok_{f_F} := (\mathring{f}_F, f^s_F, \tilde{f}_F) : X &\to ([-1, 0] \times [0, 1]) \times 2^U \times [[0, 1]], \\ x &\mapsto \begin{cases} ((t^-, t^+), \alpha, \tilde{a}) & \text{if } x \in F, \\ ((0, 0),\, \emptyset,\, \tilde{0}) & \text{otherwise,} \end{cases} \end{aligned} \tag{25}$$

where $t^- \neq 0 \neq t^+$, $\alpha \neq \emptyset$ *and* $\tilde{a} \neq \tilde{0} := [0,0]$. *Then* $Dok_{f_F} := (\mathring{f}_F,\ f^s_F,\ \tilde{f}_F)$ *is a Dokdo filter of* (U, X) *if and only if* F *is a filter of* $\mathcal{X} := (X, |)$. *Moreover, we have*

$$F = X_{f_F} := \{x \in X \mid \mathring{f}_F(x) = \mathring{f}_F(1),\ f^s_F(x) = f^s_F(1),\ \tilde{f}_F(x) = \tilde{f}_F(1)\}.$$

Proof. Assume that $Dok_{f_F} := (\mathring{f}_F,\ f^s_F,\ \tilde{f}_F)$ is a Dokdo filter of (U, X). Then $Dok_{f_F}(1) = ((t^-, t^+), \alpha, \tilde{a})$ by (19), and so $1 \in F$. Let $x, y \in X$ be such that $y \in F$. Then $\mathring{f}^-_F(y) = t^-$, $\mathring{f}^+_F(y) = t^+$, $f^s_F(y) = \alpha$, and $\tilde{f}_F(y) = \tilde{a}$. It follows from (20) that $\mathring{f}^-_F(x|(y|y)) \leq \mathring{f}^-_F(y) = t^-$, $\mathring{f}^+_F(x|(y|y)) \geq \mathring{f}^+_F(y) = t^+$, $f^s_F(x|(y|y)) \supseteq f^s_F(y) = \alpha$ and $\tilde{f}_F(x|(y|y)) \trianglerighteq \tilde{f}_F(y) = \tilde{a}$. Hence $\mathring{f}_F(x|(y|y)) = (t^-, t^+)$, $f^s_F(x|(y|y)) = \alpha$ and $\tilde{f}_F(x|(y|y)) = \tilde{a}$. This shows that $x|(y|y) \in F$. Let $y, z \in F$. Then $\mathring{f}^-_F(y) = t^- = \mathring{f}^-_F(z)$, $\mathring{f}^+_F(y) = t^+ = \mathring{f}^+_F(z)$, $f^s_F(y) = \alpha = f^s_F(z)$, and $\tilde{f}_F(y) = \tilde{a} = \tilde{f}_F(z)$. Using (21), we have

$$\mathring{f}^-_F((x|(y|z))|(y|z)) \leq \max\{\mathring{f}^-_F(y), \mathring{f}^-_F(z)\} = t^-,$$
$$\mathring{f}^+_F((x|(y|z))|(y|z)) \geq \min\{\mathring{f}^+_F(y), \mathring{f}^+_F(z)\} = t^+,$$

$f^s_F((x|(y|z))|(y|z)) \supseteq f^s_F(y) \cap f^s_F(z) = \alpha$ and $\tilde{f}_F((x|(y|z))|(y|z)) \trianglerighteq \text{rmin}\{\tilde{f}_F(y), \tilde{f}_F(z)\} = \tilde{a}$. It follows that $\mathring{f}_F((x|(y|z))|(y|z)) = (t^-, t^+)$, $f^s_F((x|(y|z))|(y|z)) = \alpha$ and

$$\tilde{f}_F((x|(y|z))|(y|z)) = \tilde{a}.$$

Hence $(x|(y|z))|(y|z) \in F$. Therefore F is a filter of $\mathcal{X} := (X, |)$.

