EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 3, No. 5, 2010, 903-917
ISSN 1307-5543 – www.ejpam.com

Subordination Results for Certain Subclasses of Uniformly Starlike and Convex Functions Defined by Convolution

M. K. Aouf[1,*], R. M. El-Ashwah[2] and S. M. El-Deeb[2]

[1] *Department of Mathematics, Faculty of Science, University of Mansoura, Mansoura 33516, Egypt*
[2] *Department of Mathematics, Faculty of Science at Damietta, University of Mansoura, New Damietta 34517, Egypt*

Abstract. In this paper we derive several subordination results for certain subclasses of uniformly starlike and convex functions defined by convolution.

2000 Mathematics Subject Classifications: 30C45
Key Words and Phrases: Analytic, univalent, uniformly, convolution, subordinating factor sequence.

---

1. Introduction

Let A denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

that are analytic and univalent in the open unit disk $U = \{z : |z| < 1\}$. Let $f \in A$ be given by (1) and $\Phi \in A$ be given by

$$\Phi(z) = z + \sum_{k=2}^{\infty} c_k z^k. \tag{2}$$

Definition 1 (Hadamard Product or Convolution)**.** *Given two functions f and Φ in the class A, where $f(z)$ is given by (1) and $\Phi(z)$ is given by (2) the Hadamard product (or convolution) $f * \Phi$ of f and Φ is defined (as usual) by*

$$(f * \Phi)(z) = z + \sum_{k=2}^{\infty} a_k c_k z^k = (\Phi * f)(z). \tag{3}$$

We also denote by K the class of functions $f(z) \in A$ that are convex in U.

---

*Corresponding author.

Email addresses: `mkaouf127@yahoo.com` (M. Aouf), `r_elashwah@yahoo.com` (R. El-Ashwah), `shezaeldeeb@yahoo.com` (S. El-Deeb)

Following Goodman ([9] and [10]), Ronning ([19] and [20]) introduced and studied the following subclasses:

(i) A function $f(z)$ of the form (1) is said to be in the class $S_p(\alpha,\beta)$ of uniformly β-starlike functions if it satisfies the condition:

$$\operatorname{Re}\left\{\frac{zf^{'}(z)}{f(z)}-\alpha\right\}>\beta\left|\frac{zf^{'}(z)}{f(z)}-1\right| \ (z\in U), \tag{4}$$

where $-1\leq\alpha<1$ and $\beta\geq 0$.

(ii) A function $f(z)$ of the form (1) is said to be in the class $UCV(\alpha,\beta)$ of uniformly β-convex functions if it satisfies the condition:

$$\operatorname{Re}\left\{1+\frac{zf^{''}(z)}{f^{'}(z)}-\alpha\right\}>\beta\left|\frac{zf^{''}(z)}{f^{'}(z)}\right| \ (z\in U), \tag{5}$$

where $-1\leq\alpha<1$ and $\beta\geq 0$.

It follows from (4) and (5) that

$$f(z)\in UCV(\alpha,\beta)\iff zf^{'}(z)\in S_p(\alpha,\beta). \tag{6}$$

For $-1\leq\alpha<1$, $0\leq\gamma\leq 1$ and $\beta\geq 0$, we let $S_\gamma(f,g;\alpha,\beta)$ be the subclass of A consisting of functions $f(z)$ of the form (1) and functions $g(z)$ given by

$$g(z)=z+\sum_{k=2}^{\infty}b_kz^k \ (b_k\geq 0), \tag{7}$$

and satisfying the analytic criterion:

$$\operatorname{Re}\left\{\frac{z(f*g)^{'}(z)+\gamma z^2(f*g)^{''}(z)}{(1-\gamma)(f*g)(z)+\gamma z(f*g)^{'}(z)}-\alpha\right\}>\beta\left|\frac{z(f*g)^{'}(z)+\gamma z^2(f*g)^{''}(z)}{(1-\gamma)(f*g)(z)+\gamma z(f*g)^{'}(z)}-1\right|. \tag{8}$$

We note that:

(i) $S_0(f,\Phi(z);\alpha,\beta)=H\left(\Phi,\alpha,\beta\right)$ $(-1\leq\alpha<1,\ \beta\geq 0)$ (see Raina and Bansal [18]), where

$$\Phi(z)=z+\sum_{k=2}^{\infty}\mu_kz^k \ (\mu\geq 0);$$

(ii) $S_0(f,\frac{z}{(1-z)};\alpha,1)=S_p(\alpha)$ and
$S_0(f,\frac{z}{(1-z)^2};\alpha,1)=S_1(f,\frac{z}{(1-z)};\alpha,1)=UCV(\alpha)(-1\leq\alpha<1)$ (see Bharati et al. [5]);

(iii) $S_1(f,\frac{z}{(1-z)};0,\beta)=UCV(\beta)\left(\beta\geq 0\right)$ (see Subramanian et al. [24]);

(iv) $S_0(f, z + \sum_{k=2}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}} z^k; \alpha, \beta) = S(\alpha, \beta)\,(-1 \le \alpha < 1, \beta \ge 0, c \ne 0, -1, -2, \ldots)$ (see Murugusundaramoorthy and Magesh [14,15]);

(v) $S_0(f, z + \sum_{k=2}^{\infty} k^n z^k; \alpha, \beta) = S(n, \alpha, \beta)\,(-1 \le \alpha < 1, \beta \ge 0,\ n \in N_0 = N \cup \{0\}, N = \{1, 2, ...\})$ (see Rosy and Murugusundaramoorthy [21]);

(vi) $S_0(f, z + \sum_{k=2}^{\infty} [1 + \lambda(k-1)]^n z^k; \alpha, \beta) = S_\lambda(n, \alpha, \beta)\,(-1 \le \alpha < 1, \beta \ge 0, \lambda \ge 0, n \in N_0)$ (see Aouf and Mostafa [2]);

