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Grundy Total Hop Dominating Sequences in Graphs

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Abstract. Let $G = (V(G), E(G))$ be an undirected graph with $\gamma(C) \neq 1$ for each component C of G. Let $S = (v_1, v_2, \cdots, v_k)$ be a sequence of distint vertices of a graph G, and let $\hat{S} = \{v_1, v_2, \ldots, v_k\}$. Then S is a *legal open hop neighborhood sequence* if $N^2_G(v_i) \setminus \bigcup_{j=1}^{i-1} N^2_G(v_j) \neq \varnothing$ for every $i \in \{2, \ldots, k\}$. If, in addition, $\hat{S}$ is a total hop dominating set of G, then S is a *Grundy total hop dominating sequence.* The maximum length of a Grundy total hop dominating sequence in a graph G, denoted by $\gamma^{th}_{gr}(G)$, is the *Grundy total hop domination number* of G. In this paper, we show that the Grundy total hop domination number of a graph G is between the total hop domination number and twice the Grundy hop domination number of G. Moreover, determine values or bounds of the Grundy total hop domination number of some graphs.

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Key Words and Phrases: total hop domination, total hop domination number, open hop neighborhood sequence, Grundy total hop dominating sequence, Grundy total hop domination number

1. Introduction

Domination has attracted many researchers because of its nice applications in various fields and in networks. A number of variations of the domination concept (see for example, [7, 18, 19, 21]) have been introduced and studied. Recently, hop domination was defined and studied by Natarajan and Ayyaswamy in [17]. From then on a lot of investigations of the concept and some of its variants have been done (see [1, 2, 8–15, 20]).

In 2014, Bresar et al. [4] introduced another concept called Grundy dominating sequence in a graph. The newly defined concept has subsequently attracted other researchers in the area to study and generate more interesting results (see [6, 16]) on it.

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In 2016, the concept of Grundy total domination in graphs was investigated by Bresar et al. [5]. Bresar [3] studied further the concept on the product of graphs.

In this study, the concept of Grundy total hop domination number of a graph will be introduced and investigated. Its relationship with total hop domination, and Grundy hop domination numbers of a graph will be given. Bounds for the parameter will be determined for the shadow graph as well as the join and the corona of two graphs.

2. Terminology and Notation

Two vertices u, v of a graph G are *adjacent*, or *neighbors*, if uv is an edge of G. Moreover, an edge uv of G is *incident* to two vertices u, v of G. The set of neighbors of a vertex u in G, denoted by $N_G(u)$, is called the *open neighborhood* of u in G. The *closed neighborhood* of u in G is the set $N_G[u] = N_G(u) \cup \{u\}$. If $X \subseteq V(G)$, the *open neighborhood* of X in G is the set $N_G(X) = \bigcup_{u \in X} N_G(u)$. The *closed neighborhood* of X in G is the set $N_G[X] = N_G(X) \cup X$.

Let G be a graph. A set $D \subseteq V(G)$ is a *total dominating set* of G if for every $v \in V(G)$, there exists $u \in D$ such that $uv \in E(G)$, that is, $N_G(D) = V(G)$. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set of G. Any total dominating set with cardinality equal to $\gamma_t(G)$ is called a γ_t-set.

Let $S = (v_1, v_2, \cdots, v_k)$ be a sequence of distint vertices of a graph G, and let $\hat{S} = \{v_1, v_2, \ldots, v_k\}$. Then S is a *legal open neighborhood sequence* if $N_G(v_i) \setminus \bigcup_{j=1}^{i-1} N_G(v_j) \neq \varnothing$ for every $i \in \{2, \ldots, k\}$. If, in addition, $\hat{S}$ is a total dominating set of G, then S is called a *Grundy total dominating sequence*. The maximum length of a Grundy total dominating sequence in a graph G is called the *Grundy total domination number* of G, and is denoted by $\gamma_{gr}^t(G)$.

A vertex v in G is a *hop neighbor* of vertex u in G if $d_G(u, v) = 2$. The set $N_G^2(u) = \{v \in V(G) : d_G(v, u) = 2\}$ is called the *open hop neighborhood* of u. The *closed hop neighborhood* of u in G is given by $N_G^2[u] = N_G^2(u) \cup \{u\}$. The *open hop neighborhood* of $X \subseteq V(G)$ is the set $N_G^2(X) = \bigcup_{u \in X} N_G^2(u)$. The *closed hop neighborhood* of X in G is the set $N_G^2[X] = N_G^2(X) \cup X$.

A set $S \subseteq V(G)$ is a *hop dominating set* of G if $N_G^2[S] = V(G)$, that is, for every $v \in V(G) \setminus S$, there exists $u \in S$ such that $d_G(u, v) = 2$. The minimum cardinality among all hop dominating sets of G, denoted by $\gamma_h(G)$, is called the *hop domination number* of G. Any hop dominating set with cardinality equal to $\gamma_h(G)$ is called a γ_h-set.

Let $S = (v_1, v_2, \cdots, v_k)$ be a sequence of distinct vertices of G and let $\hat{S} = \{v_1, \cdots, v_k\}$. Then S is a *legal closed hop neighborhood sequence* of G if $N_G^2[v_i] \setminus \cup_{j=1}^{i-1} N_G^2[v_j] \neq \varnothing$ for each $i \in \{2, \cdots, k\}$. If, in addition, $\hat{S}$ is a hop dominating set of G, then S is called a *Grundy hop dominating sequence*. The maximum length of a Grundy hop dominating sequence in a graph G, denoted by $\gamma_{gr}^h(G)$, is called the *Grundy hop domination number* of G. Any Grundy hop dominating sequence S with $|\hat{S}| = \gamma_{gr}^h(G)$ is called a maximum

Grundy hop dominating sequence or a γ_{gr}^{h}-sequence of G. In this case, we call $\hat{S}$ a γ_{gr}^{h}-set of G.

A subset S of $V(G)$ is a *total hop dominating set* of G if for every $v \in V(G)$, there exists $u \in S$ such that $d_G(u,v) = 2$. The smallest cardinality of a total hop dominating set of G denoted by $\gamma_{th}(G)$, is called the *total hop domination number* of G. Any hop dominating set with cardinality equal to $\gamma_{th}(G)$ is called a γ_{th}-set.

Let G be any graph with $\gamma(C) \neq 1$ for each component C of G. Let $S = (v_1, v_2, \cdots, v_k)$ be a sequence of distint vertices of a graph G, and let $\hat{S} = \{v_1, v_2, \ldots, v_k\}$. Then S is a *legal open hop neighborhood sequence* if $N_G^2(v_i) \backslash \bigcup_{j=1}^{i-1} N_G^2(v_j) \neq \varnothing$ for every $i \in \{2, \ldots, k\}$. If, in addition, $\hat{S}$ is a total hop dominating set of G, then S is called a *Grundy total hop dominating sequence*. The maximum length of a Grundy total hop dominating sequence in a graph G is called the *Grundy total hop domination number* of G, and is denoted by $\gamma_{gr}^{th}(G)$. Any Grundy total hop dominating sequence S with $|\hat{S}| = \gamma_{gr}^{th}(G)$ is called a maximum Grundy total hop dominating sequence or a γ_{gr}^{th}-sequence of G. In this case, we call $\hat{S}$ a γ_{gr}^{th}-set of G. A legal open hop neighborhood sequence $S = (v_1, v_2, \cdots, v_k)$ with maximum length, i.e., $k = \max\{p \in \mathbb{N} : \exists$ a legal open hop neighborhood sequence $(x_1, \cdots, x_p)$ of $G\}$, will be referred to as a *maximum legal open hop neighborhood sequence*. We say that vertex v_i *hop-footprints* the vertices from $N_G^2[v_i] \setminus \cup_{j=1}^{i} N_G^2[v_j]$ $\Big($ resp. $N_G^2(v_i) \setminus \cup_{j=1}^{i} N_G^2(v_j)\Big)$, and that v_i is their *hop-footprinter*. Two sequences S and S' in G are *loh-identical* if they are legal open hop neighborhood sequences (or Grundy total hop dominating sequences) and $\hat{S} = \hat{S'}$ (i.e., one is a reaarrangement of the terms of the other).

