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J-Domination in Graphs

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Abstract. Let G be a graph. A subset $D = \{d_1, d_2, \cdots, d_m\}$ of vertices of G is called a J-set if $N_G[d_i] \setminus N_G[d_j] \neq \varnothing$ for every $i \neq j$, where $i, j \in \{1, 2, \ldots, m\}$. A J-set is called a J-dominating set of G if $D = \{d_1, d_2, \ldots, d_m\}$ is a dominating set of G. The J-domination number of G, denoted by $\gamma_J(G)$, is the maximum cardinality of a J-dominating set of G. In this paper, we introduce this new concept and we establish formulas and properties on some classes of graphs and in join of two graphs. Upper and lower bounds of J-domination parameter with respect to the order of a graph and other parameters in graph theory are obtained. In addition, we present realization result involving this parameter and the standard domination. Moreover, we characterize J-dominating sets in some classes of graphs and join of two graphs and finally determine the exact value of the parameter of each of these graphs.

2020 Mathematics Subject Classifications: 05C69

Key Words and Phrases: J-set, J-dominating set, J-domination number

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1. Introduction

Domination in a graph has been one of the most interesting and widely studied topics in Graph Theory. It has many applications in various fields and in networks.

The study of domination in graphs came about partially as a result of the study of games and recreational mathematics. In particular, mathematicians studied how chess pieces of a particular type could be placed on a chessboard in such a way that they would attack, or dominate, every square on the board.

In 1962, Oystein Ore introduced the concepts of dominating set and domination number in a graph in his book on Graph Theory [12]. A decade later, Cockayne and Hedetniemi published a survey paper, in which the notation $\gamma(G)$ was first used for the domination number of a graph G [3]. Since then, several mathematicians had studied and introduced new domination parameters in graphs. Some variants of domination were defined and further studied by researchers can be found in [1, 2, 4–11, 13, 14].

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In this paper, new variant of domination called J-domination in a graph will be introduced and investigated. Its relationships with other variants of domination and other concepts in graph theory will be determined. Moreover, characterizations of J-dominating sets in some classes of graphs and join of two of graphs will be presented. These results will be used to determine exact values or bounds of the parameter for these graphs. We believe that this study and its results will help other researchers in the field for more research directions in the future.

2. Terminology and Notation

Let $G = (V(G), E(G))$ be a simple and undirected graph. Two vertices x, y of G are *adjacent*, or *neighbors*, if xy is an edge of G. The *open neighborhood* of x in G is the set $N_G(x) = \{y \in V(G) : xy \in E(G)\}$. The *closed neighborhood* of x in G is the set $N_G[x] = N_G(x) \cup \{x\}$. If $X \subseteq V(G)$, the *open neighborhood* of X in G is the set $N_G(X) = \bigcup_{x \in X} N_G(x)$. The *closed neighborhood* of X in G is the set $N_G[X] = N_G(X) \cup X$.

A subset D of $V(G)$ is called a *dominating* of G if for every $x \in V(G) \setminus D$, there exists $y \in D$ such that $xy \in E(G)$. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set in G.

A graph G is *connected* if every pair of its vertices can be Joined by a path. Otherwise, G is *disconnected*. A maximal connected subgraph (not a subgraph of any connected subgraph) of G is called a *component* of G.

A *dominating set* D of G is called a connected dominating if the induced subgraph $\langle D \rangle$ of D is connected. The *connected domination number* of G, denoted by $\gamma_c(G)$, is the minimum cardinality of a connected dominating set in G. Any connected dominating set D with cardinality equal to $\gamma_c(G)$ is called a γ_c-set of G.

The *distance* $d_G(u, v)$ in G of two vertices u, v is the length of a shortest *u-v path* in G. The greatest distance between any two vertices in G, denoted by $diam(G)$, is called the *diameter* of G.

A subset I of $V(G)$ is called an *independent* if for every pair of distinct vertices $x, y \in I$, $d_G(x, y) \neq 1$. The maximum cardinality of an independent set in G, denoted by $\alpha(G)$, is called the *independence* number of G. Any independent set I with cardinality equal to $\alpha(G)$ is called an α-set of G.

A graph is *complete* if every pair of distinct vertices are adjacent. A complete graph of order n is denoted by K_n.

The *complement* of a graph G, denoted by $\overline{G}$, is the graph with $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \{uv : u, v \in V(G) \text{ and } uv \notin E(G)\}$.

Let G and H be any two graphs. The *join* of G and H, denoted by $G + H$ is the graph with vertex set $V(G + H) = V(G) \cup V(H)$ and edge set

$$E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}.$$

3. Results

We begin this section by introducing the concept of J-domination in a graph.

Definition 1. Let G be a simple and undirected graph and $m \in N$. A subset $D = \{d_1, d_2, \cdots, d_m\}$ of vertices of G is called a J-set if $N_G[d_i] \setminus N_G[d_j] \neq \varnothing$ for every $i \neq j$, where $i, j \in \{1, 2, \ldots, m\}$. A J-set is called a J-dominating set of G if $D = \{d_1, d_2, \ldots, d_m\}$ is a dominating set of G. The J-domination number of G, denoted by $\gamma_J(G)$, is the maximum cardinality of a J-dominating set of G. Any J-dominating set D with $|D| = \gamma_J(G)$ (resp. $|D| = \gamma(G)$), is called a γ_J-set or the maximum (resp. minimum) J-dominating set of G. Moreover, if $x \in N_G[d_i] \setminus N_G[d_j]$, then we say d_i J-footprinted a vertex x in G.