(vii) $S_\gamma(f, z + \sum_{k=2}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}} z^k; \alpha, \beta) = S(\gamma, \alpha, \beta)\,(-1 \le \alpha < 1, \beta \ge 0, 0 \le \gamma \le 1, c \ne 0, -1, -2, \ldots)$ (see Murugusundaramoorthy et al. [16]);

(viii) $S_\gamma(f, z + \sum_{k=2}^{\infty} \Gamma_k z^k; \alpha, \beta) = S_q^s(\gamma, \alpha, \beta)$ (see Ahuja et al. [1]), where

$$\Gamma_k = \frac{(\alpha_1)_{k-1} \ldots (\alpha_q)_{k-1}}{(\beta_1)_{k-1 \ldots} (\beta_s)_{k-1}} \frac{1}{(k-1)!} \tag{9}$$

for $\alpha_i > 0,\ i = 1, \ldots, q;\ \beta_j > 0,\ j = 1, \ldots, s;\ q \le s + 1;\ q, s \in N_0$.

Also we note that:

(i)

$$\begin{aligned} & S_0(f, z + \sum_{k=2}^{\infty} \binom{k + \lambda - 1}{\lambda} z^k; \alpha, \beta) = S(\alpha, \beta, \lambda) \\ = & \left\{ f \in A : Re \left\{ \frac{z(D^\lambda f(z))'}{D^\lambda f(z)} - \alpha \right\} > \beta \left| \frac{z(D^\lambda f(z))'}{D^\lambda f(z)} - 1 \right| \right. \\ & \left. (-1 \le \alpha < 1, \beta \ge 0, \lambda > -1, z \in U) \right\}, \end{aligned} \tag{10}$$

where D^λ is Ruscheweyh derivative [22], defined by

$$D^\lambda f(z) = \frac{z(z^{\lambda-1} f(z))^\lambda}{\lambda!} = \frac{z}{(1-z)^{\lambda+1}} * f(z);$$

(ii)

$$\begin{aligned} & S_\gamma(f, z + \sum_{k=2}^{\infty} k^n z^k; \alpha, \beta) = S_\gamma(n, \alpha, \beta) \\ = & \left\{ f \in A : Re \left\{ \frac{(1-\gamma) z (D^n f(z))' + \gamma z (D^{n+1} f(z))'}{(1-\gamma) D^n f(z) + \gamma D^{n+1} f(z)} - \alpha \right\} \right. \end{aligned}$$

$$> \beta \left| \frac{(1-\gamma)z(D^n f(z))' + \gamma z(D^{n+1} f(z))'}{(1-\gamma)D^n f(z) + \gamma D^{n+1} f(z)} - 1 \right|,$$

$$\left. \left(-1 \le \alpha < 1,\ \beta \ge 0,\ n \in N_0,\ z \in U\right) \right\}, \tag{11}$$

(iii)

$$S_\gamma(f, z + \sum_{k=2}^{\infty} \left(\frac{c+1}{c+k}\right) z^k; \alpha, \beta) = S_\gamma\left(c, \alpha, \beta\right)$$

$$= \left\{ f \in A : \mathrm{Re}\left\{ \frac{z(J_c f(z))' + \gamma z^2 (J_c f(z))''}{(1-\gamma)J_c f(z) + \gamma z(J_c f(z))'} - \alpha \right\} \right.$$

$$> \beta \left| \frac{z(J_c f(z))' + \gamma z^2 (J_c f(z))''}{(1-\gamma)J_c f(z) + \gamma z(J_c f(z))'} - 1 \right|,$$

$$\left. \left(0 \le \gamma \le 1,\ -1 \le \alpha < 1,\ \beta \ge 0,\ c > -1,\ z \in U\right) \right\}, \tag{12}$$

where J_c is a Bernardi operator [4], defined by

$$\begin{aligned} J_c f(z) &= \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt \\ &= z + \sum_{k=2}^{\infty} \left(\frac{c+1}{c+k}\right) a_k z^k. \end{aligned}$$

Note that the operator $J_1 f(z)$ was studied earlier by Libera [11] and Livingston [12];

(iv)

$$S_\gamma(f, z + \sum_{k=2}^{\infty} \frac{(\mu)_{k-1}}{(\lambda+1)_{k-1}} z^k; \alpha, \beta) = S_\gamma\left(\mu, \lambda; \alpha, \beta\right)$$

$$= \left\{ f \in A : \mathrm{Re}\left\{ \frac{z(I_{\lambda,\mu} f(z))' + \gamma z^2 (I_{\lambda,\mu} f(z))''}{(1-\gamma)I_{\lambda,\mu} f(z) + \gamma z(I_{\lambda,\mu} f(z))'} - \alpha \right\} \right.$$

$$> \beta \left| \frac{z(I_{\lambda,\mu} f(z))' + \gamma z^2 (I_{\lambda,\mu} f(z))''}{(1-\gamma)I_{\lambda,\mu} f(z) + \gamma z(I_{\lambda,\mu} f(z))'} - 1 \right|,$$

$$\left. \left(0 \le \gamma \le 1,\ -1 \le \alpha < 1,\ \beta \ge 0,\ \lambda > -1,\ \mu > 0,\ z \in U\right) \right\}, \tag{13}$$

where $I_{\lambda,\mu}$ is a Choi-Saigo-Srivastava operator [7], defined by

$$I_{\lambda,\mu} f(z) = z + \sum_{k=2}^{\infty} \frac{(\mu)_{k-1}}{(\lambda+1)_{k-1}} a_k z^k \ (\lambda > -1;\ \mu > 0);$$

