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On the Number of Restricted One-to-One and Onto Functions Having Integral Coordinates

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Abstract. Let N_m be the set of positive integers $1, 2, , \cdots, m$ and $S \subseteq N_m$. In 2000, J. Caumeran and R. Corcino made a thorough investigation on counting restricted functions $f_{|S}$ under each of the following conditions:

(a) $f(a) \leq a$, $\forall a \in S$;

(b) $f(a) \leq g(a)$, $\forall a \in S$ where g is any nonnegative real-valued continuous functions;

(c) $g_1(a) \leq f(a) \leq g_2(a)$, $\forall a \in S$, where g_1 and g_2 are any nonnegative real-valued continuous functions.

Several formulae and identities were also obtain by Caumeran using basic concepts in combinatorics. In this paper we count those restricted functions under condition $f(a) \leq a$, $\forall a \in S$ which is one-to-one and onto and establish some formulas and identities parallel to those obtained by J. Caumeran and R. Corcino.

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1. Introduction

Cantor [12] is the first to consider the study of counting functions when he attempted to give meaning to power of cardinal numbers. Cantor obtained that the number of possible functions from an m-set to an n-set is equal to n^m in which $(n)_m = n(n-1)(n-2)\dots(n-m+1)$ of these are one-to-one functions.Stirling number of the first and second kind was first introduced by James Stirling published in 1730 in his book Methodes Differentials. The Stirling numbers of the second kind $S(n,m)$ count the number of ways of partitioning a set containing n elements into m nonempty subsets. By making use of the classical Stirling numbers of the second kind $S(n,k)$, it is shown that the number of onto functions is $n!S(m,n)$ (see [3]). The Stirling numbers of the second kind satisfy the following recurrence relations and explicit formula:

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(i) $S(n,k) = S(n-1,k-1) + kS(n-1,k), n,k \geq 1.$
$S(n,0) = S(0,k) = 0$ except $S(0,0) = 1.n,k \geq 1.$

(ii) $$S(n,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}(k-j)^n$$
$$= \frac{1}{k!}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}(i)^n.$$

Using these identities, we can easily construct the following table of values of $S(n,k)$:

$n \backslash k$	0	1	2	3	4	5	6
0	0						
1	0	1					
2	0	1	1				
3	0	1	3	1			
4	0	1	7	6	1		
5	0	1	15	25	10	1	
6	0	1	31	90	65	15	1

Table 1: Values of $S(n,k)$ for $0 \leq n \leq 6$

The Stirling numbers of the second kind has been generalized by introducing two parameters r and β. These generalized numbers are referred to as $(r,\beta)-$ Stirling numbers, denoted by $S_{r,\beta}(n,k)$. They were introduced by R. Corcino [6] as coefficient of the explicit formula:

$$S_{r,\beta}(n,k) = \frac{1}{\beta k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(\beta j + r)^n.$$

In [2], it was obtained that the number of restricted functions $f|_S : N_m \longrightarrow N_n$ for all $S \subseteq N_m$ where $N_m = \{1,2,\ldots,m\}$ is equal to $(n+1)^m$.

R. Corcino et al. [7] established some formulas in counting restricted functions $f|_S : N_m \longrightarrow N$, $S \subseteq N_m$ under each of the following conditions:

(i) $f(a) \leq a$, $\forall a \in S$;

(ii) $f(a) \leq g(a)$, $\forall a \in S$ where g is any nonnegative real-valued continuous functions;

(iii) $g_1(a) \leq f(a) \leq g_2(a)$, $\forall a \in S$, where g_1 and g_2 are any nonnegative real-valued continuous functions.