Example 1. Consider the graph G in Figure 1. Let $D = \{a, b, c, d\}$. Observe that $a \in N_G[a] \setminus N_G[u]$, $\forall\ u \in \{b, c, d\}$, $b \in N_G[b] \setminus N_G[v]$, $\forall\ v \in \{a, c, d\}$, $c \in N_G[c] \setminus N_G[w]$, $\forall\ w \in \{a, b, d\}$ and $d \in N_G[d] \setminus N_G[x]$, $\forall\ x \in \{a, b, c\}$. It follows that $N_G[s] \setminus N_G[t] \neq \varnothing$ for every $s, t \in D, s \neq t$. Thus, D is a J-set of G. Since $N_G[D] = V(G)$, it follows that D is a dominating set of G. Consequently, D is a J-dominating set of G. Moreover, it can be verified that $\gamma_J(G) = 4$.

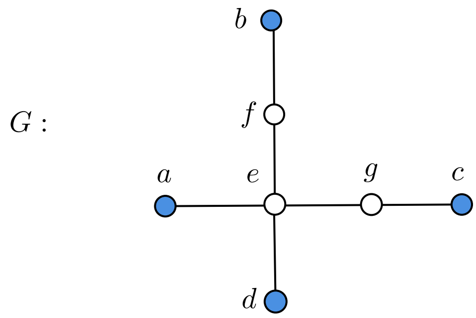

Figure 1: A graph G with $\gamma_J(G) = 4$

Theorem 1. *Let G be any graph and $m \in \mathbb{N}$. Then $T = \{t_1, t_2, \ldots, t_m\} \subseteq V(G)$ is a minimum dominating set in G if and only if T is a minimum J-dominating set in G.*

Proof. Suppose that $T = \{t_1, t_2, \ldots, t_m\} \subseteq V(G)$ is a minimum dominating set in G. Then it remains to show that T is a J-set in G. Suppose on the contrary that $T = \{t_1, t_2, \cdots, t_m\}$ is not a J-set in G. Then there exist $a, b \in T$ such that either $N_G[a] \setminus N_G[b] = \varnothing$ or $N_G[b] \setminus N_G[a] = \varnothing$. This means that either $N_G[a] \subseteq N_G[b]$ or $N_G[b] \subseteq N_G[a]$. If $N_G[a] \subseteq N_G[b]$, then $D = T \setminus \{a\}$ is a dominating set in G, contradicting to the minimality of T. Similarly, when $N_G[b] \subseteq N_G[a] = \varnothing$. Hence, T is a

J-dominating set in G. Since $\gamma(G) = |T|$, the assertion follows.

Conversely, suppose that $T = \{t_1, t_2, \ldots, t_m\}$ is a minimum J-dominating set of G. Then $|T| = \gamma(G)$ (by definition). It follows that T is a minimum dominating set in G. □

Proposition 1. *Let G be any graph and $r \in \mathbb{N}$. Then each of the following holds:*

(i) A graph G admits a J-domination.

(ii) Given any J-dominating set $D = \{d_1, d_2, \cdots, d_r\}$ of G, we have $|D| = r \leq \gamma_J(G)$.

(iii) $\gamma(G) \leq \gamma_J(G)$, and this bound is sharp.

Proof. (i) Since any graph G admits domination, the result follows from Theorem 1.

(ii) Let D be any J-dominating set of G. If D is the maximum, then $\gamma_J(G) = |D|$ and we are done. If D is not the maximum, then $\gamma_J(G) > |D|$. Consequently, $|D| \leq \gamma_J(G)$ for any J-dominating set D of G.

(iii) Let D' be a minimum dominating set of G. Then D' is a minimum J-dominating set in G by Theorem 1. Thus, by (ii), $\gamma(G) = |D'| \leq \gamma_J(G)$. To see the bound is sharp, consider P_4. Then $\gamma_J(P_4) = 2 = \gamma(P_4)$. □

Theorem 2. *Let G be any graph. Then each of the following is true.*

(i) A J-set may not be a J-dominating set. In particular, if the cardinality of a J-set D is strictly less than the domination number of G, then D cannot be a J-dominating set of G.

(ii) A vertex set $V(G)$ of G may not be a J-set in G.

(iii) Every J-dominating set in G is a dominating set but the converse is not always true.

Proof. (i) Consider the Example 1 and let $D = (b, e)$. Then $N_G[b] \setminus N_G[e] = \{b\} \neq \varnothing$ and $N_G[e] \setminus N_G[b] = \{a, d, e, g\} \neq \varnothing$. It follows that D is a J-set in G. However, D is not a J-dominating set in G since $c \notin N_G[D]$, that is, D not a dominating set in G. For the particular case, suppose on the contrary that a J-set D can be a J-dominating set of G. Then D is a dominating set (by definition). Thus, $\gamma(G) \leq |D|$, a contradiction. Therefore, D cannot be a J-dominating set of G.

(ii) Consider again the graph in Example 1. Observe that

$$N_G[c] \setminus N_G[g] = \{c, g\} \setminus \{c, e, g\} = \varnothing.$$

This shows that $V(G)$ is not a J-set of G.