(v)

$$S_\gamma(f, z+\sum_{k=2}^{\infty}\frac{(c)_{k-1}}{(a)_{k-1}}\frac{(\lambda+1)_{k-1}}{(1)_{k-1}}z^k;\alpha,\beta)=S_\gamma\left(a,c,\lambda;\alpha,\beta\right)$$
$$=\left\{f\in A:\mathrm{Re}\left\{\frac{z(I^\lambda(a,c)f(z))'+\gamma z^2(I^\lambda(a,c)f(z))''}{(1-\gamma)I^\lambda(a,c)f(z)+\gamma z(I^\lambda(a,c)f(z))'}-\alpha\right\}\right.$$
$$>\beta\left|\frac{z(I^\lambda(a,c)f(z))'+\gamma z^2(I^\lambda(a,c)f(z))''}{(1-\gamma)I^\lambda(a,c)f(z)+\gamma z(I^\lambda(a,c)f(z))'}-1\right|,$$
$$\left.\left(0\le\gamma\le 1,\ -1\le\alpha<1,\ \beta\ge 0,\ a,c\in R\backslash Z_0^-,\ \lambda>-1,\ z\in U\right)\right\}, \tag{14}$$

where $I^\lambda(a,c)$ is a Cho-Kwon-Srivastava operator [6], defined by

$$I^\lambda(a,c)f(z)=z+\sum_{k=2}^{\infty}\frac{(c)_{k-1}}{(a)_{k-1}}\frac{(\lambda+1)_{k-1}}{(1)_{k-1}}a_k z^k;$$

(vi)

$$S_\gamma(f, z+\sum_{k=2}^{\infty}\frac{(2)_{k-1}}{(n+1)_{k-1}}z^k;\alpha,\beta)=S_\gamma\left(n;\alpha,\beta\right)$$
$$=\left\{f\in A:\mathrm{Re}\left\{\frac{z(I_nf(z))'+\gamma z^2(I_nf(z))''}{(1-\gamma)I_nf(z)+\gamma z(I_nf(z))'}-\alpha\right\}\right.$$
$$>\beta\left|\frac{z(I_nf(z))'+\gamma z^2(I_nf(z))''}{(1-\gamma)I_nf(z)+\gamma z(I_nf(z))'}-1\right|,$$
$$\left.\left(0\le\gamma\le 1,\ -1\le\alpha<1,\ \beta\ge 0,\ n>-1,\ z\in U\right)\right\}, \tag{15}$$

where I_n is a Noor integral operator [17], defined by

$$I_nf(z)=z+\sum_{k=2}^{\infty}\frac{(2)_{k-1}}{(n+1)_{k-1}}a_k z^k\ (n>-1).$$

Definition 2 (Subordination Principle)**.** *For two functions f and Φ, analytic in U, we say that the function $f(z)$ is subordinate to $\Phi(z)$ in U, and write $f(z)\prec\Phi(z)$ $(z\in U)$, if there exists a Schwarz function $w(z)$, which (by definition) is analytic in U with $w(0)=0$ and $|w(z)|<1$, such that $f(z)=\Phi(w(z))$ $(z\in U)$. Indeed it is known that*

$$f(z)\prec\Phi(z)\ (z\in U)\Rightarrow f(0)=\Phi(0)\ \text{and}\ f(U)\subset\Phi(U).$$

Furthermore, if the function Φ is univalent in U, then we have the following equivalence [13, p. 4]:

$$f(z)\prec\Phi(z)\ (z\in U)\Leftrightarrow f(0)=\Phi(0)\ \text{and}\ f(U)\subset\Phi(U).$$

Definition 3 (Subordination Factor Sequence). *A Sequence $\{c_k\}_{k=1}^{\infty}$ of complex numbers is said to be a subordinating factor sequence if, whenever $f(z)$ of the form (1) is analytic, univalent and convex in U, we have the Subordination given by*

$$\sum_{k=1}^{\infty} a_k c_k z^k \prec f(z) \; (z \in U; \; a_1 = 1). \tag{16}$$

2. Main Result

To prove our main result we need the following lemmas.

Lemma 1 ([25]). *The sequence $\{c_k\}_{k=1}^{\infty}$ is a subordinating factor sequence if and only if*

$$Re\left\{1 + 2\sum_{k=1}^{\infty} c_k z^k\right\} > 0 \; (z \in U).$$

Now, we prove the following lemma which gives a sufficient condition for functions belonging to the class $S_\gamma(f, g; \alpha, \beta)$.

Lemma 2. *A function $f(z)$ of the form (1) is in $S_\gamma(f, g; \alpha, \beta)$ if*

$$\sum_{k=2}^{\infty} \left[k(1+\beta) - (\alpha+\beta)\right] \left[1 + \gamma(k-1)\right] \left|a_k\right| b_k \leq 1 - \alpha, \tag{17}$$

where $-1 \leq \alpha < 1$, $\beta \geq 0$, $0 \leq \gamma \leq 1$ and $b_k \geq b_2$ $(k \geq 2)$.