In this paper, we count those restricted functions considered by Caumeran [2] under condition (i) which is one-to-one and also onto. It is known that the number of one-to-one functions that can be formed from A to B where $|A| = n$ and $|B| = m$ is equal to

$$m(m-1)(m-2)\cdots(m-n+1).$$

On the other hand, the number of onto functions $f|_S : N_n \longrightarrow N_m$ that can be formed is $m! \cdot S(n, m)$, where $N_n = \{1, 2, 3, \cdots, n\}$ and $S(n, m)$, denotes the Stirling numbers of the second kind satisfying the relation

$$x^n = \sum_{i=0}^{m} S(n,i)(x)_i,$$

where $(x)_i = x(x-1(x-2)\cdots(x-i+1)$. It is observed that the process of counting onto functions makes use of the multiplication principle and the appropriate application of Stirling numbers of the second kind. The process of obtaining one-to-one and onto functions may be applicable in counting restricted one-to-one and onto functions. Recall that, for a finite sets A and B, a function $f : A \longrightarrow B$ is said to be onto if $f(A) = B$. Hence, in order for the function f to be onto, $|A|$ must be greater than or equal to $|B|$.

2. Number of Restricted One-to-One Functions

Let S_i be a subset of N_m and $|S_i| = i$. The number of restricted one-to-one functions $f|_S : N_m \longrightarrow N_n$ for all $S \subseteq N_m$ is

$$n(n-1)(n-2)\ldots(n-(i-1)) = (n)_i.$$

Let $\hat{\mathfrak{I}}_m = \bigcup_{i=0}^{m} \hat{\mathfrak{I}}_{i,m}$. Then

$$\hat{\mathfrak{I}}_m = \bigcup_{S_i \subseteq N_m} \{f|_{S_i} : f \text{ is a one-to-one function}\}.$$

The number of subsets of N_m containing i elements is $\binom{m}{i}$ and

$$|\hat{\mathfrak{I}}_m| = \sum_{i=0}^{m} |\hat{\mathfrak{I}}_{i,m}^{(n)}| = \sum_{i=0}^{m} \left| \bigcup_{S_i \subseteq N_m} \{f|_{S_i} : f \text{ is a one-to-one function}\} \right|$$

implying that

$$|\hat{\mathfrak{I}}_m| = \sum_{i=0}^{m} \binom{m}{i} (n)_i.$$

To state this result formally, we have the following proposition.

Proposition 1. *Let $f|_S : N_m \longrightarrow N_n$ such that $m \leq n$. If $\hat{\mathfrak{I}}_m = \bigcup_{i=0}^{m} \hat{Y}_{i,m}$ where $\hat{\mathfrak{I}}_m = \{f|_{S_i} : S_i \subseteq N_m$ and f is a one-to-one function$\}$, then*

$$|\hat{\mathfrak{I}}_m| = \sum_{i=0}^{m} \binom{m}{i} (n)_i.$$

Example 1. If $m = 3$ and $n = 4$ we have $N_3 = 1, 2, 3$ and $N_4 = 1, 2, 3, 4$.
For $i = 0$, $S_0 = \{\}$, $f|_{S_0} = \{\}$ is the only one-to-one function.
For $i = 1$, $S_i = \{1\}, \{2\}, \{3\}$, the one-to-one functions are

$$\begin{array}{cccc} \{(1,1)\} & \{(1,2)\} & \{(1,3)\} & \{(1,4)\} \\ \{(2,1)\} & \{(2,2)\} & \{(2,3)\} & \{(2,4)\} \\ \{(3,1)\} & \{(3,2)\} & \{(3,3)\} & \{(3,4)\} \end{array}$$