A sequence $S = (v_1, v_2, \cdots, v_k)$ of distinct vertices of a graph G is a *co-legal open neighborhood sequence* in G if $[V(G) \setminus N_G[v_i]] \setminus \cup_{j=1}^{i-1} [V(G) \setminus N_G[v_j]] \neq \varnothing$ for each $i \in \{2, \ldots, k\}$. A co-legal open neighborhood sequence $S = (v_1, v_2, \ldots, v_k)$ is a *co-Grundy total dominating sequence* if $V(G) = \cup_{i=1}^{k} [V(G) \setminus N_G[v_i]]$. The maximum length of a co-Grundy total dominating sequence in a graph G is called the *co-Grundy total domination number* of G, and is denoted by $\gamma_{cogr}^{t}(G)$.

Let $S_1 = (v_1, \cdots, v_n)$ and $S_2 = (u_1, \cdots, u_m)$, $n, m \geq 1$ be two sequences of distinct vertices of G. The *concatenation* of S_1 and S_2, denoted by $S_1 \oplus S_2$, is the sequence given by $S_1 \oplus S_2 = (v_1, \cdots, v_n, u_1, \cdots, u_m)$.

The *shadow graph* $S(G)$ of a graph G is constructed by taking two copies of G, say G_1 and G_2 and joining each vertex $u \in G_1$ to the neighbors of the corresponding vertex $u' \in G_2$.

Let G and H be any two graphs. The *join* of G and H, denoted by $G + H$ is the graph with vertex set $V(G+H) = V(G) \cup V(H)$ and edge set $E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$. The *corona* G and H, denoted by $G \circ H$, the graph obtained by taking one copy of G and $|V(G)|$ copies of H, and then joining the *ith* vertex of G to every vertex of the *ith* copy of H. We denote by H^v the copy of H in $G \circ H$ corresponding to the vertex $v \in G$ and write $v + H^v$ for $\langle\{v\}\rangle + H^v$.

3. Results

Theorem 1. *Let G be a graph of order n with $\gamma(C) \neq 1$ for each component C of G. Then the following statements hold.*

(i) *If $\gamma_{th}(G) = t$ and $D = \{u_1, u_2, \ldots, u_t\}$ is a minimum total hop dominating set of G, then $S = (u_1, u_2, \cdots, u_t)$ is a Grundy total hop dominating sequence. In particular, $\gamma_{th}(G) \leq \gamma_{gr}^{th}(G)$.*

(ii) *If $S = (u_1, u_2, \cdots, u_s)$ is a minimum Grundy total hop dominating sequence, then $\gamma_{th}(G) = |\hat{S}|$.*

Proof. (i) Suppose that there exists $i \in \{2, 3, \ldots, t\}$ such that $N_G^2(u_i) \setminus \cup_{j=1}^{i-1} N_G^2(u_j) = \varnothing$. Then $N_G^2(u_i) \subseteq \cup_{j=1}^{i-1} N_G^2(u_j)$. This means that $D \setminus \{u_i\}$ is a total hop dominating set of G, which is a contradiction to the minimality of D. Hence, $N_G^2(u_i) \setminus \cup_{j=1}^{i-1} N_G^2(u_j) \neq \varnothing$ for each $i \in \{2, 3, \ldots, t\}$, and so S is Grundy total hop dominating sequence. Consequently, $\gamma_{th}(G) \leq \gamma_{gr}^{th}(G)$.

(ii) From (i), every γ_{th}-set of G forms a Grundy total hop dominating sequence. Since S is a minimum Grundy total hop dominating sequence, it follows that $|\hat{S}| \leq \gamma_{th}(G)$. On the other hand, since every Grundy total hop dominating sequence forms a total hop dominating set by definition, it follows that $\gamma_{th}(G) \leq |\hat{S}|$. Consequently, $\gamma_{th}(G) = |\hat{S}|$. □

Theorem 2. *Let G be a graph of order n with $\gamma(C) \neq 1$ for each component C of G. Then $S = (u_1, u_2, \cdots, u_l)$ is a maximum legal open hop neighborhood sequence of G if and only if S is a Grundy total hop dominating sequence of G and $\gamma_{gr}^{th}(G) = l$.*

Proof. Let $S = (u_1, \cdots, u_l)$ be a maximum legal open hop neighborhood sequence of G. Suppose on the contrary that $\hat{S}$ is not a total hop dominating set of G. Then there exists $u \in V(G)$ such that $u \notin N_G^2(\hat{S})$. This means that $u \notin N_G^2(v)$ for every $v \in \hat{S}$. Since u is not hop dominated by any $v \in \hat{S}$, $u \in N_G^2(t)$ for some $t \in V(G) \setminus \hat{S}$. This means that $N_G^2(t) \setminus \bigcup_{i=1}^{k} N_G^2(u_i) \neq \varnothing$. Thus, $S' = (u_1, \cdots, u_l, t)$ is a legal open hop neighborhood sequence of G, a contradiction to the maximality of S. Therefore, $\hat{S}$ is a total hop dominating set of G. Consequently, S is a Grundy total hop dominating sequence of G and $\gamma_{gr}^{th}(G) = l$.

The converse is clear. □

The next result follows from Theorem 2.

Corollary 1. *Let G be a graph of order n with $\gamma(C) \neq 1$ for each component C of G and let $T = (x_1, \cdots, x_j)$ be a legal open hop neighborhood sequence of G. Then $|\hat{T}| = j \leq \gamma_{gr}^{th}(G)$.*

Theorem 3. *Let G be a graph of order n with $\gamma(C) \neq 1$ for each component C of G. Then $4 \leq \gamma_{gr}^{th}(G) \leq n$ and these bounds cannot be improved.*

Proof. Clearly $\gamma_{gr}^{th}(G) = 1$ is not possible. Suppose $\gamma_{gr}^{th}(G) = 2$, say, $S = (v_1, v_2)$ is a Grundy total hop dominating sequence. Since $\hat{S}$ is a total hop dominating set, $v_1 \in N_G^2(v_2)$. Let $v \in N_G(v_1) \cap N_G(v_2)$. Then $v \notin N_G^2(v_1) \cup N_G^2(v_2)$, a contradiction. Next, suppose $\gamma_{gr}^{th}(G) = 3$, say $S = (v_1, v_2, v_3)$ is a Grundy total hop dominating sequence. Since $\hat{S}$ is a total hop dominating set, v_1 is hop dominated by v_2 or v_3. Suppose $d_G(v_1, v_2) = 2$ and let $p \in N_G(v_1) \cap N_G(v_2)$. Then $p \in N_G^2(v_3)$ and $v_3 \in N_G^2(v_1) \cup N_G^2(v_2)$. Let $S^* = (p, v_1, v_2, v_3)$. Then $\hat{S^*}$ is a total hop dominating set. Moreover, observe that $v_3 \in N_G^2(p)$, $v_2 \in N_G^2(v_1) \setminus N_G^2(p)$, $v_1 \in [N_G^2(v_2) \setminus (N_G^2(v_1) \cup N_G^2(p))]$ and $p \in [N_G^2(v_3) \setminus (N_G^2(v_2) \cup N_G^2(v_1) \cup N_G^2(p))]$. Hence, S^* is a legal open hop neighborhood sequence, and so S^* is a Grundy total hop dominating sequence of G, contrary to our assumption that $\gamma_{gr}^{th}(G) = 3$. Therefore, $\gamma_{gr}^{th}(G) \geq 4$.