Proof. It suffices to show that

$$\beta \left| \frac{z(f*g)^{'}(z) + \gamma z^2 (f*g)^{''}(z)}{(1-\gamma)(f*g)(z) + \gamma z (f*g)^{'}(z)} - 1 \right| - \text{Re}\left\{ \frac{z(f*g)^{'}(z) + \gamma z^2 (f*g)^{''}(z)}{(1-\gamma)(f*g)(z) + \gamma z (f*g)^{'}(z)} - 1 \right\} \leq 1 - \alpha.$$

We have

$$\begin{aligned} &\beta \left| \frac{z(f*g)^{'}(z) + \gamma z^2 (f*g)^{''}(z)}{(1-\gamma)(f*g)(z) + \gamma z (f*g)^{'}(z)} - 1 \right| \\ &-\text{Re}\left\{ \frac{z(f*g)^{'}(z) + \gamma z^2 (f*g)^{''}(z)}{(1-\gamma)(f*g)(z) + \gamma z (f*g)^{'}(z)} - 1 \right\} \\ &\leq (1+\beta) \left| \frac{z(f*g)^{'}(z) + \gamma z^2 (f*g)^{''}(z)}{(1-\gamma)(f*g)(z) + \gamma z (f*g)^{'}(z)} - 1 \right| \\ &\leq \frac{(1+\beta)\sum\limits_{k=2}^{\infty}(k-1)\left[1+\gamma(k-1)\right]\left|a_k\right| b_k}{1 - \sum\limits_{k=2}^{\infty}\left[1+\gamma(k-1)\right]\left|a_k\right| b_k}. \end{aligned}$$

This last expression is bounded above by $(1-\alpha)$ if

$$\sum_{k=2}^{\infty}\left[k(1+\beta)-(\alpha+\beta)\right]\left[1+\gamma(k-1)\right]\left|a_k\right| b_k \leq 1-\alpha,$$

and hence the proof is completed.

Let $S_{\gamma}^{*}(f,g;\alpha,\beta)$ denote the class of $f(z)\in A$ whose coefficients satisfy the condition (17). We note that $S_{\gamma}^{*}(f,g;\alpha,\beta)\subseteq S_{\gamma}(f,g;\alpha,\beta)$.

Employing the technique used earlier by Attiya [3] and Srivastava and Attiya [23], we prove:

Theorem 1. *Let $f(z)\in S_{\gamma}^{*}(f,g;\alpha,\beta)$. Then*

$$\frac{(2+\beta-\alpha)(1+\gamma)b_2}{2\left[(2+\beta-\alpha)(1+\gamma)b_2+(1-\alpha)\right]}(f*h)(z)\prec h(z)\ (z\in U), \tag{18}$$

for every function h in K, and

$$Re(f(z))>-\frac{\left[(2+\beta-\alpha)(1+\gamma)b_2+(1-\alpha)\right]}{(2+\beta-\alpha)(1+\gamma)b_2},\ (z\in U). \tag{19}$$

The constant factor $\frac{(2+\beta-\alpha)(1+\gamma)b_2}{2\left[(2+\beta-\alpha)(1+\gamma)b_2+(1-\alpha)\right]}$ in the subordination result (18) cannot be replaced by a larger one.

Proof. Let $f(z)\in S_{\gamma}^{*}(f,g;\alpha,\beta)$ and let $h(z)=z+\sum\limits_{k=2}^{\infty}c_kz^k\in K$. Then we have

$$\frac{(2+\beta-\alpha)(1+\gamma)b_2}{2\left[(2+\beta-\alpha)(1+\gamma)b_2+(1-\alpha)\right]}(f*h)(z)=$$
$$\frac{(2+\beta-\alpha)(1+\gamma)b_2}{2\left[(2+\beta-\alpha)(1+\gamma)b_2+(1-\alpha)\right]}\left(z+\sum_{k=2}^{\infty}a_kc_kz^k\right). \tag{20}$$

Thus, by Definition 3, the subordination result (18) will hold true if the sequence

$$\left\{\frac{(2+\beta-\alpha)(1+\gamma)b_2}{2\left[(2+\beta-\alpha)(1+\gamma)b_2+(1-\alpha)\right]}a_k\right\}_{k=1}^{\infty} \tag{21}$$

is a subordinating factor sequence, with $a_1=1$. In view of Lemma 1, this is equivalent to the following inequality:

$$\mathrm{Re}\left\{1+\sum_{k=1}^{\infty}\frac{(2+\beta-\alpha)(1+\gamma)b_2}{\left[(2+\beta-\alpha)(1+\gamma)b_2+(1-\alpha)\right]}a_kz^k\right\}>0\ (z\in U). \tag{22}$$

Now, since

$$\Psi(k)=\left[k(1+\beta)-(\alpha+\beta)\right]\left[1+\gamma(k-1)\right]b_k$$

is an increasing function of k $(k \geq 2)$, we have

$$\begin{aligned}
& \operatorname{Re}\left\{1+\sum_{k=1}^{\infty}\frac{(2+\beta-\alpha)(1+\gamma)b_2}{\left[(2+\beta-\alpha)(1+\gamma)b_2+(1-\alpha)\right]}a_k z^k\right\} \\
= {} & \operatorname{Re}\left\{1+\frac{(2+\beta-\alpha)(1+\gamma)b_2}{\left[(2+\beta-\alpha)(1+\gamma)b_2+(1-\alpha)\right]}z+\right. \\
& \left.\frac{1}{\left[(2+\beta-\alpha)(1+\gamma)b_2+(1-\alpha)\right]}\sum_{k=2}^{\infty}(2+\beta-\alpha)(1+\gamma)b_2 a_k z^k\right\} \\
\geq {} & 1-\frac{(2+\beta-\alpha)(1+\gamma)b_2}{\left[(2+\beta-\alpha)(1+\gamma)b_2+(1-\alpha)\right]}r- \\
& \frac{1}{\left[(2+\beta-\alpha)(1+\gamma)b_2+(1-\alpha)\right]}\sum_{k=2}^{\infty}\left[k(1+\beta)-(\alpha+\beta)\right]\left[1+\gamma(k-1)\right]b_k\left|a_k\right|r^k \\
> {} & 1-\frac{(2+\beta-\alpha)(1+\gamma)b_2}{\left[(2+\beta-\alpha)(1+\gamma)b_2+(1-\alpha)\right]}r-\frac{(1-\alpha)}{\left[(2+\beta-\alpha)(1+\gamma)b_2+(1-\alpha)\right]}r \\
= {} & 1-r>0 \ (|z|=r<1),
\end{aligned}$$