For $i = 2$, $S_i = \{1, 2\}, \{1, 3\}, \{2, 3\}$, the one-to-one functions are

$$\begin{array}{cccc} \{(1,1),(2,2)\} & \{(1,2),(2,1)\} & \{(1,3),(2,1)\} & \{(1,4),(2,1)\} \\ \{(1,1),(2,3)\} & \{(1,2),(2,3)\} & \{(1,3),(2,2)\} & \{(1,4),(2,2)\} \\ \{(1,1),(2,4)\} & \{(1,2),(2,4)\} & \{(1,3),(2,4)\} & \{(1,4),(2,3)\} \\ \{(1,1),(3,2)\} & \{(1,2),(3,1)\} & \{(1,3),(3,1)\} & \{(1,4),(3,1)\} \\ \{(1,1),(3,3)\} & \{(1,2),(3,3)\} & \{(1,3),(3,2)\} & \{(1,4),(3,2)\} \\ \{(1,1),(3,4)\} & \{(1,2),(3,4)\} & \{(1,3),(3,4)\} & \{(1,4),(3,3)\} \\ \{(2,1),(3,2)\} & \{(2,2),(3,1)\} & \{(2,3),(3,1)\} & \{(2,4),(3,1)\} \\ \{(2,1),(3,3)\} & \{(2,2),(3,3)\} & \{(2,3),(3,2)\} & \{(2,4),(3,2)\} \\ \{(2,1),(3,4)\} & \{(2,2),(3,4)\} & \{(3,3),(3,4)\} & \{(2,4),(3,3)\} \end{array}$$

For $i = 3$, $S_i = \{1, 2, 3\}, \{1, 3\}, \{2, 3\}$, the one-to-one functions are

$$\begin{array}{cccc} \{(1,1),(2,2),(3,3)\} & \{(1,2),(2,1),(3,3)\} & \{(1,3),(2,1),(3,2)\} & \{(1,4),(2,1),(3,2)\} \\ \{(1,1),(2,2),(3,4)\} & \{(1,2),(2,1),(3,4)\} & \{(1,3),(2,1),(3,4)\} & \{(1,4),(2,1),(3,3)\} \\ \{(1,1),(2,3),(3,4)\} & \{(1,2),(2,3),(3,1)\} & \{(1,3),(2,2),(3,1)\} & \{(1,4),(2,2),(3,1)\} \\ \{(1,1),(2,3),(3,2)\} & \{(1,2),(2,3),(3,4)\} & \{(1,3),(2,2),(3,4)\} & \{(1,4),(2,2),(3,3)\} \\ \{(1,1),(2,4),(3,2)\} & \{(1,2),(2,4),(3,3)\} & \{(1,3),(2,4),(3,1)\} & \{(1,4),(2,3),(3,1)\} \\ \{(1,1),(2,4),(3,3)\} & \{(1,2),(2,4),(3,4)\} & \{(1,3),(2,4),(3,2)\} & \{(1,4),(2,3),(3,2)\} \end{array}$$

Thus, the total number of restricted one-to-one function is 73. Using Proposition 1, with $m = 3$ and $n = 4$, we have

$$\begin{aligned} |\hat{\mathfrak{I}}_3| = \sum_{i=0}^{3} \binom{3}{i} (4)_i &= \binom{3}{0}(4)_0 + \binom{3}{1}(4)_1 + \binom{3}{2}(4)_2 + \binom{3}{3}(4)_3 \\ &= 1 + 3(4) + 3(4)(3) + 1(4)(3)(2) \\ &= 73. \end{aligned}$$

The next proposition counts the number of restricted one-to-one functions f with the condition that $f(a) \leq a, \forall a \in S$.

Proposition 2. *Let* $f|_S : N_m \longrightarrow N_n$ *such that* $m \leq n$ *and* $f(a) \leq a, \forall a \in S$. *If* $\hat{Y}_{(i,m)} = \left| \bigcup \{f|_{S_i} : f \textit{ is one to one and } |S_i| = i\} \right|$,*then*

$$|\hat{Y}(i,m)| = \sum_{1 \leq j_1 < j_2 < \ldots < j_i \leq m} \prod_{k=1}^{i} (j_k - k + 1).$$