For tightness of the bounds, consider $G = C_4$ and $H = P_8$. Then $\gamma_{gr}^{th}(G) = 4$ and $\gamma_{gr}^{th}(H) = 8$. □

Theorem 4. *Let G be a connected graph of order n such that $\gamma(G) \neq 1$. If $n = 2m, m \geq 2$ and the vertices of G can be labeled as $u_1, \ldots, u_m, v_1, \ldots, v_m$ in such a way that*

(i) $d_G(u_i, v_i) = 2$ *for each* i,

(ii) $\{u_1, \ldots, u_m\}$ *is a hop independent set of* G, *and*

(iii) $d_G(u_i, v_j) = 2$ *implies that* $i \geq j$,

then $\gamma_{gr}^{th}(G) = n$.

Proof. Suppose the vertices of G can be labeled as described. Clearly, $\hat{S} = \{u_1, \cdots, u_m, v_m, \cdots, v_1\}$ is a total hop dominating set of G. Observe that $v_i \in N_G^2(u_i) \setminus \bigcup_{j=1}^{i-1} N_G^2(u_j)$ for each $i \in \{2, \ldots, m\}$ by (i) and (iii) and $u_m \in N_G^2(v_m) \setminus \bigcup_{j=1}^{m} N_G^2(u_j)$ by (i) and (ii). Now, for any $k < m$,

$$u_k \in N_G^2(v_k) \setminus \left[\left(\bigcup_{j=1}^{m} N_G^2(u_j)\right) \cup \left(\bigcup_{i=m}^{k+1} N_G^2(v_i)\right)\right] \text{ by } (i), (ii), \text{ and } (iii).$$

Therefore, S is a Grundy total hop dominating sequence, showing that $\gamma_{gr}^{th}(G) = n$. □

Proposition 1. *Let n and m be any positive integers such that $n \geq 4$. Then*

$$\gamma_{gr}^{th}(P_n) = \begin{cases} n-2 \text{ if } n = 4m+2 \\ n-1 \text{ if } n \geq 5 \text{ and odd} \\ n \text{ if } n = 4m \end{cases}$$

Proof. Let $P_n = [v_1, v_2, \ldots, v_n]$. Clearly, $\gamma_{gr}^{th}(P_6) = 4$. For $n = 4m + 2 \geq 10$, let $S = (v_1, v_2, \ldots, v_{n-2})$. Clearly, S is a Grundy total hop dominating sequence of P_n. Thus,

$\gamma_{gr}^{th}(P_n) \geq n-2$. On the other hand, let $S' = (w_1, w_2, \ldots, w_k)$ be a Grundy total hop dominating sequence of P_n. Notice that one of the vertices $v_1, v_5, v_9, \ldots, v_{n-5}, v_{n-1}$ is not in $\hat{S}'$. Suppose all the vertices $v_1, v_5, v_9, \ldots, v_{n-5}, v_{n-1}$ are in $\hat{S}$. If v_5 comes before v_1, then $N^2_{P_n}(v_1) \subseteq N^2_{P_n}(v_5)$, a contradiction. So, v_5 comes after v_1. Next, suppose v_9 comes before v_5, then $N^2_{P_n}(v_5) \subseteq N^2_{P_n}(v_1) \cup N^2_{P_n}(v_9)$, a contradiction. Thus, v_9 comes after v_5. Continuing in this manner, we find that the following order of appearance (not necessarily consecutive) of the given vertices in the Grundy total hop dominating sequence S': $v_1, v_5, v_9, \ldots, v_{n-5}, v_{n-1}$. However, $N^2_{P_n}(v_{n-1}) \subseteq N_{P_n}(v_{n-5})$. Hence, $v_{n-1} \notin \hat{S}'$, a contradiction. Similarly, one of the vertices $v_2, v_6, v_{10}, \ldots, v_{n-4}, v_n$ is not in $\hat{S}'$. Therefore, $\gamma_{gr}^{th}(P_n) = k \leq n-2$. Consequently, $\gamma_{gr}^{th}(P_n) = n-2$ for all $n = 4m+2$.

Next, let $n \geq 5$ and odd. Clearly, $\gamma_{gr}^{th}(P_5) = 4$. Suppose $n \geq 7$ and odd. For $n \in \{7, 11, 15, \ldots\}$, let

$$S_1 = (v_1, v_2, v_5, v_6, \cdots, v_{n-6}, v_{n-5}, v_{n-2}, v_n, v_{n-3}, v_{n-4}, v_{n-7}, v_{n-8}, \cdots, v_4, v_3).$$

Then S_1 is a Grundy total hop dominating sequence of P_n. Hence,

$$\gamma_{gr}^{th}(P_n) \geq n-1.$$

Next, for $n \in \{9, 13, 17, \ldots\}$, let

$$S_2 = (v_1, v_2, v_5, v_6, \cdots, v_{n-4}, v_{n-3}, v_{n-2}, v_{n-1}, v_{n-6}, v_{n-5}, \cdots, v_3, v_4).$$

Observe that S_2 is a Grundy total hop dominating sequence of P_n. Hence,

$$\gamma_{gr}^{th}(P_n) \geq n-1.$$

Suppose $\gamma_{gr}^{th}(P_n) = n$, say $S_0 = (w_1, w_2, \cdots, w_n)$ is a Grundy total hop dominating sequence of P_n. Observe that for $n \in \{7, 11, 15, \ldots\}$, one of the vertices $v_2, v_6, v_{10}, \ldots, v_{n-4}, v_n$ is not in $\hat{S}_0$. Suppose all vertices $v_2, v_6, v_{10}, \ldots, v_{n-4}, v_n$ are in $\hat{S}_0$. If v_6 comes before v_2, then $N^2_{P_n}(v_2) \subseteq N^2_{C_n}(v_6)$, a contradiction. So, v_6 comes after v_2. Next, suppose v_{10} comes before v_6, then $N^2_{P_n}(v_6) \subseteq N^2_{P_n}(v_2) \cup N^2_{P_n}(v_{10})$, a contradiction. Thus, v_{10} comes after v_6. Continuing in this manner, we find that the following order of appearance (not necessarily consecutive) of the given vertices in the Grundy total hop dominating sequence S_0: $v_2, v_6, v_{10}, \ldots, v_{n-4}, v_n$. However, $N^2_{P_n}(v_n) \subseteq N_{P_n}(v_{n-4})$. Hence, $v_n \notin \hat{S}_0$, a contradiction. Similarly, for $n \in \{9, 13, 17, \ldots\}$, one of the vertices $v_1, v_5, v_9, \ldots, v_{n-5}, v_{n-1}$ is not in the Grundy total hop dominating sequence, say S''. Thus, $\gamma_{gr}^{th}(P_n) \leq n-1$. Consequently, $\gamma_{gr}^{th}(P_n) = n-1$ for all $n \geq 5$ and odd.