where we have also made use of assertion (17) of Lemma 2. Thus (22) holds true in U. this proves the inequality (18). the inequality (19) follows from (18) by taking the convex function $h(z)=\frac{z}{1-z}=z+\sum_{k=2}^{\infty}z^k$. To prove the sharpness of the constant $\frac{(2+\beta-\alpha)(1+\gamma)b_2}{2\left[(2+\beta-\alpha)(1+\gamma)b_2+(1-\alpha)\right]}$, we consider the function $f_0(z)\in S_\gamma^*(f,g;\alpha,\beta)$ given by

$$f_0(z)=z-\frac{1-\alpha}{(2+\beta-\alpha)(1+\gamma)b_2}z^2. \tag{23}$$

Thus from (18), we have

$$\frac{(2+\beta-\alpha)(1+\gamma)b_2}{2\left[(2+\beta-\alpha)(1+\gamma)b_2+(1-\alpha)\right]}f_0(z)\prec\frac{z}{1-z}\ (z\in U). \tag{24}$$

Moreover, it can easily be verified for the function $f_0(z)$ given by (23) that

$$\min_{|z|\leq r}\left\{\operatorname{Re}\frac{(2+\beta-\alpha)(1+\gamma)b_2}{2\left[(2+\beta-\alpha)(1+\gamma)b_2+(1-\alpha)\right]}f_0(z)\right\}=-\frac{1}{2}. \tag{25}$$

This shows that the constant $\frac{(2+\beta-\alpha)(1+\gamma)b_2}{2\left[(2+\beta-\alpha)(1+\gamma)b_2+(1-\alpha)\right]}$ is the best possible.

Remark 1.

(i) Taking $g(z)=z+\sum_{k=2}^{\infty}\frac{(a)_{k-1}}{(c)_{k-1}}z^k$ $(c\neq 0,-1,-2,\ldots)$ *and* $\gamma=0$ *in Theorem 1, we obtain the result obtained by Frasin [8, Theorem 2.1];*

(ii) *Taking $g(z)=\frac{z}{1-z}$ and $\gamma=0$ in Theorem 1, we obtain the result obtained by Frasin [8, Corollary 2.2];*

(iii) *Taking $g(z)=\frac{z}{1-z}$ and $\beta=\gamma=0$ in Theorem 1, we obtain the result obtained by Frasin [8, Corollary 2.3];*

(iv) *Taking $g(z)=\frac{z}{1-z}$ and $\alpha=\beta=\gamma=0$ in Theorem 1, we obtain the result obtained by Frasin [8, Corollary 2.4];*

(v) *Taking $g(z)=\frac{z}{(1-z)^2}$ and $\gamma=0$ in Theorem 1, we obtain the result obtained by Frasin [8, Corollary 2.5];*

(vi) *Taking $g(z)=\frac{z}{(1-z)^2}$ and $\beta=\gamma=0$ in Theorem 1, we obtain the result obtained by Frasin [8, Corollary 2.6];*

(vii) *Taking $g(z)=\frac{z}{(1-z)^2}$ and $\alpha=\beta=\gamma=0$ in Theorem 1, we obtain the result obtained by Frasin [8, Corollary 2.7];*

(viii) *Taking $g(z)=z+\sum_{k=2}^{\infty}\mu_k z^k$ and $\gamma=0$ in Theorem 1, we obtain the result obtained by Raina and Bansal [18, Theorem 5.2]. Also, we establish subordination results for the associated sub classes, $S_p^*(\alpha)$, $UCV^*(\alpha)$, $UCV^*(\beta)$, $S^*\left(n,\alpha,\beta\right)$, $S^*\left(\alpha,\beta,\lambda\right)$, $S_\lambda^*\left(n,\alpha,\beta\right)$, $S_q^{s,*}(\gamma,\alpha,\beta)$, $S^*\left(\gamma,\alpha,\beta\right)$, $S_\gamma^*\left(n,\alpha,\beta\right)$, $S_\gamma^*\left(c,\alpha,\beta\right)$, $S_\gamma^*\left(\mu,\lambda;\alpha,\beta\right)$, $S_\gamma^*\left(a,c,\lambda;\alpha,\beta\right)$, $S_\gamma^*\left(n;\alpha,\beta\right)$, whose coefficients satisfy the (17) in the special cases as mentioned in p. 905 - 907.*

Putting $g(z)=\frac{z}{(1-z)}$, $\gamma=0$ and $\beta=1$ in Theorem 1, we have

Corollary 1. *Let the function $f(z)$ defined by (1) be in the class $S_p^*(\alpha)$ and suppose that $h(z)\in K$. Then*

$$\frac{3-\alpha}{2(4-2\alpha)}(f*h)(z)\prec h(z)\ (z\in U) \tag{26}$$

and

$$Re(f(z))>-\frac{4-2\alpha}{3-\alpha}\ (z\in U). \tag{27}$$

The constant factor $\frac{3-\alpha}{2(4-2\alpha)}$ in the subordination result (26) cannot be replaced by a larger one.

Putting $g(z)=\frac{z}{(1-z)^2}$, $\gamma=0$ and $\beta=1$ in Theorem 1, we have

Corollary 2. *Let the function $f(z)$ defined by (1) be in the class $UCV^*(\alpha)$ and suppose that $h(z)\in K$. Then*

$$\frac{3-\alpha}{7-3\alpha}(f*h)(z)\prec h(z)\ (z\in U) \tag{28}$$

and

$$Re(f(z))>-\frac{7-3\alpha}{2(3-\alpha)}\ (z\in U). \tag{29}$$

The constant factor $\frac{3-\alpha}{7-3\alpha}$ in the subordination result (28) cannot be replaced by a larger one.