Proof. Let $f : N_m \longrightarrow N_n$ such that $m \leq n$ and $f(a) \leq a, \forall a \in N_m$. Consider $S_i \subseteq N_m$, say $S_i = \{j_1, j_2, j_3, \cdots, j_i\}$, such that $j_1 \leq j_2 \leq j_3, \cdots, j_i$. To form a restricted one-to-one function, $f|_{S_i}$, consider the following sequence of events

$$\begin{array}{c} E_1 \text{ be an event of mapping } j_1 \text{ to } N_m \text{ such that } f(j_1) \leq j_1. \\ E_2 \text{ be an event of mapping } j_2 \text{ to } N_m \text{ such that } f(j_2) \leq j_2. \\ \vdots \\ E_i \text{ be an event of mapping } j_i \text{ to } N_m \text{ such that } f(j_i) \leq j_i. \end{array}$$

As $f|_{S_i}$ is one-to-one,

$$\big|E_1\big| = j_1, \big|E_2\big| = j_2 - 1, \big|E_3\big| = j_3 - 2, \cdots, \big|E_i\big| = j_i - i - 1.$$

By Multiplication Principle(MP), the number of restricted one-to-one functions, $f|_{S_i}$ such that $f(j_i) \leq j_i$ is

$$\begin{aligned} \prod_{t=1}^{i} |E_i| &= j_1(j_2 - 1)(j_3 - 2) \cdots (j_i - (i - 1)) \\ &= \prod_{k=1}^{i} (j_k - (k - 1)). \end{aligned}$$

Then

$$\begin{aligned} |\hat{Y}(i, m)| &= \Big|\bigcup \{f|_{S_i} : f \text{ is one to one and } |S_i| = i\}\Big| \\ &= \sum_{1 \leq j_1 < j_2 < \ldots < j_i \leq m} \big|\{f|_{S_i} : f \text{ is one to one}\}\big| \\ &= \sum_{1 \leq j_1 < j_2 < \ldots < j_i \leq m} \prod_{k=1}^{i} (j_k - (k - 1)). \end{aligned}$$

Example 2. If $i = 2, m = 3, S_2 = \{1, 2\}, \{1, 3\}, \{2, 3\}, f|_{S_2} : S_2 \longrightarrow N_3$ such that $f(a) \leq a, \forall a \in S_2$. The one-to-one functions are

$$\begin{array}{lll} \{(1,1),(2,2)\} & \{(2,1),(3,2)\} & \{(2,2),(3,2)\} \\ \{(1,1),(3,2)\} & \{(2,1),(3,3)\} & \{(2,2),(3,3)\} \\ \{(1,1),(3,3)\} & & \end{array}$$

The number of restricted one-to-one functions from S_2 to N_3 is 7.

Using Proposition 2, with $i = 2, m = 3$,

$$\begin{aligned} \hat{Y}(i, m) &= \sum_{1 \leq j_1 \leq j_2 \leq 3} j_1(j_2 - 1)) \\ &= 1(2 - 1) + 1(3 - 1) + 2(3 - 1) = 7. \end{aligned}$$

2.1. A Recurrence Relation of the Number $\hat{Y}(i,m)$

For quick computation of the first values of $\hat{Y}(i,m)$, the following recurrence relation will be useful

Proposition 3. *The following recurrence relation holds:*

$$\hat{Y}(i,m+1) = \hat{Y}(i,m) + (m+2-i)\hat{Y}(i-1,m)$$

with initial conditions $\hat{Y}(0,0) = 1, \hat{Y}(i,m) = 0$ *with* $i > m$ *and* $\hat{Y}(i,m) = 0$ *when* $i > 0$.