Lastly, assume that $n = 4m$. Clearly, $\gamma_{gr}^{th}(P_4) = 4$. For $n = 4m \geq 8$, let

$$C = (v_1, v_2, v_5, v_6, \cdots, v_{n-3}, v_{n-2}, v_{n-1}, v_n, v_{n-5}, v_{n-4}, \cdots, v_3, v_4).$$

Then C is a Grundy total hop dominating sequence of P_n. Thus, $\gamma_{gr}^{th}(P_n) = n$ for all $n = 4m$. □

Proposition 2. *Let G be a graph of order n with $\gamma(C) \neq 1$ for each component C of G. If $|N_G^2(v)| \geq m$ for every $v \in V(G)$, then $\gamma_{gr}^{th}(G) \leq n-(m-1)$.*

Proof. Suppose $\gamma_{gr}^{th}(G) = k$, say $S = (w_1, w_2, \cdots, w_k)$ is a Grundy total hop dominating sequence of G. Assume $w_1 = v_i$ for some $i \in \{1, \ldots, n\}$. Then $|N_G^2(w_1)| = |N_G^2(v_i)| \geq m$ for some $i \in \{1, \ldots, n\}$. It follows that there are at most $n-m$ remaining vertices of G that could be hop footprinted by the next terms of S. Therefore,

$$\gamma_{gr}^{th}(G) = k \leq n - m + |\{v_i\}| = n - m + 1 = n - (m-1). \quad \square$$

The next result follows from Proposition 2.

Corollary 2. *Let G be a graph of order n with $\gamma(C) \neq 1$ for each component C of G. If $|N_G^2(u)| \geq 2$ for every $u \in V(G)$, then $\gamma_{gr}^{th}(G) \leq n-1$.*

Proposition 3. *Let n and m be any positive integers such that $n \geq 4$. Then*

$$\gamma_{gr}^{th}(C_n) = \begin{cases} 4 \text{ if } n = 4,5,6 \\ 6 \text{ if } n = 8 \\ n-4 \text{ if } n = 4m \geq 12 \\ n-2 \text{ if } n = 4m+2 \geq 10 \\ n-1 \text{ if } n \geq 7 \text{ and odd} \end{cases}$$

Proof. Clearly for $n = 4,5,6$ and $n = 8$, $\gamma_{gr}^{th}(C_n) = 4$ and $\gamma_{gr}^{th}(C_n) = 6$, respectively. For $n = 4m \geq 12$, let $V(C_n) = \{v_1, v_2, \ldots, v_n\}$. Observe that $S = (v_1, v_2, \cdots, v_{n-4})$ is a Grundy total hop dominating sequence of C_n. Thus, $\gamma_{gr}^{th}(C_n) \geq n-4$. On the other hand, let $S' = (w_1, w_2, \ldots, w_k)$ be a Grundy total hop dominating sequence of C_n. Notice that one of the $v_1, v_5, v_9, \ldots, v_{n-7}, v_{n-3}$ is not in $\hat{S'}$. Suppose all the vertices $v_1, v_5, v_9, \ldots, v_{n-7}, v_{n-3}$ are in $\hat{S'}$. WLOG, assume that $w_1 = v_1$. If v_9 comes before v_5, then $N_{C_n}^2(v_5) \subseteq N_{C_n}^2(v_1) \cup N_{C_n}^2(v_9)$, a contradiction. So, v_9 comes after v_5. Next, suppose v_{13} comes before v_9, then $N_{C_n}^2(v_9) \subseteq N_{C_n}^2(v_1) \cup N_{C_n}^2(v_5) \cup N_{C_n}^2(v_{13})$, a contradiction. Thus, v_{13} comes after v_9. Continuing in this manner, we find that the following order of appearance (not necessarily consecutive) of the given vertices in the Grundy total hop dominating sequence S': $v_1, v_5, v_9, \ldots, v_{n-7}, v_{n-3}$. However, $N_{C_n}^2(v_{n-3}) \subseteq N_{C_n}^2(v_1) \cup N_{C_n}(v_{n-7})$. Hence, $v_{n-3} \notin \hat{S'}$, a contradiction. Similarly, one of the vertices $v_2, v_6, v_{10}, \ldots, v_{n-6}, v_{n-2}$, $v_3, v_7, v_{11}, \ldots, v_{n-5}, v_{n-1}$, and $v_4, v_8, v_{12}, \ldots, v_{n-4}, v_n$, respectively, is not in $\hat{S'}$. Therefore, $\gamma_{gr}^{th}(P_n) = k \leq n-4$. Consequently, $\gamma_{gr}^{th}(P_n) = n-4$ for all $n = 4m$.

Next, for $n = 4m+2 \geq 10$, let

$$S_1 = (v_1, v_5, \ldots, v_{n-5}, v_{n-1}, v_3, v_7, \ldots, v_{n-7}, v_2, v_6, \ldots, v_{n-4}, v_n, v_4, v_8, \ldots, v_{n-6}).$$

Then S_1 is a Grundy total hop dominating sequence of C_n. Hence, $\gamma_{gr}^{th}(C_n) \geq n-2$. On the other hand, let $S = (w_1, w_2, \ldots, w_k)$ be a Grundy total hop dominating sequence of C_n. Then applying the same arguments with the first part, one can show that one of the

$v_1, v_5, v_9, \ldots, v_{n-5}, v_{n-1}, v_3, v_7, \ldots, v_{n-7}, v_{n-3}$ and $v_2, v_6, v_{10}, \ldots, v_{n-4}, v_n, v_4, v_8, \ldots, v_{n-6}, v_{n-2}$ is not in $\hat{S}_1$, respectively. Hence, $\gamma_{gr}^{th}(C_n) = k \leq n-2$. Consequently, $\gamma_{gr}^{th}(C_n) = n-2$.

Let $n \geq 7$ and odd. Clearly, $\gamma_{gr}^{th}(C_7) = 6$. Suppose $n \geq 9$ and odd. For $n \in \{9, 13, 17, \ldots\}$, let

$$S_2 = (v_1, v_5, \ldots, v_n, v_4, \ldots, v_{n-1}, v_3, \ldots, v_{n-2}, v_2, \ldots, v_{n-7}).$$

Then S_2 is a Grundy total hop dominating sequence of C_n. Hence, $\gamma_{gr}^{th}(C_n) \geq n-1$. Next, for $n \in \{11, 15, 19, \ldots\}$, let

$$S_3 = (v_1, v_5, \ldots, v_{n-2}, v_2, \ldots, v_{n-1}, v_3, \ldots, v_n, v_4, \ldots, v_{n-7}).$$

Then S_3 is a Grundy total hop dominating sequence of C_n. Hence, $\gamma_{gr}^{th}(C_n) \geq n-1$. On the other hand, since $|N_{C_n}(v)| = 2$ for every $v \in V(C_n)$, it follows that $\gamma_{gr}^{th}(C_n) \leq n-1$ by Corollary 2. Therefore, $\gamma_{gr}^{th}(C_n) = n-1$. □