(iii) Let G be a graph and let D be a J-dominating set. Then D is a dominating set in G (by definition). To see that the converse is not true, consider the graph in Example 1 and let $\tilde{A} = \{d, e, f, g\}$. Then $N_G[\tilde{A}] = V(G)$, and so $\tilde{A}$ is a dominating set of G. However,

$N_G[d] \setminus N_G[e] = \{d, e\} \setminus \{a, d, e, f, g\} = \varnothing$, showing that $\tilde{A}$ is not a J-set in G. Thus, $\tilde{A}$ cannot be a J-dominating set in G. □

Theorem 3. *Let G be any graph and $k \in \mathbb{N}$. Then $T = \{v_1, v_2, \cdots, v_k\}$ is a maximum J-set of G if and only if T is a maximum J-dominating set of G. In particular, $\gamma_J(G) = k$.*

Proof. Let $T = \{v_1, \cdots, v_k\}$ be a maximum J-set of G. Suppose T is not a dominating set of G. Then there exists $a \in V(G) \setminus T$ such that $a \notin N_G[T]$. This implies that $a \notin N_G[b]$ for every $b \in T$. Let $T_0 = \{v_0, v_1, \cdots, v_k\}$, where $v_0 = a$. Since T is a J-set and $v_0 \in N_G[v_0]$, it follows that $N_G[v_i] \setminus N_G[v_j] \neq \varnothing$ for every $i \neq j$, where $i, j \in \{0, 1, \ldots, k\}$. Hence, T_0 is a J-set in G, contradicting the maximality of T. Therefore, T is a dominating set of G. Since T is a maximum J-set in G, it follows that T is a maximum J-dominating set in G. Consequently, $\gamma_J(G) = k$.

The converse is clear. □

The following result follows from Theorem 3.

Corollary 1. *Let G be a graph and let $D = \{x_1, x_2, \cdots, x_s\}$ be a J-set of G. Then $|D| = s \leq \gamma_J(G)$.*

Theorem 4. *Let G be any graph and D be any J-dominating set of G. Then*

(i) $u \in D$ if and only if $N_G[u] \nsubseteq N_G[v]$ and $N_G[v] \nsubseteq N_G[u]$ $\forall$ $v \in D \setminus \{u\}$; and

(ii) a pendant vertex x is in D if and only if its neighbor y is not in D.

Proof. (i) Let D be a J-dominating set of G and $u \in D$. Since D is a J-set, we have $N_G[u] \setminus N_G[v] \neq \varnothing$ and $N_G[v] \setminus N_G[u] \neq \varnothing$ $\forall$ $v \in D \setminus \{u\}$. It follows that $N_G[u] \nsubseteq N_G[v]$ and $N_G[v] \nsubseteq N_G[u]$ $\forall$ $v \in D \setminus \{u\}$.

Conversely, suppose that $N_G[u] \nsubseteq N_G[v]$ and $N_G[v] \nsubseteq N_G[u]$ $\forall$ $v \in D \setminus \{u\}$. Then $N_G[u] \setminus N_G[v] \neq \varnothing$ and $N_G[v] \setminus N_G[u] \neq \varnothing$ $\forall$ $v \in D \setminus \{u\}$. This means that $u \in D$.

(ii) Let $x \in D$ be a pendant vertex of G and let y be a neighbor of x. Then $N_G[x] \subseteq N_G[y]$. Suppose on the contrary that $y \in D$. Then by (i), $N_G[x] \nsubseteq N_G[y]$ and $N_G[y] \nsubseteq N_G[x]$ $\forall$ $y \in D \setminus \{x\}$, a contradiction. Thus, $y \notin D$.

Conversely, suppose on the contrary that $x \notin D$. Then by (i), $N_G[x] \subseteq N_G[a]$ or $N_G[a] \subseteq N_G[x]$ for some $a \in D$. Since its neighbor y is not in D, we have $x \notin N_G[w]$ for every $w \in D$. Thus, $N_G[D] \neq V(G)$, showing that D is not a dominating set in G. However, this is a contradiction to the fact that D is a J-dominating set in G. Therefore, $x \in D$. □

Theorem 5. *Let G be any graph and let $S \subseteq V(G)$. Then*

(i) every independent set S is a J-set; and

(ii) every maximum independent set S is a J-dominating set. Moreover, $\alpha(G) \leq \gamma_J(G)$.

Proof. (i) Let $S = \{s_1, s_2, \ldots, s_k\}$ be an independent set in G. Then $d_G(s_i, s_j) \geq 2$ for every $i \neq j$, where $i, j \in \{1, 2, \ldots, k\}$. It follows that $s_i \in N_G[s_i] \setminus N_G[s_j]$ for every $i \neq j$. Thus, $N_G[s_i] \setminus N_G[s_j] \neq \varnothing$ for every $i \neq j$. This shows that S is a J-set of G.

(ii) Next, let $S' = \{a_1, a_2, \ldots, a_m\}$ be a maximum independent set of G. Then S' is a J-set in G by (i). Now, suppose on the contrary that S' is not a dominating set of G. Then there exists $x \in V(G) \setminus S'$ such that $x \notin N_G[y]\ \forall\ y \in S'$. This means that $d_G(x, y) \geq 2$ for all $y \in S'$. Thus, $S^* = \{x\} \cup S'$ is an independent set in G, contradicting the maximality of S'. Therefore, S' is a dominating set of G, and so S' is a J-dominating in G. Consequently, $\alpha(G) \leq \gamma_J(G)$ by Proposition 1. □

Remark 1. *The converse of the Theorem 5 is not true. Moreover, the sharpness of the bound is attainable.*