Proof. We know that $\hat{Y}(i,m+1)$ counts the number of restricted one-to-one functions $f|_{S_i}$ overall $S_i \subseteq N_{m+1}$. Forming such restricted one-to-one functions can also be done by considering the following disjoint cases:
Case 1. Forming those functions $f|_{S_i}$ overall $S_i \subseteq N_{m+1}$ such that $m+1 \notin S_i$. Then the number of such restricted one-to-one functions is equal to the number of restricted one-to-one functions $f|_{S_i}$ overall $S_i \subseteq N_{m+1}$. By definition, there are $\hat{Y}(i,m)$ such functions.
Case 2. Forming those functions $f|_{S_i}$ overall $S_i \subseteq N_{m+1}$ such that $m+1 \notin S_i$. This event can be decomposed into the following sequence of events:

E_1 : Event of forming those restricted one-to-one functions $f|_{S_{i-1}}$ overall $S_{i-1} \subseteq N_m$.
E_2 : Event of inserting $m+1$ to S_{i-1} so that every $S_i = S_{i-1} \cup \{m+1\}$ contains $m+1$ and then mapping $m+1$ to N_{m+1} so that one-to-oneness of f will be preserved.

Note that $|E_1| = \hat{Y}(i-1,m)$ and $|E_2| = m+1-(i-1)$. By Multiplication Principle, the number of such restricted one-to-one functions $f|_{S_i} = f|_{S_{i-1}\cup\{m+1\}}$ overall $S_i \subseteq N_{m+1}$ is equal to

$$|E_1||E_2| = \hat{Y}(i-1,m)(m+2-i).$$

Since any of these cases gives the desired restricted one-to-one functions, by Addition Principle,

$$\hat{Y}(i,m+1) = \hat{Y}(i,m) + (m+2-i)\hat{Y}(i-1,m).$$

Example 3. From Example 2, $\hat{Y}(2,3) = 7$ and using Proposition 3,

$$\hat{Y}(1,3) = \sum_{j_i=1}^{3} j_i = 1+2+3 = 6.$$

Then, by applying Proposition 3, with $i = 2, m = 3$, we have

$$\begin{aligned}\hat{Y}(2,4) &= \hat{Y}(2,3) + (3+2-2)\hat{Y}(1,3). \\ &= 7 + 3(6) = 25.\end{aligned}$$

Using Proposition 2, we have

$$\begin{aligned}\hat{Y}(2,4) &= \sum_{1\le j_1\le j_2\le 3} j_1(j_2-1)) \\ &= 1(2-1)+1(3-1)+1(4-1)+2(3-1)+2(4-1)+3(4-1) \\ &= 25.\end{aligned}$$

Note that

$$\hat{Y}(0,1) = \hat{Y}(0,1) + (0+2-0)\hat{Y}(-1,0) = 1$$
$$\hat{Y}(1,1) = \hat{Y}(1,0) + (0+2-1)\hat{Y}(0,0) = 1$$
$$\hat{Y}(1,2) = \hat{Y}(0,1) + (1+2-0)\hat{Y}(-1,1) = 1$$
$$\hat{Y}(1,2) = \hat{Y}(0,1) + (1+2-0)\hat{Y}(-1,1) = 1.$$

The following table of values for $\hat{Y}(i,m)$ can be constructed using Proposition 3.

m \ i	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	3	1				
3	1	6	7	1			
4	1	10	25	15	1		
5	1	15	65	90	31	1	
6	1	21	140	350	301	63	1

Table 2: Values of $\hat{Y}(i,m)$ for $0 \le i \le 6, 0 \le m \le 6$

Remark 1. We know from Proposition 1, that the total number of restricted one-to-one functions $f|_{S_i} : N_m \longrightarrow N_n, \forall S \subseteq N_m$ is

$$|\hat{\mathfrak{I}}_m| = \sum_{i=0}^{m} \binom{m}{i} (n)_i \tag{1}$$

and, from Proposition 2, the number of restricted one-to-one functions $f|_{S_i} : N_m \longrightarrow N_n, \forall S_i \subseteq N_m, |S_i| = i$ such that $f(a) \le a, \forall a \in N_m$ is

$$\hat{Y}(i,m) = \sum_{1\le j_1<j_2<\ldots<j_i\le m} \prod_{k=1}^{i} (j_k - k + 1) \tag{2}$$