Theorem 5. *Let G be a graph of order n with $\gamma(C) \neq 1$ for each component C of G. Then $\gamma_{gr}^{th}(G) \leq 2\gamma_{gr}^{h}(G)$.*

Proof. Let $S = (v_1, v_2, \cdots, v_k)$ be a Grundy total hop dominating sequence of G, where $k = \gamma_{gr}^{th}(G)$. We will prove that at most $k/2$ vertices can be removed from S in such a way the resulting sequence S' forms a legal closed hop neighborhood sequence of G. Notice that a vertex $v_i \in \hat{S}$ prevents S from being a legal closed hop neighborhood sequence only if $N_G^2[v_i] \setminus \bigcup_{j=1}^{i-1} N_G^2[v_j] = \varnothing$ for each $i \in \{1, \ldots, k\}$. Since S is a Grundy total hop dominating sequence, v_i hop footprinted only vertices from S that precedes v_i. That is, $h_S^{-1}(v_i) \subseteq \{v_1, \ldots, v_{i-1}\}$, where $h_S : V(G) \to \hat{S}$ is a hop footprinter function, mapping each vertex to its hop footprinter. Set $T = \{v_i \in \hat{S} : h_S^{-1}(v_i) \subseteq \{v_1, \ldots, v_{i-1}\}\}$. Since $v_1 \notin T$, $T \neq \hat{S}$. Suppose that $h_S^{-1}(v_j) \cap T \neq \varnothing$ for some $v_j \in T$. Let $v_i \in h_S^{-1}(v_j) \cap T$. Since $v_j \in T$, the vertex v_i that is hop footprinted by v_j satisties $i < j$. Since $v_i \in T$, v_i hop footprints some vertex v_t, where $t < i$. This means that $h_S(v_t) = v_i$, where $1 \leq t \leq i-1$. It follows that $v_i \notin N_G^2(v_j) \setminus \cup_{k=1}^{j-1} N_G^2(v_k)$, that is, $h_S(v_i) \neq v_j$, contrary to the assumption that $v_i \in h_S^{-1}(v_j)$. Therefore, $h_S^{-1}(v_j) \cap T = \varnothing$ for every vertex $v_j \in T$. Now, suppose that $v_i, v_j \in T$, where $i < j$. By definition,

$$h_S^{-1}(v_i) \subseteq \{v_1, \ldots, v_{i-1}\} \text{ and } h_S^{-1}(v_j) \subseteq \{v_1, \ldots, v_{j-1}\}.$$

Since every vertex is hop footprinted by a unique vertex in $\hat{S}$, it follows that $h_S^{-1}(v_i) \cap h_S^{-1}(v_j) = \varnothing$. Since $h_S^{-1}(v_j) \cap T = \varnothing$ for every vertex $v_j \in T$, $\{h_S^{-1}(v_i) : v_i \in T\}$ forms a partition of a subset of $\hat{S} \setminus T$. Note that for each $v_i \in T$, $|h_S^{-1}(v_i)| \geq 1$, and so $|T| \leq |\bigcup_{v_i \in T} h_S^{-1}(v_i)| \leq |\hat{S}| - |T|$ implying that $2|T| \leq |\hat{S}| = k$. Hence, $|T| \leq \frac{k}{2}$. Let S' be a sequence obtained from S by deleting vertices from T. Then S' is a legal closed hop neighborhood sequence of G. Thus, $\gamma_{gr}^{h}(G) \geq |S'| = k - |T| \geq k - \frac{k}{2} = \frac{k}{2} = \frac{1}{2}\gamma_{gr}^{th}(G)$. Consequently, $\gamma_{gr}^{th}(G) \leq 2\gamma_{gr}^{h}(G)$. □

Remark 1. *The bound given in Theorem 5 is tight.*

To see this, consider C_4 in Fig. 1. Let $S = (u_1, u_2, u_3, u_4)$. Then S is a γ_{gr}^{th}-sequence of C_4. Thus, $\gamma_{gr}^{th}(C_4) = 4$. Next, let $S^* = (u_1, u_2)$. Then S^* is a γ_{gr}^{h}-sequence of C_4. Hence, $\gamma_{gr}^{h}(C_4) = 2$. Consequently, $\gamma_{gr}^{th}(C_4) = 4 = 2\gamma_{gr}^{h}(C_4)$.

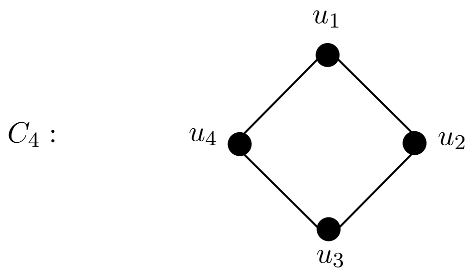

Figure 1: A graph C_4 with $\gamma_{gr}^{th}(C_4) = 4 = 2\gamma_{gr}^{h}(C_4)$

Remark 2. *Let G be a graph of order n with $\gamma(C) \neq 1$ for each component C of G. Then $\gamma_{gr}^{th}(G) \geq \gamma_{gr}^{h}(G)$ does not hold in general.*

To see this, consider $K_5 \circ K_2$ in Fig. 2. Let $S = (v_1, v_2, \ldots, v_{10})$. Then S is a γ_{gr}^{h}-sequence of $K_5 \circ K_2$, that is, $\gamma_{gr}^{h}(K_5 \circ K_2) = 10$. Next, let $S' = (v_1, v_{12}, v_3, v_{13})$. Then S' is a γ_{gr}^{th}-sequence of $K_5 \circ K_2$. Thus, $\gamma_{gr}^{th}(K_5 \circ K_2) = 4$.

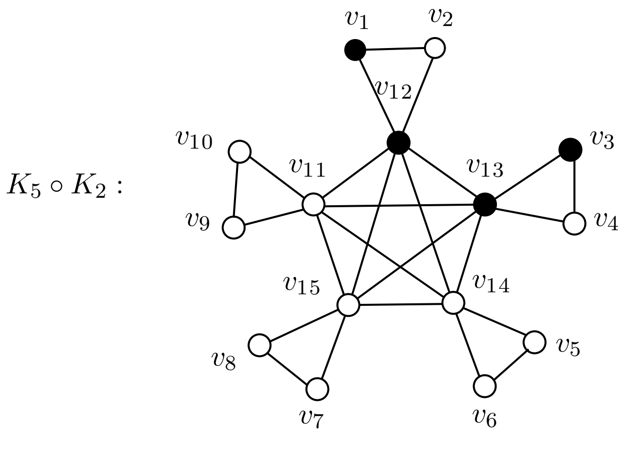

Figure 2: A graph $K_5 \circ K_2$ with $\gamma_{gr}^{th}(K_5 \circ K_2) = 4 < 10 = \gamma_{gr}^{h}(K_5 \circ K_2)$

Lemma 1. *Let G be a non-trivial connected graph and let G_1 and G_2 be two copies of G in the graph $S(G)$. If $v \in V(G_1)$ and $v' \in V(G_2)$ is the corresponding vertex of v, then* *(i)* $N^2_{S(G)}(v) = N^2_{G_1}(v) \cup N^2_{G_2}[v']$ *and*

(ii) $N^2_{S(G)}(v') = N^2_{G_1}[v] \cup N^2_{G_2}(v')$.