To see this, consider the graph G in Figure 2. Let $C = \{a, e, h\}$. Then $N_G[a] = \{a, b\}$, $N_G[e] = \{b, c, d, e, f, g, h\}$ and $N_G[h] = \{e, g, h, i\}$. Thus, $N_G[a] \setminus N_G[e] = \{a\} \neq \varnothing$, $N_G[a] \setminus N_G[h] = N_G[a] \neq \varnothing$, $N_G[e] \setminus N_G[a] = \{c, d, e, f, g, h\} \neq \varnothing$, $N_G[e] \setminus N_G[h] = \{b, c, d, f\} \neq \varnothing$, $N_G[h] \setminus N_G[a] = N_G[h] \neq \varnothing$ and $N_G[h] \setminus N_G[e] = \{i\} \neq \varnothing$. Therefore, C is a J-set in G. However, C is not an independent set in G since $d_G(e, h) = 1$. For the sharpness, consider again the graph G in Figure 2 and let $T = \{a, c, d, f, g, i\}$. Then T is an α-set of G, and so $\alpha(G) = 6$. Now, since $x \in N_G[x] \setminus N_G[y]\ \forall\ x, y \in T$, it follows that T is a J-set in G. Moreover, observe that $N_G[T] = V(G)$. Hence, T is J-dominating set in G. Since a, d, f and i are pendant vertices of G, it follows that T is the maximum J-dominating of G by Theorem 4. Consequently, $\alpha(G) = 6 = \gamma_J(G)$.

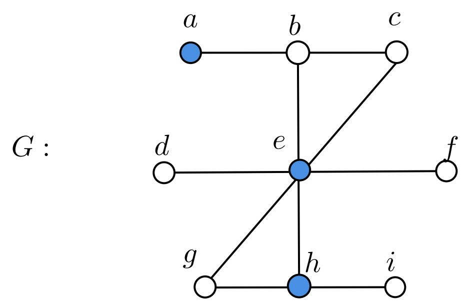

Figure 2: A graph G with a J-set C which is not an independent set in G

Theorem 6. *Let G be any graph on $n \geq 1$ vertices. Then each of the following statements holds.*

(i) $1 \leq \gamma_J(G) \leq n$.

(ii) $\gamma_J(G) = 1$ *if and only if G is complete.*

(iii) *If $\gamma_J(G) \leq n-1$, then $|N_G[v]| \geq 2$ for some $v \in V(G)$. However, the converse is not true.*

(iv) *If $G = \overline{K}_n$, then $\gamma_J(G) = |V(G)| = n$. However, the converse is not true.*

Proof. (i) Since any singleton set $\{a\}$ is a J-set for any $a \in V(G)$, the lower bound follows by Corollary 1. Moreover, since any γ_J-set D is always a subset of $V(G)$, the upper bound follows. Consequently, $1 \leq \gamma_J(G) \leq n$.

(ii) Assume that $\gamma_J(G) = 1$. Suppose G is non-complete graph. Then there exist $a, b \in V(G)$ such that $d_G(a, b) \geq 2$. Let $C = \{a, b\}$. Since $d_G(a, b) \geq 2$, it follows that $a \in N_G[a] \setminus N_G[b]$ and $b \in N_G[b] \setminus N_G[a]$. Thus, $N_G[a] \setminus N_G[b] \neq \varnothing$ and $N_G[b] \setminus N_G[a] \neq \varnothing$ C, showing that C is a J-set in G. By Corollary 1, $\gamma_J(G) \geq 2$ which is a contradiction. Hence, G is complete.

Conversely, suppose that G is complete. Then $N_G[a] = V(G)$ for any $a \in V(G)$. It follows that for every two distinct vertices $x, y \in V(G)$, $N_G[x] \setminus N_G[y] = V(G) \setminus V(G) = \varnothing$. Thus, $\gamma_J(G) \geq 2$ is impossible. Hence, $\gamma_J(G) = 1$ by (i).

(iii) Assume that $\gamma_J(G) \leq n-1$. Suppose on the contrary that $|N_G[v]| < 2$ for every $v \in V(G)$. This means that $|N_G[v]| = 1$ for every $v \in V(G)$. It follows that $G = \overline{K}_n$. Observe that $\gamma(\overline{K}_n) = n$. Thus, $\gamma(\overline{K}_n) = n \leq \gamma_J(G)$ by Proposition 1, a contradiction. To see that the converse is not true, consider the graph G in Figure 3. Let $K = V(G) = \{a, b, c, d, e, f\}$. Then $N_G[a] \setminus N_G[b] = \{d\}, N_G[a] \setminus N_G[c] = \{a\}, N_G[a] \setminus N_G[d] = \{b\}$, $N_G[a] \setminus N_G[e] = \{a, b\}, N_G[a] \setminus N_G[f] = \{a, d\}$, $N_G[b] \setminus N_G[a] = \{c, f\}$, $N_G[b] \setminus N_G[c] = \{a, f\}, N_G[b] \setminus N_G[d] = \{b, f\}, N_G[b] \setminus N_G[e] = \{a, b, c\}$, $N_G[b] \setminus N_G[f] = \{a, c\}$, $N_G[c] \setminus N_G[a] = \{c\}, N_G[c] \setminus N_G[b] = \{d\}$, $N_G[c] \setminus N_G[d] = \{b\}, N_G[c] \setminus N_G[e] = \{b, c\}, N_G[c] \setminus N_G[f] = \{d, c\}$, $N_G[d] \setminus N_G[a] = \{c, e\}, N_G[d] \setminus N_G[b] = \{d, e\}, N_G[d] \setminus N_G[c] = \{a, e\}$, $N_G[d] \setminus N_G[e] = \{a, c\}, N_G[d] \setminus N_G[f] = \{a, c, d\}, N_G[e] \setminus N_G[a] = \{e, f\}$, $N_G[e] \setminus N_G[b] = \{e, d\}, N_G[e] \setminus N_G[c] = \{e, f\}, N_G[e] \setminus N_G[d] = \{f\}$, $N_G[e] \setminus N_G[f] = \{d\}, N_G[f] \setminus N_G[a] = \{e, f\}, N_G[f] \setminus N_G[b] = \{e\}$, $N_G[f] \setminus N_G[c] = \{e, f\}, N_G[f] \setminus N_G[d] = \{b, f\}, N_G[f] \setminus N_G[e] = \{b\}$. Thus, K is a J-dominating set of G, and so $\gamma_G(G) = |V(G| = 6$.