Conversely, let F be a filter of $\mathcal{X} := (X, |)$. Since $1 \in F$, we have $\mathring{f}^-(1) = t^- \leq \mathring{f}^-(x)$ and $\mathring{f}^+(1) = t^+ \geq \mathring{f}^+(x)$, and so $\frac{1}{(x,x)} \in \mathring{f}(M, m)$ for all $x \in X$. Also $f^s(1) = \alpha \supseteq f^s(x)$ and $\tilde{f}(1) = \tilde{a} \trianglerighteq \tilde{f}(x)$ for all $x \in X$. Let $x, y \in X$. If $y \in F$, then $x|(y|y) \in F$, and thus $\mathring{f}^-(x|(y|y)) = t^- = \mathring{f}^-(y)$ and $\mathring{f}^+(x|(y|y)) = t^+ = \mathring{f}^+(y)$. Hence $\frac{x|(y|y)}{(y,y)} \in \mathring{f}(M, m)$. Also we get $f^s(x|(y|y)) = \alpha = f^s(y)$ and $\tilde{f}(x|(y|y)) = \tilde{a} = \tilde{f}(y)$. If $y \notin F$, then it is clear that $\frac{x|(y|y)}{(y,y)} \in \mathring{f}(M, m)$, $f^s(x|(y|y)) \supseteq f^s(y)$ and $\tilde{f}(x|(y|y)) \trianglerighteq \tilde{f}(y)$. Let $x, y, z \in X$. It is obvious that if $y \notin F$ or $z \notin F$, $\frac{(x|(y|z))|(y|z)}{(y,z)} \in \mathring{f}(M, m)$, $f^s((x|(y|z))|(y|z)) \supseteq f^s(y) \cap f^s(z)$ and $\tilde{f}((x|(y|z))|(y|z)) \trianglerighteq \text{rmin}\{\tilde{f}(y), \tilde{f}(z)$. Suppose that $y, z \in F$. Then $(x|(y|z))|(y|z) \in F$. Thus $\mathring{f}^-((x|(y|z))|(y|z)) = t^- = \max\{\mathring{f}^-(y), \mathring{f}^-(z)\}$ and $\mathring{f}^+((x|(y|z))|(y|z)) = t^+ = \min\{\mathring{f}^+(y), \mathring{f}^+(z)\}$. Hence $\frac{(x|(y|z))|(y|z)}{(y,z)} \in \mathring{f}(M, m)$. Also we get $f^s((x|(y|z))|(y|z)) = \alpha = f^s(y) \cap f^s(z)$ and $\tilde{f}((x|(y|z))|(y|z)) = \tilde{a} = \text{rmin}\{\tilde{f}(y), \tilde{f}(z)\}$. Consequently, $Dok_{f_F} := (\mathring{f}_F,\ f^s_F,\ \tilde{f}_F)$ is a Dokdo filter of (U, X). Since F is a filter of $\mathcal{X} := (X, |)$, we get

$$\begin{aligned} X_{f_F} &= \{x \in X \mid \mathring{f}_F(x) = \mathring{f}_F(1),\ f^s_F(x) = f^s_F(1),\ \tilde{f}_F(x) = \tilde{f}_F(1)\} \\ &= \{x \in X \mid \mathring{f}_F(x) = (t^-, t^+),\ f^s_F(x) = \alpha,\ \tilde{f}_F(x) = \tilde{a}\} \\ &= \{x \in X \mid x \in F\} = F. \end{aligned}$$

This completes the proof.

Definition 5. *A Dokdo structure* $Dok_f := (\mathring{f},\ f^s,\ \tilde{f})$ *is called a Dokdo deductive system of* $(U, \mathcal{X})$ *if it satisfies* (19) *and*

$$(\forall x, y \in X) \left(\begin{array}{l} \frac{y}{(x, x|(y|y))} \in \mathring{f}(M, m), \\ f^s(y) \supseteq f^s(x) \cap f^s(x|(y|y)), \\ \tilde{f}(y) \trianglerighteq \text{rmin}\{\tilde{f}(x), \tilde{f}(x|(y|y))\} \end{array} \right). \tag{26}$$

Example 4. *Consider the Sheffer stroke Hilbert algebra* $\mathcal{X} := (X, |)$ *in Example 1 and let* $Dok_f := (\mathring{f},\ f^s,\ \tilde{f})$ *be a Dokdo structure in* $(X, U = \mathbb{Z})$ *which is given by Table 7.*