Proof. (i) Let $a \in N^2_{S(G)}(v)$. Then $d_{S(G)}(a, v) = 2$. If $a \in V(G_1)$, then $a \in N^2_{G_1}(v)$. Suppose $a \in V(G_2)$. By assumption, it follows that $d_{G_2}(a, v') = 2$. Thus, $a \in N^2_{G_2}(v')$. Hence, $N^2_{S(G)}(v) \subseteq N^2_{G_1}(v) \cup N^2_{G_2}[v']$. Clearly, $N^2_{G_1}(v) \cup N^2_{G_2}[v'] \subseteq N^2_{S(G)}(v)$. Consequently, $N^2_{S(G)}(v) = N^2_{G_1}(v) \cup N^2_{G_2}[v']$.

(ii) can be proved similarly. □

Theorem 6. *Let G be a graph of order n with $\gamma(C) \neq 1$ for each component C of G. If S is a Grundy total hop dominating sequence of G_1 or G_2, then S is a Grundy total hop dominating sequence of $S(G)$. Moreover, $\gamma^{th}_{gr}(G) \leq \gamma^{th}_{gr}(S(G))$.*

Proof. Let G_1 and G_2 be two copies of G. Let $S = (v_1, v_2, \ldots, v_k)$ be a Grundy total hop dominating sequence in G_1 and let $v' \in V(G_2)$. Then

$$N^2_{G_1}(v_i) \setminus \bigcup_{j=1}^{i-1} N^2_{G_1}(v_j) \neq \varnothing \text{ for each } i \in \{2, 3, \ldots, k\}.$$

Thus, by Lemma 1

$$\begin{aligned}
&N^2_{S(G)}(v_i) \setminus \bigcup_{j=1}^{i-1} N^2_{S(G)}(v_j) \\
&= \left[N^2_{G_1}(v_i) \cup N^2_{G_2}[v'_i]\right] \setminus \bigcup_{j=1}^{i-1} \left[N^2_{G_1}(v_j) \cup N^2_{G_2}[v'_j]\right] \\
&= \left[\left(N^2_{G_1}(v_i) \cup N^2_{G_2}[v'_i]\right) \setminus \bigcup_{j=1}^{i-1} N^2_{G_1}(v_j)\right] \cup \left[\left(N^2_{G_1}(v_i) \cup N^2_{G_2}[v'_i]\right) \setminus \bigcup_{j=1}^{i-1} N^2_{G_2}[v'_j]\right] \\
&\neq \varnothing \text{ for each } i \in \{2, \ldots, k\}.
\end{aligned}$$

Since $\hat{S}$ is a total hop dominating set of G_1, there exists $w \in \hat{S} \cap N^2_{G_1}(v)$. By Lemma 1, $w \in \hat{S} \cap N^2_{S(G)}(v')$, i.e., $w \in \hat{S}$ and $d_{S(G)}(v', w) = 2$. Consequently, $\hat{S}$ is a Grundy total hop dominating sequence of $S(G)$. □

Remark 3. *The bound given in Theorem 6 is tight. Moreover, strict inequality can also be attained.*

For equality, consider C_4. Then $\gamma^{th}_{gr}(C_4) = 4$. Now, consider the shadow graph of C_4 given in Fig. 3. Let $S = (a, a', b, b')$. Observe that S is a Grundy total hop dominating sequence of $S(C_4)$. Moreover, it can be verified that $\gamma^{th}_{gr}(S(C_4)) = 4$. Consequently, $\gamma^{th}_{gr}(S(C_4)) = 4 = \gamma^{th}_{gr}(S(C_4))$.

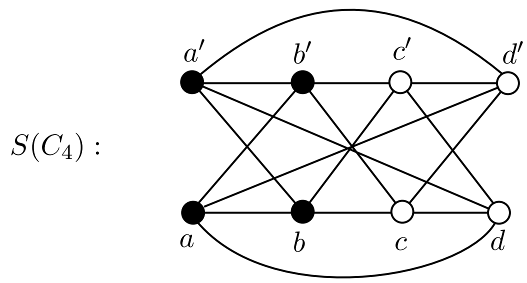

Figure 3: A graph C_4 with $\gamma_{gr}^{th}(C_4) = \gamma_{gr}^{th}(S(C_4))$

For strict inequality, consider P_5. Then $\gamma_{gr}^{th}(P_5) = 4$. Now, consider the shadow graph of P_5 given in Fig. 4. Let $S = (a, a', e, e', d, d')$. Observe that S is a Grundy total hop dominating sequence of $S(P_5)$. Moreover, it can be verified that $\gamma_{gr}^{th}(S(P_5)) = 6$. Hence, $\gamma_{gr}^{th}(S(P_5)) = 4 < 6 = \gamma_{gr}^{th}(S(P_5))$.

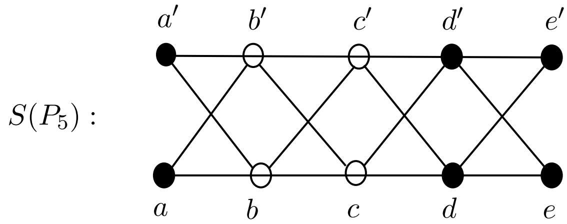

Figure 4: A graph G with $\gamma_{gr}^{th}(G) < \gamma_{gr}^{th}(S(G))$

Lemma 2. *Let G be a graph of order n with no isolated vertices. If $|N_G(v)| \geq l$ for every $v \in V(G)$, then $\gamma_{gr}^{t}(G) \leq n - (l-1)$.*

Proof. Suppose $\gamma_{gr}^{t}(G) = k$, say $S = (w_1, w_2, \cdots, w_k)$ is a Grundy total dominating sequence of G. Assume $w_1 = v_i$ for some $i \in \{1, \ldots, n\}$. Then $|N_G(w_1)| = |N_G(v_i)| \geq l$ for some $i \in \{1, \ldots, n\}$. It follows that there are at most $n - l$ remaining vertices of G that could be footprinted by the next terms of S. Therefore,

$$\gamma_{gr}^{t}(G) = k \leq n - l + |\{v_i\}| = n - l + 1 = n - (l-1). \quad \square$$

Proposition 4. *Let $n \geq 4$ be any positive integer. Then*

$$\gamma_{gr}^{t}(\overline{P}_n) = 4 = \gamma_{gr}^{t}(\overline{C}_n).$$

Proof. Clearly, $\gamma^t_{gr}(\overline{P}_4) = 4$. For $n \geq 5$, let $P_n = [u_1, u_2, \cdots, u_n]$ and $S = (u_3, u_2, u_1, u_n)$. Notice that $\hat{S}$ is a total dominating set of $\overline{P}_n$. Now, observe that $v_4 \in N_{\overline{P}_n}(v_2) \setminus N_{\overline{P}_n}(v_3)$, $v_3 \in N_{\overline{P}_n}(v_1) \setminus (N_{\overline{P}_n}(v_2) \cup N_{\overline{P}_n}(v_3))$, and $v_2 \in N_{\overline{P}_n}(v_n) \setminus (N_{\overline{P}_n}(v_1) \cup N_{\overline{P}_n}(v_2) \cup N_{\overline{P}_n}(v_3))$. It follow that S is a Grundy total dominating sequence of $\overline{P}_n$. Hence, $\gamma^t_{gr}(\overline{P}_n) \geq 4$. On the other hand, suppose $\gamma^t_{gr}(\overline{P}_n) = k$, say $S = (w_1, w_2, \cdots, w_k)$ is a Grundy total dominating sequence of $\overline{P}_n$. Since $|N_{\overline{P}_n}(v_j)| \geq n-3$ for every $j \in \{1, \ldots, n\}$, it follows that $\gamma^t_{gr}(\overline{P}_n) = k \leq n - (n-3-1) = 4$ by Lemma 2. Therefore, $\gamma^t_{gr}(\overline{P}_n) = 4$.