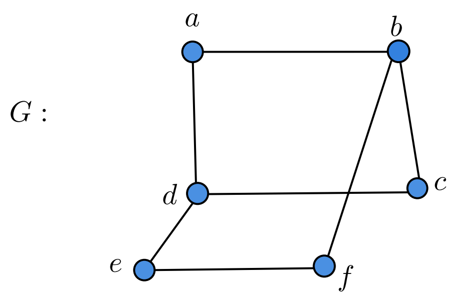

Figure 3: A graph G with $|N_G[v]| \geq 2$ for some $v \in V(G)$, however, $\gamma_J(G) = |V(G)|$

(iv). Let $G = \overline{K}_n$ and $D = \{v_1, v_2, \ldots, v_n\}$ be a vertex set of G. Then $N_G[v_i] = \{v_i\}$ for each $i \in \{1, 2, \ldots, n\}$. Thus, $v_i \in N_G[v_i] \setminus N_G[v_j]$ for each $i \neq j$, where $i, j \in \{1, 2, \ldots, n\}$. This means that $N_G[v_i] \setminus N_G[v_j] \neq \varnothing$ for each $i \neq j$. Thus, $V(G)$ is a J-set of G. Since $V(G)$ is a dominating set of G, $V(G)$ is a J-dominating set in G. Hence, $\gamma_J(G) = |V(G)| = n$ by (i). The converse follows by considering the graph in Figure 3. □

Theorem 7. *Let G be any non-complete graph. Then every dominating vertex v of G is not in γ_J-set D of G.*

Proof. Suppose G is non-complete graph. Then $\gamma_J(G) \geq 2$ by Theorem 6(ii). Now, let $v \in V(G)$ be a dominating vertex of G. Suppose on the contrary that $v \in D$. Since v is a dominating vertex of G, we have $N_G[v] = V(G)$. Thus,

$$N_G[u] \setminus N_G[v] = N_G[u] \setminus V(G) = \varnothing \ \forall \ u \in D \setminus \{v\}.$$

It follows that $\{v\}$ is a maximum J-set of G. By Theorem 3, $\gamma_J(G) = 1$, a contradiction. Therefore, $v \notin D$. □

The next result is a realization result involving J-domination number and domination number of a graph.

Theorem 8. *Let m and n be positive integers such that $2 \leq a \leq b$. Then there exists a connected graph G such that $\gamma(G) = a$ and $\gamma_J(G) = b$. That is, $\gamma_J(G) - \gamma(G)$ can be made arbitrarily large.*

Proof. For equality, consider the graph $\overline{K}_a$. Since $\gamma(\overline{K}_s) = s$, it follows that $\gamma_J(\overline{K}_a) = a = \gamma(\overline{K}_a)$ by Theorem 6(iv).

For the inequality($a < b$), consider the following cases:
Case 1: a=2
Let $m = b - 2$ and consider consider the graph G in Figure 4. Let $D_1 = \{y_1, y_2\}$ and

$D_2 = \{x_1, x_2, z_1, z_2, \ldots, z_m\}$. Since for any $x \in V(G)$ is not a dominating vertex, it follows that D_1 is a γ-set of G, that is, $\gamma(G) = 2$. Now, observe that $x_1, y_1 \in N_G[x_1] \setminus N_G[u]\ \forall\ u \in D_2 \setminus \{x_1\}$, $x_2 \in N_G[x_2] \setminus N_G[v]\ \forall\ v \in D_2 \setminus \{x_2\}$, and $z_i \in N_G[z_i] \setminus N_G[w]\ \forall\ w \in D_2 \setminus \{z_i\}$, $i \in \{1, 2, \ldots, m\}$. Thus, D_2 is a J-set of G. Since D_2 is a dominating set, D_2 is a J-dominating of G. Since x_i and z_j are pendant vertices for each $i \in \{1, 2\}$ and $j \in \{1, 2, \ldots, m\}$, it follows that D_2 is a maximum J-dominating set of G by Theorem 4. Hence, $\gamma_J(G) = m + 2 = b$ by Theorem 3. Consequently, $\gamma(G) = a < b = \gamma_J(G)$.