Hence, the number of restricted one-to-one functions $f|_S, \forall S \subseteq N_m$ such that $f(a) \le a, \forall a \in N_m$ is

$$\tilde{Y}_m = \sum_{i=0}^{m} \hat{Y}(i,m). \tag{3}$$

The number of restricted one-to-one functions $f|_S, \forall S \subseteq N_m$ such that $f(a) \le a, \forall a \in N_m$ is

$$\tilde{Y}_m = |\hat{\mathfrak{I}}_m| - \tilde{Y}_m$$

$$= \sum_{i=0}^{m} \binom{m}{i} (n)_i - \sum_{i=0}^{m} \hat{Y}(i,m)$$
$$= \sum_{i=0}^{m} \left\{ \binom{m}{i} (n)_i - \hat{Y}(i,m) \right\}. \tag{4}$$

Geometrically, the integral points involved in the counting of one-to-one functions in (1) are those points bounded by $1 \leq x \leq m$ and $1 \leq y \leq n$ as shown in the Figure 1. The integral points involved in (2) and (3) are those points inside the region bounded by $1 \leq y \leq x$ and $1 \leq x \leq m$ and the integral points involved in (4) are those points bounded by $1 \leq x \leq m$ and $x + 1 \leq y \leq n$.

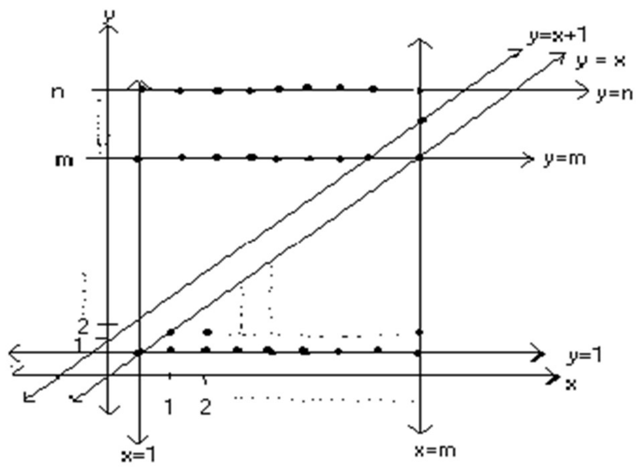

Figure 1: Graphs of $y = n, y = m, y = x, y = x + 1$

3. Number of Restricted Onto Functions

Consider a set $S_i \subseteq N_m, |S_i| = i, i \leq n$. To count the number of restricted onto functions $f|_{S_i}$, let us consider the following sequence of events:

E_1 : event of choosing a subset S_1 of N_m such that $|S_i| = i$.
E_2 : event of forming a restricted onto function $f|_{S_i} : N_m \longrightarrow N_n$.

Since $E_1 = \binom{m}{i}$ and $E_2 = n!S(i,n)$, by multiplication principle the number of restricted onto functions $f|_{S_i}$ over all $S_i \subseteq N_m$ with $|S_i| = i$ is

$$E_1 \cdot E_2 = \binom{m}{i} \cdot n! \cdot S(i,n).$$

This result will be stated formally in the following Proposition.

Proposition 4. *Let* $f|_{S_i} : N_m \longrightarrow N_n$ *such that* $m \leq n, i \leq n$. *If* $\dot{\mathfrak{I}}_{i,m}(n) = \bigcup_{S_i \subseteq N_m} \{f|_{S_i} : |S_i| = i \text{ and} f \text{ is onto}\}$, *then*

$$\left|\dot{\mathfrak{I}}_{i,m}(n)\right| = \binom{m}{i} n! S(i,n).$$

Example 4. If $i = 3, m = 4$, and $n = 2, N_4 = \{1,2,3,4\}$ and $N_2 = \{1,2\}$. $S_3 = \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}$. The possible onto functions