Putting $g(z) = \frac{z}{(1-z)}$, $\gamma = 1$ and $\alpha = 0$ in Theorem 1, we have

Corollary 3. *Let the function $f(z)$ defined by (1) be in the class $UCV^*(\beta)$ and suppose that $h(z) \in K$. Then*

$$\frac{2+\beta}{5+2\beta}(f*h)(z) \prec h(z) \ (z \in U) \tag{30}$$

and

$$Re(f(z)) > -\frac{5+2\beta}{2(2+\beta)} \ (z \in U). \tag{31}$$

The constant factor $\frac{2+\beta}{5+2\beta}$ in the subordination result (30) cannot be replaced by a larger one.

Putting $g(z) = z + \sum_{k=2}^{\infty} k^n z^k$ $(n \in N_0)$ and $\gamma = 0$ in Theorem 1, we have

Corollary 4. *Let the function $f(z)$ defined by (1) be in the class $S^*\left(n, \alpha, \beta\right)$ and suppose that $h(z) \in K$. Then*

$$\frac{2^n(2-\alpha+\beta)}{2\left[2^n(2-\alpha+\beta)+(1-\alpha)\right]}(f*h)(z) \prec h(z) \ (z \in U) \tag{32}$$

and

$$Re(f(z)) > -\frac{\left[2^n(2-\alpha+\beta)+(1-\alpha)\right]}{2^n(2-\alpha+\beta)} \ (z \in U). \tag{33}$$

The constant factor $\frac{2^n(2-\alpha+\beta)}{2\left[2^n(2-\alpha+\beta)+(1-\alpha)\right]}$ in the subordination result (32) cannot be replaced by a larger one.

Putting $g(z) = z + \sum_{k=2}^{\infty} \binom{k+\lambda-1}{\lambda} z^k$ $(\lambda > -1)$ and $\gamma = 0$ in Theorem 1, we have

Corollary 5. *Let the function $f(z)$ defined by (1) be in the class $S^*\left(\alpha, \beta, \lambda\right)$ and suppose that $h(z) \in K$. Then*

$$\frac{(2-\alpha+\beta)(1+\lambda)}{2\left[2\lambda+3-\alpha(\lambda+2)+(1+\lambda)\beta\right]}(f*h)(z) \prec h(z) \ (z \in U) \tag{34}$$

and

$$Re(f(z)) > -\frac{\left[2\lambda+3-\alpha(\lambda+2)+(1+\lambda)\beta\right]}{(2-\alpha+\beta)(1+\lambda)} \ (z \in U). \tag{35}$$

The constant factor $\frac{(2-\alpha+\beta)(1+\lambda)}{2\left[2\lambda+3-\alpha(\lambda+2)+(1+\lambda)\beta\right]}$ in the subordination result (34) cannot be replaced by a larger one.

Putting $g(z) = z + \sum_{k=2}^{\infty} \left[1+\lambda(k-1)\right]^n z^k$ $(\lambda \geq 0, n \in N_0)$ and $\gamma = 0$ in Theorem 1, we have

Corollary 6. *Let the function $f(z)$ defined by (1) be in the class $S_{\lambda}^{*}\left(n,\alpha,\beta\right)$ and suppose that $h(z)\in K$. Then*

$$\frac{(2-\alpha+\beta)(1+\lambda)^{n}}{2\left[(2-\alpha+\beta)(1+\lambda)^{n}+(1-\alpha)\right]}(f*h)(z)\prec h(z)\ (z\in U) \tag{36}$$

and

$$Re(f(z))>-\frac{\left[(2-\alpha+\beta)(1+\lambda)^{n}+(1-\alpha)\right]}{(2-\alpha+\beta)(1+\lambda)^{n}}\ (z\in U). \tag{37}$$

The constant factor $\frac{(2-\alpha+\beta)(1+\lambda)^{n}}{2\left[(2-\alpha+\beta)(1+\lambda)^{n}+(1-\alpha)\right]}$ in the subordination result (36) cannot be replaced by a larger one.

Putting $g(z)=z+\sum\limits_{k=2}^{\infty}\Gamma_{k}z^{k}$ where Γ_{k} is defined by (9) in Theorem 1, we have

Corollary 7. *Let the function $f(z)$ defined by (1) be in the class $S_{q}^{s,*}(\gamma,\alpha,\beta)$ and suppose that $h(z)\in K$. Then*

$$\frac{(2-\alpha+\beta)(1+\gamma)\Gamma_{2}}{2\left[(2-\alpha+\beta)(1+\gamma)\Gamma_{2}+(1-\alpha)\right]}(f*h)(z)\prec h(z)\ (z\in U) \tag{38}$$

where Γ_{2} defined by (8), and

$$Re(f(z))>-\frac{\left[(2-\alpha+\beta)(1+\gamma)\Gamma_{2}+(1-\alpha)\right]}{(2-\alpha+\beta)(1+\gamma)\Gamma_{2}}\ (z\in U). \tag{39}$$

The constant factor $\frac{(2-\alpha+\beta)(1+\gamma)\Gamma_{2}}{2\left[(2-\alpha+\beta)(1+\gamma)\Gamma_{2}+(1-\alpha)\right]}$ in the subordination result (38) cannot be replaced by a larger one.