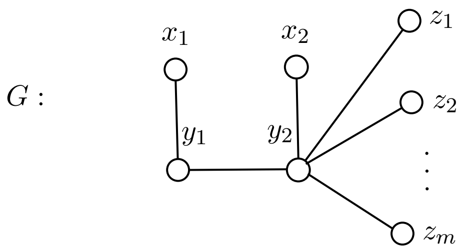

Figure 4: A graph G with $\gamma(G) < \gamma_J(G)$

Case 2: $a \geq 3$

Let $m = b - a$ and consider the graph G' in Figure 5. Let $D' = \{y_1, y_2, \ldots, y_a\}$ and $D^* = (x_1, x_2, \cdots, x_a, z_1, z_2, \cdots, z_m)$. Then D' is a γ-set of G. Thus, $\gamma(G) = a$. Observe that $x_i \in N_G[x_i] \setminus N_G[u]\ \forall\ u \in D^* \setminus \{x_i\}$, $z_{k-1} \in N_G[z_k] \setminus N_G[z_l]\ \forall\ l > k$, $z_{s+1} \in N_G[z_s] \setminus N_G[z_t]\ \forall\ s > t$, and $z_q \in N_G[z_q] \setminus N_G[x_r]$, where $i, r \in \{1, 2, \ldots, a\}$, and $i, k, l, s, t, q \in \{1, 2, \ldots, m\}$. Thus, D^* is a J-set of G. Since D^* is a dominating set, D^* is a J-dominating of G. Since x_i is a pendant vertex for each $i \in \{1, 2, \ldots, a\}$, it follows that D^* is a maximum J-dominating set of G by Theorem 4. Hence, $\gamma_J(G) = m + a = b$ by Theorem 3. Therefore, $\gamma(G) = a < b = \gamma_J(G)$.

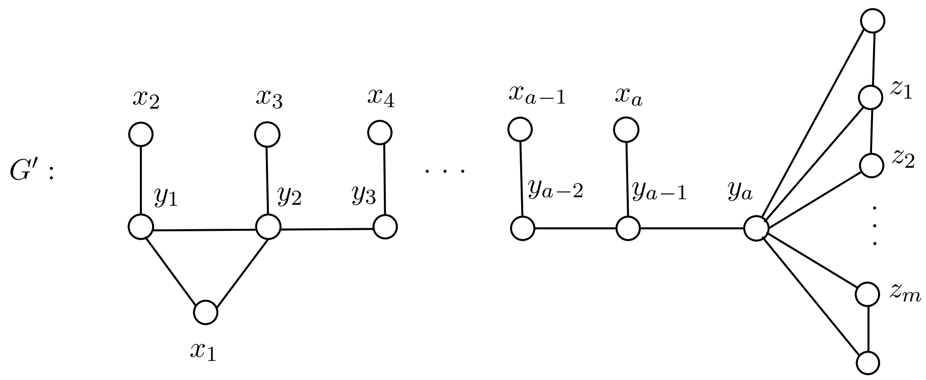

Figure 5: A graph G' with $\gamma(G') < \gamma_J(G')$

$\square$

Proposition 2. *Given any positive integer $n \geq 1$, we have*

$$(i)\ \ \gamma_J(P_n) = \begin{cases} 1 \textit{ if } n = 1, 2 \\ 2 \textit{ if } n = 3, 4 \\ n - 2 \textit{ if } n \geq 5, \end{cases}$$

$$(ii)\ \ \gamma_J(C_n) = \begin{cases} 1 \textit{ if } n = 3 \\ n \textit{ if } n \geq 4. \end{cases}$$

Proof. (i) Clearly, $\gamma_J(P_n) = 1$ for $n = 1, 2$ and $\gamma_J(P_n) = 2$ for $n = 3, 4$. Suppose $n \geq 5$. Let $P_n = G = [v_1, v_2, \ldots, v_n]$, $D = \{v_2, v_3 \cdots, v_{n-2}, v_{n-1}\}$ and let $i, j \in \{2, 3, \ldots, n-1\}$. Then $v_{i-1} \in N_G[v_i] \setminus N_G[v_j]$ and $v_{j+1} \in N_G[v_j] \setminus N_G[v_i]$ for all $j > i$, $i, j \in \{2, 3, \ldots, n-1\}$. Thus, $N_G[v_i] \setminus N_G[v_j] \neq \varnothing$ for all $i \neq j, i, j \in \{2, 3, \ldots, n-1\}$, showing that D is a J-set in G. Therefore, $\gamma_J(P_n) \geq n - 2$ by Corollary 1. If $\gamma_J(P_n) = n$, then $V(P_n)$ is the maximum J-dominating set in P_n. However, $N_G[v_n] \subseteq N_G[v_{n-1}]$. Thus, $N_G[v_n] \setminus N_G[v_{n-1}] = \varnothing$, a contradiction. Next, suppose that $\gamma_J(G) = n - 1$, say $A = \{a_1, a_2, \cdots, a_{n-1}\}$ is a J-dominating set of G. Then either $\langle A \rangle$ is connected or disconnected. Assume that $\langle A \rangle$ is connected. Then either v_1 or v_n is not in A. If $v_1 \notin A$, then $v_{n-1}, v_n \in A$. However, $N_G[v_n] \setminus N_G[v_{n-1}] = \{v_n, v_{n-1}\} \setminus \{v_n, v_{n-1}, v_{n-2}\} = \varnothing$, a contradiction. If $v_n \notin A$, then $v_1, v_2 \in A$. However, $N_G[v_1] \setminus N_G[v_2] = \{v_1, v_2\} \setminus \{v_1, v_2, v_3\} = \varnothing$, a contradiction. Next, suppose that $\langle A \rangle$ is disconnected. Then there exists $v_j \in V(G)$ such that $v_j \notin A$ for some $j \in \{2, 3, \ldots, n-1\}$. If $v_2 \notin A$, then $v_{n-1}, v_n \in A$ and $N_G[v_n] \setminus N_G[v_{n-1}] = \{v_n, v_{n-1}\} \setminus \{v_n, v_{n-1}, v_{n-2}\} = \varnothing$, a contradiction. If $v_{n-1} \notin A$, then $v_1, v_2 \in A$ and $N_G[v_1] \setminus N_G[v_2] = \{v_1, v_2\} \setminus \{v_1, v_2, v_3\} = \varnothing$, a contradiction. Lastly, suppose that $v_j \notin A$, where $2 < j < n-1$. Then $v_{n-1}, v_n \in A$ and $v_1, v_2 \in A$, a contradiction by the preceding arguments. Therefore, $\gamma_J(G) = n - 1$ is impossible. Consequently,