Table 7: Tabular representation of $Dok_f := (\mathring{f},\ f^s,\ \tilde{f})$

X	$\mathring{f}(x)$	$f^s(x)$	$\tilde{f}(x)$
0	$(-0.37, 0.48)$	$8\mathbb{N}$	$[0.19, 0.54]$
2	$(-0.57, 0.67)$	$4\mathbb{N}$	$[0.24, 0.62]$
3	$(-0.37, 0.48)$	$8\mathbb{Z}$	$[0.19, 0.54]$
4	$(-0.37, 0.48)$	$8\mathbb{N}$	$[0.19, 0.54]$
5	$(-0.61, 0.72)$	$4\mathbb{Z}$	$[0.26, 0.63]$
6	$(-0.57, 0.67)$	$4\mathbb{N}$	$[0.24, 0.62]$
7	$(-0.37, 0.48)$	$8\mathbb{N}$	$[0.19, 0.54]$
1	$(-0.69, 0.79)$	$2\mathbb{Z}$	$[0.41, 0.87]$

It is routine to verify that $Dok_f := (\mathring{f},\ f^s,\ \tilde{f})$ *is a Dokdo deductive system of* $(X, U = \mathbb{Z})$*.*

Theorem 5. *A Dokdo structure* $Dok_f := (\mathring{f},\ f^s,\ \tilde{f})$ *in* (X, U) *is a Dokdo deductive system of* (X, U) *if and only if it is a Dokdo filter of* (X, U)*.*

Proof. Assume that $Dok_f := (\mathring{f},\ f^s,\ \tilde{f})$ is a Dokdo deductive system of (X, U) and let $x, y, z \in X$. Using (1) and (5) induces $y|((x|(y|y))|(x|(y|y))) = 1$. It follows from (19) and (26) that

$$\begin{aligned} \mathring{f}^-(x|(y|y)) &\leq \max\{\mathring{f}^-(y), \mathring{f}^-(y|((x|(y|y))|(x|(y|y))))\} \\ &= \max\{\mathring{f}^-(y), \mathring{f}^-(1)\} = \mathring{f}^-(y), \end{aligned}$$

$$\begin{aligned} \mathring{f}^+(x|(y|y)) &\geq \min\{\mathring{f}^+(y), \mathring{f}^+(y|((x|(y|y))|(x|(y|y))))\} \\ &= \min\{\mathring{f}^+(y), \mathring{f}^+(1)\} = \mathring{f}^+(y), \end{aligned}$$

that is, $\frac{x|(y|y)}{(y,y)} \in \mathring{f}(M, m)$, and

$$f^s(x|(y|y)) \supseteq f^s(y) \cap f^s(y|((x|(y|y))|(x|(y|y)))) = f^s(y) \cap f^s(1)\} = f^s(y)$$

and $\tilde{f}(x|(y|y)) \trianglerighteq \text{rmin}\{\tilde{f}(y), \tilde{f}(y|((x|(y|y))|(x|(y|y))))\} = \text{rmin}\{\tilde{f}(y), \tilde{f}(1)\} = \tilde{f}(y)$. Note that

$$\begin{aligned} y|(((y|z)|z)|((y|z)|z)) &= y|(((y|z)|((z|z)|(z|z)))|((y|z)|((z|z)|(z|z)))) \\ &= (y|z)|((y|((z|z)|(z|z)))|(y|((z|z)|(z|z)))) \\ &= (y|z)|((y|z)|(y|z)) = 1 \end{aligned}$$

by (s2), (2) and (8). Using (19) and (26), we have

$$\begin{aligned} \mathring{f}^{-}((y|z)|z) &\leq \max\{\mathring{f}^{-}(y), \mathring{f}^{-}(y|(((y|z)|z)|((y|z)|z)))\} \\ &= \max\{\mathring{f}^{-}(y), \mathring{f}^{-}(1)\} = \mathring{f}^{-}(y), \end{aligned}$$

$$\begin{aligned} \mathring{f}^{+}((y|z)|z) &\geq \min\{\mathring{f}^{+}(y), \mathring{f}^{+}(y|(((y|z)|z)|((y|z)|z)))\} \\ &= \min\{\mathring{f}^{+}(y), \mathring{f}^{+}(1)\} = \mathring{f}^{+}(y), \end{aligned}$$