Next, let $C_n = [v_1, v_2, \cdots, v_n, v_1]$ and $S = (v_1, v_2, v_3, v_4)$. Clearly, S is a total dominating set of $\overline{C}_n$. Observe that $v_n \in N_{\overline{C}_n}(v_2) \setminus N_{\overline{C}_n}(v_1)$, $v_1 \in N_{\overline{C}_n}(v_3) \setminus (N_{\overline{C}_n}(v_2) \cup N_{\overline{C}_n}(v_1))$, and $v_2 \in N_{\overline{C}_n}(v_4) \setminus (N_{\overline{C}_n}(v_3) \cup N_{\overline{C}_n}(v_2) \cup N_{\overline{C}_n}(v_1))$. Thus, S is a Grundy total dominating sequence of $\overline{C}_n$ and $\gamma^t_{gr}(\overline{C}_n) \geq 4$. On the other hand, suppose $\gamma^t_{gr}(\overline{C}_n) = k$, say $S = (u_1, u_2, \cdots, u_k)$ is a Grundy total dominating sequence of $\overline{C}_n$. We may assume that $u_1 = v_1$. Then $|N_{\overline{C}_n}(u_1)| = |N_{\overline{C}_n}(v_1)| = n-3$. Thus, $\gamma^t_{gr}(\overline{C}_n) = k \leq n - (n-3-1) = 4$ by Lemma 2. Therefore, $\gamma^t_{gr}(\overline{C}_n) = 4$. □

Throughout, $[n] = \{1, 2, \ldots, n\}$ for each positive integer n.

Lemma 3. *Let G be a graph such that $\gamma(G) \neq 1$. A sequence S is a co-legal open neighborhood sequence in G if and only if S is a legal open neighborhood sequence in $\overline{G}$. Moreover, S is a co-Grundy total dominating sequence in G if and only if it is a Grundy total dominating sequence in $\overline{G}$. In particular, $\gamma^t_{cogr}(G) = \gamma^t_{gr}(\overline{G})$.*

Proof. Let $S = (u_1, \cdots, u_t)$ be a sequence in G. Notice that $V(G) \setminus N_G[u_i] = N_{\overline{G}}(u_i)$ for each $i \in [t]$. So,

$$[V(G) \setminus N_G[u_i]] \setminus \cup_{j=1}^{i-1}[V(G) \setminus N_G[u_j]] = N_{\overline{G}}(u_i) \setminus \cup_{j=1}^{i-1} N_{\overline{G}}(u_j).$$

Hence, S is a co-legal open neighborhood sequence in G if and only if it is a legal open neighborhood sequence in $\overline{G}$. Clearly, a co-legal open neighborhood sequence in G is a co-Grundy total dominating sequence if and only if it is a Grundy total dominating sequence in $\overline{G}$. Consequently, $\gamma^t_{cogr}(G) = \gamma^t_{gr}(\overline{G})$. □

Theorem 7. *Let G and H be any two graphs such that $\gamma(G) \neq 1$ and $\gamma(H) \neq 1$. A sequence S of distinct vertices of $G + H$ is a legal open hop neighborhood sequence if and only if one of the following holds:*

(i) S is a co-legal open neighborhood sequence in G (legal open neighborhood sequence in $\overline{G}$).

(ii) S is a co-legal open neighborhood sequence in H (legal open neighborhood sequence in $\overline{H}$).

(iii) S is loh-identical to $S' = S_G \oplus S_H$, where S_G and S_H are co-legal open neighborhood sequences in G and H, respectively.

Proof. Suppose that $S = (u_1, \cdots, u_t)$ is a legal open hop neighborhood sequence in $G + H$ and let $\hat{S}$ be the corresponding set of S. Suppose further that $\hat{S} \subseteq V(G)$. Then by the legality condition in S, we have

$$N^2_{G+H}(u_i) \setminus \cup_{j=1}^{i-1} N^2_{G+H}(u_j) \neq \varnothing \text{ for each } \ i \in \{2, 3, \ldots, t\}.$$

Since $N^2_{G+H}(u_i) = V(G) \setminus N_G[u_i]$ for each $i \in [t]$, it follows that

$$[V(G) \setminus N_G[u_i]] \setminus \cup_{j=1}^{i-1}[V(G) \setminus N_G[u_j]] \neq \varnothing \text{ for each } \ i \in \{2, 3, \ldots, t\}.$$

Hence, S is a co-legal open neighborhood sequence in G, and so (i) holds. Similarly, (ii) holds if $\hat{S} \subseteq V(H)$.

Next, suppose that $\hat{S} \cap V(G) \neq \varnothing$ and $\hat{S} \cap V(H) \neq \varnothing$. Since $N^2_{G+H}(u_j) \subseteq V(G)$ for all $u_j \in \hat{S} \cap V(G)$ and $N^2_{G+H}(u_s) \subseteq V(H)$ for all $u_s \in \hat{S} \cap V(H)$, S is loh-identical to $S' = S_G \oplus S_H$, where $\hat{S_G} = \hat{S} \cap V(G) = \{u_1, u_2, \ldots, u_m\}$, $\hat{S_H} = \hat{S} \cap V(H) = \{w_1, w_2, \ldots, w_t\}$, $|\hat{S}| = m+t$, and the orders of appearances of the terms of both sequences in S are retained. Since S is a legal open hop neighborhood sequence, it follows that

$$[V(G) \setminus N_G[u_i]] \setminus \cup_{j=1}^{i-1}[V(G) \setminus N_G[u_j]] = N^2_{G+H}(u_i) \setminus \cup_{j=1}^{i-1} N^2_{G+H}(u_j) \neq \varnothing$$

for each $i \in \{2, 3, \ldots, m\}$. Thus, S_G is a co-legal open neighborhood sequence in G. Similarly, S_H is a co-legal open neighborhood sequence in H, and so (iii) holds.

The converse is clear. □

The next result follows from Lemma 3 and Theorem 7.