$\gamma_J(P_n) = n-2$ for all $n \geq 5$.

(ii) Clearly, $\gamma_J(C_3) = 1$. Suppose $n \geq 4$. Let $C_n = G = [v_1, v_2, \ldots, v_n, v_1]$ and let $D^* = V(G) = \{v_1, v_2, \ldots, v_n\}$. Then $v_n \in N_G[v_1] \setminus N_G[v_j]$ for all $j \in \{2, 3, \ldots, n-2\}$, $v_1 \in N_G[v_1] \setminus N_G[v_{n-1}]$, $v_2 \in N_G[v_1] \setminus N_G[v_n]$, $v_1 \in N_G[v_n] \setminus N_G[v_s]$ for all $s \in \{3, 4, \ldots, n-1\}$, $v_n \in N_G[v_n] \setminus N_G[v_2]$, $v_{n-1} \in N_G[v_n] \setminus N_G[v_1]$, $v_{i+1} \in N_G[v_i] \setminus N_G[v_j]\ \forall\ i, j \in \{1, 2, \ldots, n-1\}, i > j$, and $v_{k-1} \in N_G[v_k] \setminus N_G[v_l]\ \forall\ k, l \in \{2, 3, \ldots, n\}$, $k < l$. It follows that $N_G[v_q] \setminus N_G[v_r] \neq \varnothing$ for every $q \neq r, q, r \in \{1, 2, \ldots, n\}$. Therefore, $D^* = V(G)$ is a J-dominating set in G, showing that $\gamma_J(C_n) = n$ for all $n \geq 4$. □

Theorem 9. *Let G and H be two non-complete graphs. A subset D of distinct vertices of $G + H$ is a J-dominating set in $G + H$ if and only if one of the following condition holds:*

(*i*) *D is a J-dominating set of G.*

(*ii*) *D is a J-dominating set of H.*

(*iii*) *$D = D_G \cup D_H$, where D_G and D_H are J-sets in G and H, respectively.*

Proof. Suppose that D is J-dominating set in $G + H$. If $D_H = \varnothing$, then $D = D_G$ is a J-dominating set in G, and so (i) holds. If $D_G = \varnothing$, then $D = D_H$ is a J-dominating set in H, showing that (ii) holds. Now, assume that D_G and D_H are both non-empty. Suppose on the contrary that D_G is not a J-set in G. Then there exist $a, b \in D_G \subseteq D$ such that either $N_G[a] \setminus N_G[b] = \varnothing$ or $N_G[b] \setminus N_G[a] = \varnothing$, a contradiction. Therefore, D_G is a J-set in G. Similarly, D_H is a J-set in H. Consequently, (iii) holds.

Conversely, suppose that (i) holds. Then D is a J-dominating set in $G+H$. Similarly, if (ii) holds, then D is a J-dominating set in $G + H$. Now, assume that (iii) holds. Since D_G and D_H are both non-empty, it follows that D is a dominating set in $G+H$. It remains to show that D is a J-set in $G+H$. Let $a, b \in D$. If $a, b \in D_G \subseteq D$, then $N_G[a] \setminus N_G[b] \neq \varnothing$ and $N_G[b] \setminus N_G[a] \neq \varnothing$ by assumption. This implies that $N_{G+H}[a] \setminus N_{G+H}[b] \neq \varnothing$ and $N_{G+H}[b] \setminus N_{G+H}[a] \neq \varnothing$, and we are done. Similarly, if $a, b \in D_H \subseteq D$, then D is a J-set in $G + H$. Now, assume that $a \in D_G$ and $b \in D_H$. If a is a dominating vertex of G, then $D_H = \varnothing$, a contradiction. Thus, a is not a dominating vertex of G. Similarly, b is not a dominating vertex of H. Let $x \in V(G)$ and $y \in V(H)$, where $x \notin N_G[a]$ and $y \notin N_H[b]$. Then $y \in N_{G+H}[a] \setminus N_{G+H}[b] \neq \varnothing$ and $x \in N_{G+H}[b] \setminus N_{G+H}[a]$. Thus, D is a J-set in $G + H$. Consequently, D is a J-dominating set in $G + H$. □

Corollary 2. *Let G and H be two non-complete graphs. Then*

$$\gamma_J(G + H) = \gamma_J(G) + \gamma_J(H).$$

In particular, each of the following holds:

(*i*) $\gamma_J(K_{m,n}) = \gamma_J(\overline{K}_m) + \gamma_J(\overline{K}_n) = m + n$ *for all* $m, n \geq 2$.

$$(ii)\ \gamma_J(P_n+P_m)=\begin{cases}4 \text{ if } n=3, m=3\\ m \text{ if } n=3, m\geq 4\\ n \text{ if } n\geq 4\geq m=3\\ n+m-4 \text{ if } n\geq 4, m\geq 4.\end{cases}$$

$$(iii)\ \gamma_J(P_n+C_m)=\begin{cases}m+2 \text{ if } n=3\geq m\geq 4\\ n+m-2 \text{ if } n\geq 4, m\geq 4.\end{cases}$$

(iv) $\gamma_J(C_n+C_m)=n+m$ *for all* $n,m\geq 4$.