Putting $g(z)=z+\sum\limits_{k=2}^{\infty}\frac{(a)_{k-1}}{(c)_{k-1}}z^{k}$ $(c\neq 0,-1,-2,\ldots)$ in Theorem 1, we have

Corollary 8. *Let the function $f(z)$ defined by (1) be in the class $S^{*}\left(\gamma,\alpha,\beta\right)$ and suppose that $h(z)\in K$. Then*

$$\frac{(2-\alpha+\beta)(1+\gamma)a}{2\left[(2-\alpha+\beta)(1+\gamma)a+(1-\alpha)c\right]}(f*h)(z)\prec h(z)\ (z\in U) \tag{40}$$

and

$$Re(f(z))>-\frac{\left[(2-\alpha+\beta)(1+\gamma)a+(1-\alpha)c\right]}{(2-\alpha+\beta)(1+\gamma)a}\ (z\in U). \tag{41}$$

The constant factor $\frac{(2-\alpha+\beta)(1+\gamma)a}{2\left[(2-\alpha+\beta)(1+\gamma)a+(1-\alpha)c\right]}$ in the subordination result (40) cannot be replaced by a larger one.

Putting $g(z)=z+\sum\limits_{k=2}^{\infty}k^{n}z^{k}$, $(n\in N_{0})$ in Theorem 1, we have

Corollary 9. *Let the function $f(z)$ defined by (1) be in the class $S^*_\gamma\left(n,\alpha,\beta\right)$ and suppose that $h(z)\in K$. Then*

$$\frac{2^n(2-\alpha+\beta)(1+\gamma)}{2\left[2^n(2-\alpha+\beta)(1+\gamma)+(1-\alpha)\right]}(f*h)(z)\prec h(z)\ (z\in U) \tag{42}$$

and

$$Re(f(z))>-\frac{\left[2^n(2-\alpha+\beta)(1+\gamma)+(1-\alpha)\right]}{2^n(2-\alpha+\beta)(1+\gamma)}\ (z\in U). \tag{43}$$

The constant factor $\frac{2^n(2-\alpha+\beta)(1+\gamma)}{2\left[2^n(2-\alpha+\beta)(1+\gamma)+(1-\alpha)\right]}$ *in the subordination result (42) cannot be replaced by a larger one.*

Putting $g(z)=z+\sum\limits_{k=2}^{\infty}\left(\frac{c+1}{c+k}\right)z^k\ (c>-1)$ in Theorem 1, we have

Corollary 10. *Let the function $f(z)$ defined by (1) be in the class $S^*_\gamma\left(c,\alpha,\beta\right)$ and suppose that $h(z)\in K$. Then*

$$\frac{(2-\alpha+\beta)(1+\gamma)(c+1)}{2\left[(2-\alpha+\beta)(1+\gamma)(c+1)+(1-\alpha)(c+2)\right]}(f*h)(z)\prec h(z)\ (z\in U) \tag{44}$$

and

$$Re(f(z))>-\frac{\left[(2-\alpha+\beta)(1+\gamma)(c+1)+(1-\alpha)(c+2)\right]}{(2-\alpha+\beta)(1+\gamma)(c+1)}\ (z\in U). \tag{45}$$

The constant factor $\frac{(2-\alpha+\beta)(1+\gamma)(c+1)}{2\left[(2-\alpha+\beta)(1+\gamma)(c+1)+(1-\alpha)(c+2)\right]}$ *in the subordination result (44) cannot be replaced by a larger one.*

Putting $g(z)=z+\sum\limits_{k=2}^{\infty}\frac{(\mu)_{k-1}}{(\lambda+1)_{k-1}}z^k\ (\lambda>-1,\ \mu>0)$ in Theorem 1, we have

Corollary 11. *Let the function $f(z)$ defined by (1) be in the class $S^*_\gamma\left(\mu,\lambda;\alpha,\beta\right)$ and suppose that $h(z)\in K$. Then*

$$\frac{(2-\alpha+\beta)(1+\gamma)\mu}{2\left[(2-\alpha+\beta)(1+\gamma)\mu+(1-\alpha)(\lambda+1)\right]}(f*h)(z)\prec h(z)\ (z\in U) \tag{46}$$

and

$$Re(f(z))>-\frac{\left[(2-\alpha+\beta)(1+\gamma)\mu+(1-\alpha)(\lambda+1)\right]}{(2-\alpha+\beta)(1+\gamma)\mu}\ (z\in U). \tag{47}$$

The constant factor $\frac{(2-\alpha+\beta)(1+\gamma)\mu}{2\left[(2-\alpha+\beta)(1+\gamma)\mu+(1-\alpha)(\lambda+1)\right]}$ *in the subordination result (46) cannot be replaced by a larger one.*

Putting $g(z)=z+\sum\limits_{k=2}^{\infty}\frac{(c)_{k-1}}{(a)_{k-1}}\frac{(\lambda+1)_{k-1}}{(1)_{k-1}}z^k\ (a,c\in R\backslash Z_0^-,\ \lambda>-1)$ in Theorem 1, we have

Corollary 12. *Let the function $f(z)$ defined by (1) be in the class $S_{\gamma}^{*}\left(a,c,\lambda;\alpha,\beta\right)$ and suppose that $h(z)\in K$. Then*

$$\frac{(2-\alpha+\beta)(1+\gamma)(\lambda+1)c}{2\left[(2-\alpha+\beta)(1+\gamma)(\lambda+1)c+(1-\alpha)a\right]}(f*h)(z)\prec h(z)\ (z\in U) \tag{48}$$

and

$$Re(f(z))>-\frac{\left[(2-\alpha+\beta)(1+\gamma)(\lambda+1)c+(1-\alpha)a\right]}{(2-\alpha+\beta)(1+\gamma)(\lambda+1)c}\ (z\in U). \tag{49}$$

The constant factor $\frac{(2-\alpha+\beta)(1+\gamma)(\lambda+1)c}{2\left[(2-\alpha+\beta)(1+\gamma)(\lambda+1)c+(1-\alpha)a\right]}$ in the subordination result (48) cannot be replaced by a larger one.