Corollary 3. *Let G and H be any two graphs such that $\gamma(G) \neq 1$ and $\gamma(H) \neq 1$. A sequence S of distinct vertices of $G + H$ is a Grundy total hop dominating sequence in $G + H$ if and only if S is loh-identical to $S' = S_G \oplus S_H$, where S_G and S_H are co-Grundy total dominating sequences in G and H, respectively (Grundy total dominating sequences in $\overline{G}$ and $\overline{H}$, respectively). Moreover,*

$$\gamma^{th}_{gr}(G + H) = \gamma^t_{cogr}(G) + \gamma^t_{cogr}(H) = \gamma^t_{gr}(\overline{G}) + \gamma^t_{gr}(\overline{H}).$$

In particular, we have

(i) $\gamma^{th}_{gr}(K_{m,n}) = \gamma^{th}_{gr}(\overline{K}_m + \overline{K}_n) = \gamma^t_{gr}(K_m) + \gamma^t_{gr}(K_n) = 4$ *for any* $m, n \geq 2$,

(ii) $\gamma^{th}_{gr}(\overline{K}_n + P_m) = \gamma^t_{gr}(K_n) + \gamma^t_{gr}(\overline{P}_m) = 6$ *for any* $n \geq 2$ *and* $m \geq 4$,

(iii) $\gamma^{th}_{gr}(\overline{K}_n + C_m) = \gamma^t_{gr}(K_n) + \gamma^t_{gr}(\overline{C}_m) = 6$ *for any* $n \geq 2$ *and* $m \geq 4$,

(iv) $\gamma^{th}_{gr}(P_n + P_m) = \gamma^t_{gr}(\overline{P}_n) + \gamma^t_{gr}(\overline{P}_m) = 8$ *for any* $n, m \geq 4$,

(v) $\gamma^{th}_{gr}(P_n + C_m) = \gamma^t_{gr}(\overline{P}_n) + \gamma^t_{gr}(\overline{C}_m) = 8$ *for any* $n, m \geq 4$, *and*

(vi) $\gamma^{th}_{gr}(C_n + C_m) = \gamma^t_{gr}(\overline{C}_n) + \gamma^t_{gr}(\overline{C}_m) = 8$ *for any* $n, m \geq 4$.

Theorem 8. *Let G be a non-trivial connected graph on n vertices and let H be any graph such that $\gamma(H) \neq 1$. Then $\gamma_{gr}^{th}(G \circ H) \geq n \cdot \gamma_{cogr}^{t}(H) = n \cdot \gamma_{gr}^{t}(\overline{H})$.*

Proof. Let $V(G) = \{u_1, u_2, \ldots, u_n\}$ and let $S_{u_i} = (w_{u_i}^1, w_{u_i}^2, \cdots, w_{u_i}^k)$ be a co-Grundy total dominating sequence in H^{u_i} for each $i \in [n]$, where $k = \gamma_{cogr}^{t}(H)$. Let $S = S_{u_1} \oplus S_{u_2} \oplus \cdots \oplus S_{u_n}$. Let $v \in V(G \circ H) \setminus \hat{S}$ and let $u_t \in V(G)$ such that $v \in V(u_t + H^{u_t})$ for some $t \in [n]$. Suppose first that $v = u_t$. Let $u_s \in N_G(u_t)$ and pick any $w_{u_s}^j \in \hat{S}_{u_s}$ for some $s \in [n]$. Then $w_{u_s}^j \in \hat{S} \cap N_{G\circ H}^2(u_t)$. Suppose $v \neq u_t$. Then $v \in V(H^{u_t}) \setminus \hat{S}_{u_t}$. Since $\hat{S}_{u_t}$ is a co-Grundy total dominating sequence in H^{u_t}, it follows that there exists $w_{u_t}^l \in \hat{S}_{u_t} \subseteq \hat{S}$ such that $d_{H^{u_t}}(v, w_{u_t}^l) \neq 1$. It follows that $d_{G\circ H}(v, w_{u_t}^l) = 2$. Therefore, $\hat{S}$ is a total hop dominating set in $G \circ H$. Now, we relabel the terms in S, say $S = (v_1, v_2, \cdots, v_k, \cdots, v_{nk})$. Let $i \in [nk] \setminus \{1\}$ and let $v_i = w_{u_r}^t$ for some $r \in [n]$ and $t \in [k]$. Then

$$\begin{aligned} N_{G\circ H}^2(v_i) \setminus \cup_{j=1}^{i-1} N_{G\circ H}^2(v_j) &= N_{G\circ H}^2(w_{u_r}^t) \setminus [(\cup_{s=1}^{t-1} N_{G\circ H}^2(w_{u_r}^s)) \cup \\ &\quad (\cup\{N_{G\circ H}^2(w_{u_q}^p) : p \in [k] \text{ and } 1 \leq q \leq r-1\})]. \end{aligned}$$

If $t = 1$, then $N_{G\circ H}^2(w_{u_r}^t) \setminus (\cup_{s=1}^{t-1} N_{G\circ H}^2(w_{u_r}^s)) = N_{G\circ H}^2(w_{u_r}^t)$. Clearly,

$$w_{u_r}^t \in N_{G\circ H}^2(w_{u_r}^t) \setminus [\cup\{N_{G\circ H}^2(w_{u_q}^p) : p \in [k] \text{ and } 1 \leq q \leq r-1\}].$$

Suppose $t \neq 1$. Since S_{u_r} is a co-legal open neighborhood sequence in H^{u_r},

$$\begin{aligned} N_{G\circ H}^2(w_{u_r}^t) \setminus (\cup_{s=1}^{t-1} N_{G\circ H}^2(w_{u_r}^s)) &= [V(H^{u_r}) \setminus N_{H^{u_r}}(w_{u_r}^t)] \setminus \\ &\quad [\cup_{s=1}^{t-1}(V(H^{u_r}) \setminus N_{H^{u_r}}(w_{u_r}^s))] \\ &\neq \varnothing. \end{aligned}$$

Observe that

$$N_{G\circ H}^2(w_{u_r}^t) \setminus (\cup_{s=1}^{t-1} N_{G\circ H}^2(w_{u_r}^s)) \cap [\cup\{N_{G\circ H}^2(w_{u_q}^p) : p \in [k] \text{ and } 1 \leq q \leq r-1\}] = \varnothing.$$

Hence, $N_{G\circ H}^2(v_i) \setminus \cup_{j=1}^{i-1} N_{G\circ H}^2(v_j) \neq \varnothing$ for all $i \in [nk] \setminus \{1\}$ and so S is a Grundy total hop dominating sequence in $G \circ H$. Consequently,

$$\gamma_{gr}^{th}(G \circ H) \geq |\hat{S}| = \sum_{i=1}^{n} |\hat{S}_{v_i}| = n \cdot \gamma_{cogr}^{t}(H) = n \cdot \gamma_{gr}^{t}(\overline{H}). \quad \square$$

Remark 4. *The bound given in Theorem 8 is tight.*

To see this, consider the graph $K_5 \circ P_4$ in Fig. 5. Let $S = (a_1, a_2, \cdots, a_{20})$. Then S is a Grundy total hop dominating sequence of $K_5 \circ P_4$. Moreover, it can be verified that $\gamma_{gr}^{th}(K_5 \circ P_4) = 20$. Since $\gamma_{gr}^{t}(\overline{P}_4) = 4$, the assertion follows.

$K_5 \circ P_4$:

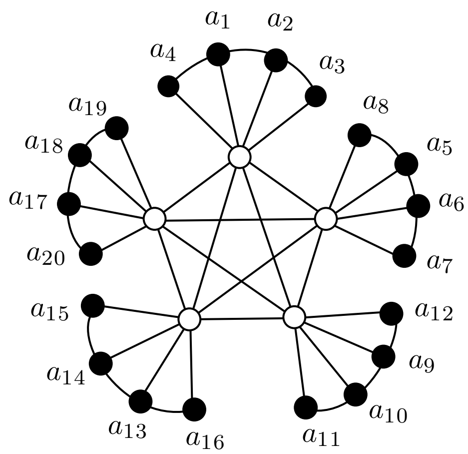

Figure 5: A graph $K_5 \circ P_4$ with $\gamma_{gr}^{th}(K_5 \circ P_4) = |K_5|\gamma_{gr}^{t}(\overline{P}_4)$.

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