Proof. Let $D=D_G\cup D_H$ be a γ_J-set of $G+H$. Then by Theorem 9, D_G and D_H are J-sets in G and H, respectively. It follows that

$$\gamma_J(G+H)=|D|=|D_G|+|D_H|\leq\gamma_J(D_G)+\gamma_J(D_H) \text{ by Corollary 1.}$$

On the other hand, suppose that $D=D_G\cup D_H$, where D_G and D_H are maximum J-sets in G and H, respectively. Then by Theorem 9, D is a J-dominating set in $G+H$. Thus, $\gamma_J(G)+\gamma_J(H)=|D_G|+|D_H|=|D|\leq\gamma_J(G+H)$ by Proposition 1. Consequently, $\gamma_J(G+H)=\gamma_J(G)+\gamma_J(H)$. Moreover, particular cases follow from Theorem 6 and Proposition 2. □

Theorem 10. *Let G and H be any complete and any non-complete graphs, respectively. A subset $D=D_G\cup D_H$ of distinct vertices of $G+H$ is a J-dominating set in $G+H$ if and only if one of the following conditions holds:*

(i) $D_H=\varnothing$ *and* D_G *is a* J*-dominating set of* G.

(ii) $D_G=\varnothing$ *and* D_H *is a* J*-dominating set of* H.

Proof. Suppose that D is J-dominating set in $G+H$. Since G is complete, both $D_G\neq\varnothing$ and $D_H\neq\varnothing$ are impossible. It follows that either $D_G\neq\varnothing$ and $D_H=\varnothing$ or $D_H\neq\varnothing$ and $D_G=\varnothing$. If $D_H=\varnothing$, then $D=D_G$ is a J-dominating set in G, and so (i) holds. If $D_G=\varnothing$, then $D=D_H$ is a J-dominating set in H, and so (ii) holds.

Conversely, suppose that $D_H=\varnothing$ and D_G is a J-dominating set of G. Then $D=D_G$ is a J-set in $G+H$. Since D_G is dominating set in G, $N_{G+H}[D]=V(G+H)$. Thus, D is a dominating set in $G+H$, showing that D is a J-dominating set in $G+H$. Similarly, if (ii) holds, then D is a J-dominating set in $G+H$. □

Corollary 3. *Let G and H be complete and non-complete graphs, respectively. Then*

$$\gamma_J(G+H)=\gamma_J(H).$$

In particular, each of the following holds:

(i) $\gamma_J(K_{1,n})=n$ *for all* $n\geq 1$.

(ii) $\gamma_J(W_n) = \gamma_J(K_1 + C_n) = n$ *for all* $n \geq 4$.

(iii) $\gamma_J(F_n) = \gamma_J(K_1 + P_n) = n - 2$ *for all* $n \geq 3$.

Proof. Let $D = D_G \cup D_H$ be a γ_J-set of $G + H$. Then by Theorem 10, either $D_H = \varnothing$ and D_G is J-dominating set in G or $D_G = \varnothing$ and D_H is J-dominating set in H. It follows that

$$\gamma_J(G + H) = |D| = |D_G| + |D_H| = |D_G| \leq \gamma_J(G)$$

or

$$\gamma_J(G + H) = |D| = |D_G| + |D_H| = |D_H| \leq \gamma_J(D_H) \text{ by Proposition 1.}$$

Thus,

$$\gamma_J(G + H) \leq max\{\gamma_J(G), \gamma_J(D_H)\}.$$

Since G is complete, it follows that $\gamma_J(G + H) \leq \gamma_J(D_H)\}$.

On the other hand, suppose that $D = D_G \cup D_H$, where either $D_H = \varnothing$ and $D_G = D$ is γ_J- set in G or $D_G = \varnothing$ and $D_H = D$ is γ_J-set in H. Then by Theorem 10, D is a J-dominating set in $G + H$. It follows that $\gamma_J(G) = |D_G| \leq \gamma_J(G + H)$ or $\gamma_J(H) = |D_H| \leq \gamma_J(G + H)$ by Proposition 1. Thus, $max\{\gamma_J(G), \gamma_J(H)\} \leq \gamma_J(G + H)$. Since G is complete, we have $\gamma_J(H) \leq \gamma_J(G + H)$. Consequently, $\gamma_J(G + H) = \gamma_J(H)$. Moreover, particular cases follow from Theorem 6 and Proposition 2. □

4. Conclusion

The concept of J-domination has been introduced and investigated in this study. Its bounds with respect to the order of a graph and other parameters have been determined. It was shown that any graph G admits a J-domination and was found out that a graph G satisfying $\gamma_J(G) = |V(G)|$ is not unique. Moreover, characterizations of J-dominating sets in some graphs have been used to determine exact values of parameter of some graphs. Some graphs that were not considered in this study could be an interesting to investigate further for the concept. In addition, researchers may consider the complexity and algorithm, and real life application of the concept.

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