Putting $g(z)=z+\sum_{k=2}^{\infty}\frac{(2)_{k-1}}{(n+1)_{k-1}}z^{k}$ $(n>-1)$ in Theorem 1, we have

Corollary 13. *Let the function $f(z)$ defined by (1) be in the class $S_{\gamma}^{*}\left(n;\alpha,\beta\right)$ and suppose that $h(z)\in K$. Then*

$$\frac{(2-\alpha+\beta)(1+\gamma)}{\left[2(2-\alpha+\beta)(1+\gamma)+(1-\alpha)(n+1)\right]}(f*h)(z)\prec h(z)\ (z\in U) \tag{50}$$

and

$$Re(f(z))>-\frac{\left[2(2-\alpha+\beta)(1+\gamma)+(1-\alpha)(n+1)\right]}{2(2-\alpha+\beta)(1+\gamma)}\ (z\in U). \tag{51}$$

The constant factor $\frac{(2-\alpha+\beta)(1+\gamma)}{\left[2(2-\alpha+\beta)(1+\gamma)+(1-\alpha)(n+1)\right]}$ in the subordination result (50) cannot be replaced by a larger one.

References

[1] O. Ahuja, G. Murugusundaramoorthy and N. Magesh, Integral means for uniformly convex and starlike functions associated with generalized hypergeometric functions, J. Inequal. Pure Appl. Math. 8, no. 4, Art. 118, 1-9. 2007.

[2] M. K. Aouf and A. O. Mostafa, Some properties of a subclass of uniformly convex functions with negative coefficients, Demonstration Math. 2, 353-370. 2008.

[3] A. A. Attiya, On some application of a subordination theorems, J. Math. Anal. Appl. 311, 489-494. 2005.

[4] S. D. Bernardi, Convex and starlike univalent functions, Trans. Amer. Math. Soc., 135, 429-446. 1969.

[5] R. Bharati, R. Parvatham and A. Swaminathan, On subclasses of uniformly convex functions and corresponding class of starlike functions, Tamakang J. Math. 28, 17-32. 1997.

[6] N. E. Cho, O. S. Kwon and H. M. Srivastava, Inclusion relationships and argument properties for certain subclasses of multivalent functions associated with a family of linear operators, J. Math. Anal. Appl. 292, 470-483. 2004.

[7] J. H. Choi, M. Saigo, H. M. Srivastava, Some inclusion properties of a certain family of integral operators, J. Math. Anal. Appl. 276, 432-445. 2002.

[8] B. A. Frasin, Subordination results for a class of analytic functions defined by a linear operator, J. Inequal. Pure Appl. Math. 7, no. 4, Art. 134, 1-7. 2006.

[9] A. W. Goodman, On uniformly convex functions, Ann. Polon. Math. 56, 87-92. 1991.

[10] A. W. Goodman, On uniformly starlike functions, J. Math. Anal. Appl. 155, 364-370. 1991.

[11] R. J. Libera, Some classes of regular univalent functions, Proc. Amer. Math. Soc. 16, 755-758. 1965.

[12] A. E. Livingston, On the radius of univalence of certain analytic functions, Proc. Amer. Math. Soc. 17, 352-357. 1966.

[13] S. S. Miller and P. T. Mocanu, Differenatial Subordinations: theory and Applications, Series on Monographs and Textbooks in Pure and Appl. Math. No. 255 Marcel Dekker, Inc., New York, 2000.

[14] G. Murugusundaramoorthy and N. Magesh, A new subclass of uniformly convex functions and corresponding subclass of starlike functions with fixed second coefficient, J. Inequal. Pure Appl. Math. 5, no. 4, Art. 85, 1-10.2004.

[15] G. Murugusundaramoorthy and N. Magesh, Linear operators associated with a subclass of uniformly convex functions, Internat. J. Pure Appl. Math. Sci. 3, no. 2, 113-125. 2006.

[16] G. Murugusundaramoorthy, T. Rosy and K. Muthunagai, Carlson-Shaffer operator and their applications to certain subclass of uniformly convex function, General Math. 15, no. 4, 131-143. 2007.

[17] K. I. Noor, On new classes of integral operators, J. Natur. Geem. 16, 71-80. 1999.

[18] R. K. Raina and Deepak Bansal, Some properties of a new class of analytic functions defined in terms of a hadamard product, J. Inequal. Pure Appl. Math. 9, no. 1, Art. 22, 1-9. 2008.

[19] F. Ronning, On starlike functions associated with parabolic regions, Ann. Univ. Mariae-Curie-Sklodowska, Sect. A 45, 117-122. 1991.

[20] F. Ronning, Uinformly convex functions and a corresponding class of starlike functions, Proc. Amer. Math. Soc. 118, 189-196. 1993.

[21] T. Rosy and G. Murugusundaramoorthy, Fractional calculus and their applications to certain subclass of uniformly convex functions, Far East J. Math. Sci. (FJMS), 15, no. 2, 231-242. 2004.

[22] St. Ruscheweyh, New criteria for univalent functions, Proc. Amer. Math. Soc. 49, 109–115. 1975.

[23] H. M. Srivastava and A. A. Attiya, Some subordination results associated with certain subclass of analytic functions, J. Inequal. Pure Appl. Math. 5, no. 4, Art. 82, 1-6. 2004.

[24] K. G. Subramanian, G. Murugusundaramoorthy, P. Balasubrahma and H. Silverman, Subclasses of uniformly convex and uniformly starlike functions, Math. Japon. 42, no. 3, 517-522. 1995.

[25] H. S. Wilf, Subordinating factor sequence for convex maps of the unit circle, Proc. Amer. Math. Soc. 12, 689-693. 1961.