$f^s((y|z)|z) \supseteq f^s(y) \cap f^s(y|(((y|z)|z)|((y|z)|z))) = f^s(y) \cap f^s(1)\} = f^s(y)$, and

$$\tilde{f}((y|z)|z) \trianglerighteq \text{rmin}\{\tilde{f}(y), \tilde{f}(y|(((y|z)|z)|((y|z)|z)))\} = \text{rmin}\{\tilde{f}(y), \tilde{f}(1)\} = \tilde{f}(y).$$

Since $z|(((y|z)|(y|z))|((y|z)|(y|z))) \overset{(s2)}{=} z|(y|z) \overset{(s1)}{=} (y|z)|z$, we obtain

$$\begin{aligned} \mathring{f}^{-}((y|z)|(y|z)) &\leq \max\{\mathring{f}^{-}(z), \mathring{f}^{-}(z|(((y|z)|(y|z))|((y|z)|(y|z))))\} \\ &= \max\{\mathring{f}^{-}(z), \mathring{f}^{-}((y|z)|z)\} \\ &\leq \max\{\mathring{f}^{-}(z), \mathring{f}^{-}(y)\}, \end{aligned}$$

$$\begin{aligned} \mathring{f}^{+}((y|z)|(y|z)) &\geq \min\{\mathring{f}^{+}(z), \mathring{f}^{+}(z|(((y|z)|(y|z))|((y|z)|(y|z))))\} \\ &= \min\{\mathring{f}^{+}(z), \mathring{f}^{+}((y|z)|z)\} \\ &\geq \min\{\mathring{f}^{+}(z), \mathring{f}^{+}(y)\}, \end{aligned}$$

$$\begin{aligned} f^s((y|z)|(y|z)) &\supseteq f^s(z) \cap f^s(z|(((y|z)|(y|z))|((y|z)|(y|z)))) \\ &= f^s(z) \cap f^s((y|z)|z)\} \supseteq f^s(z) \cap f^s(y)\}, \end{aligned}$$

and

$$\begin{aligned} \tilde{f}((y|z)|(y|z)) &\trianglerighteq \text{rmin}\{\tilde{f}(z), \tilde{f}(z|(((y|z)|(y|z))|((y|z)|(y|z))))\} \\ &= \text{rmin}\{\tilde{f}(z), \tilde{f}((y|z)|z)\} \\ &\trianglerighteq \text{rmin}\{\tilde{f}(z), \tilde{f}(y)\}. \end{aligned}$$

Therefore

$$\begin{aligned} &\mathring{f}^{-}((x|(y|z))|(y|z)) \\ &= \mathring{f}^{-}((x|(((y|z)|(y|z))|((y|z)|(y|z))))|(((y|z)|(y|z))|((y|z)|(y|z)))) \\ &\leq \mathring{f}^{-}((y|z)|(y|z)) \leq \max\{\mathring{f}^{-}(z), \mathring{f}^{-}(y)\} \end{aligned}$$

and

$$\begin{aligned}&\mathring{f}^{+}((x|(y|z))|(y|z))\\&=\mathring{f}^{+}((x|(((y|z)|(y|z))|((y|z)|(y|z))))|(((y|z)|(y|z))|((y|z)|(y|z))))\\&\geq \mathring{f}^{+}((y|z)|(y|z))\geq \min\{\mathring{f}^{+}(z),\mathring{f}^{+}(y)\},\end{aligned}$$

that is, $\frac{(x|(y|z))|(y|z)}{(y,z)}\in \mathring{f}(M,m)$, and

$$\begin{aligned}&f^{s}((x|(y|z))|(y|z))\\&=f^{s}((x|(((y|z)|(y|z))|((y|z)|(y|z))))|(((y|z)|(y|z))|((y|z)|(y|z))))\\&\supseteq f^{s}((y|z)|(y|z))\supseteq f^{s}(z)\cap f^{s}(y),\end{aligned}$$

and

$$\begin{aligned}&\tilde{f}((x|(y|z))|(y|z))\\&=\tilde{f}((x|(((y|z)|(y|z))|((y|z)|(y|z))))|(((y|z)|(y|z))|((y|z)|(y|z))))\\&\trianglerighteq \tilde{f}((y|z)|(y|z))\trianglerighteq \mathrm{rmin}\{\tilde{f}(z),\tilde{f}(y)\}.\end{aligned}$$

Consequently, $Dok_f := (\mathring{f},\ f^s,\ \tilde{f})$ is a Dokdo filter of (X, U).

Conversely, suppose that $Dok_f := (\mathring{f},\ f^s,\ \tilde{f})$ is a Dokdo filter of (X, U). For every $x, y \in X$, we have

$$\begin{aligned}y&=((x|x)|(1|1))|(y|y)=((x|x)|((y|(y|y))|(y|(y|y))))|(y|y)\\&=((((x|x)|y)|((x|x)|y))|(y|y))|(y|y)\\&=(y|((x|x)|y))|((x|x)|y)\\&=((((x|x)|y)|y)|y)|(((x|x)|y)|y)\\&=(y|(x|(x|(y|y))))|(x|(x|(y|y)))\end{aligned}$$

by (s1), (s2), (s3), (2), (3), (4) (6) and (7). It follows from (21) that

$$\begin{aligned}\mathring{f}^{-}(y)&=\mathring{f}^{-}((y|(x|(x|(y|y))))|(x|(x|(y|y))))\leq \max\{\mathring{f}^{-}(x),\mathring{f}^{-}(x|(y|y))\},\\\mathring{f}^{+}(y)&=\mathring{f}^{+}((y|(x|(x|(y|y))))|(x|(x|(y|y))))\geq \min\{\mathring{f}^{+}(x),\mathring{f}^{+}(x|(y|y))\},\end{aligned}$$

that is, $\frac{y}{(x,x|(y|y))}\in \mathring{f}(M,m)$, and

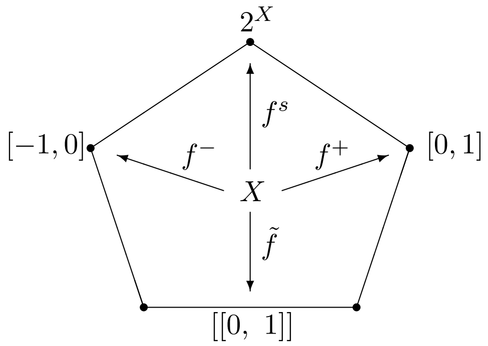
$$f^{s}(y)=f^{s}((y|(x|(x|(y|y))))|(x|(x|(y|y))))\supseteq f^{s}(x)\cap f^{s}(x|(y|y)),$$

and $\tilde{f}(y)=\tilde{f}((y|(x|(x|(y|y))))|(x|(x|(y|y))))\trianglerighteq \mathrm{rmin}\{\tilde{f}(x),\tilde{f}(x|(y|y))\}$. Therefore $Dok_f := (\mathring{f},\ f^s,\ \tilde{f})$ is a Dokdo deductive system of (X, U).

Remark 1. *By Theorem 5, it can be seen that all the results for the Dokdo filter covered above can be handled in the same way using the Dokdo deductive system.*

4. Conclusion

The Dokdo structure $Dok_f := (\mathring{f},\ f^s,\ \tilde{f})$ in a set X consists of a combination of soft set, bipolar fuzzy set, and interval-value fuzzy set, and it can be shaped into a pentagon as shown in the figure below.



where f^s is a soft set of X, $\mathring{f} := (X; f^-, f^+)$ is a bipolar fuzzy set in X, and $\tilde{f} : X \to [[0, 1]]$ is an interval-valued fuzzy set in X. What this paper intends to do is look at filters and deductive systems in Sheffer stroke Hilbert algebras using the Dokdo structure. We defined the concept of Dokdo filter and Dokdo deductive system, and investigated several properties. We formed a Dokdo filter by attaching appropriate conditions to a given Dokdo structure. We studied characterizations of Dokdo filters. We constructed Dokdo filters that are associated with filters. Finally, we showed that the Dokdo filter is consistent with the Dokdo deductive system, which means that all the results covered using the Dokdo filter can be treated using Dokdo deductive system. As an application of the Dokdo structure in the future, we will apply it to other logical algebras such as BL-algebras, MTL-algebras, hoops, Sheffer stroke Hilbert algebras, Sheffer stroke BE-algebras, etc. Based on these studies, we think the possibility of application to decision-making theory, pattern recognition, and medical diagnosis systems, etc. will open up.

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