$\{(1,1),(2,1),(3,2)\}$	$\{(1,1),(3,2),(4,1)\}$	$\{(1,2),(2,2),(3,1)\}$	$\{(2,1),(3,1),(4,1)\}$
$\{(1,1),(2,1),(4,2)\}$	$\{(1,1),(3,2),(4,2)\}$	$\{(1,2),(2,2),(4,1)\}$	$\{(2,1),(3,1),(4,2)\}$
$\{(1,1),(2,2),(3,1)\}$	$\{(1,2),(2,1),(3,1)\}$	$\{(1,2),(2,2),(4,2)\}$	$\{(2,1),(3,2),(4,2)\}$
$\{(1,1),(2,2),(3,2)\}$	$\{(1,2),(2,1),(3,2)\}$	$\{(1,2),(3,1),(4,1)\}$	$\{(2,2),(3,1),(4,1)\}$
$\{(1,1),(2,2),(4,1)\}$	$\{(1,2),(2,1),(4,1)\}$	$\{(1,2),(3,1),(4,2)\}$	$\{(2,2),(3,2),(4,2)\}$
$\{(1,1),(2,2),(4,2)\}$	$\{(1,2),(2,1),(4,2)\}$	$\{(1,2),(3,2),(4,1)\}$	$\{(2,2),(3,2),(4,1)\}$

Then there are 24 such restricted onto functions. It can easily be verified using Proposition 6, with $i = 3, m = 4, n = 2$,

$$\begin{aligned} \left|\dot{\mathfrak{I}}_{3,4}(2)\right| &= \binom{4}{i} 2! S(3,2) \\ &= 4(2)(3) = 24, \end{aligned}$$

where the value of $S(3,2)$ is taken from Table 1.

The total number of restricted onto functions $f|_S$ over all $S \subseteq N_m$ is given in the following Proposition.

Proposition 5. *If* $\dot{\mathfrak{I}}_m(n) = \bigcup_{i=1}^m \dot{\mathfrak{I}}_{i,m}(n)$ *where* $\dot{\mathfrak{I}}_{i,m}(n) = \{f|_{S_i} : |S_i| = i \text{ and} f \text{ is onto}\}$, *then*

$$\left|\dot{\mathfrak{I}}_m(n)\right| = \sum_{i=0}^m \binom{m}{i} n! S(i,n).$$

Proof. Let $\left|\dot{\mathfrak{I}}_m(n)\right| = \bigcup_{i=1}^m \dot{\mathfrak{I}}_{i,m}(n) = \sum_{i=0}^m \left|\dot{\mathfrak{I}}_{i,m}(n)\right|$. From Proposition 4, we have

$$\left|\dot{\mathfrak{I}}_m(n)\right| = \sum_{i=n}^m \binom{m}{i} n! S(i,n).$$

Since $S(i,n) = 0$ when $i = 0, 1, 2, \cdots, n-1$,

$$\left|\dot{\mathfrak{I}}_m(n)\right| = \sum_{i=0}^m \binom{m}{i} n! S(i,n).$$

Example 5. The total number of restricted onto functions $f|_S : N_4 \longrightarrow N_2$ is given by

$$\begin{aligned} \left|\dot{\mathfrak{I}}_4(2)\right| &= \sum_{i=0}^{4} \binom{4}{i} 2!S(i,2) \\ &= \binom{4}{0} 2!S(0,2) + \binom{4}{1} 2!S(1,2) + \binom{4}{2} 2!S(2,2) + \binom{4}{3} 2!S(3,2) + \binom{4}{4} 2!S(4,2) \\ &= 6(2)(1) + 4(2)(3) + 1(2)(7) = 50. \end{aligned}$$

3.1. Some Corollaries

Using the explicit formula of $S(i,n)$, we can rewrite the formula in Proposition 4, as follows:

Corollary 1. $\left|\dot{\mathfrak{I}}_{i,m}(n)\right| = \sum_{j=0}^{n} (-1)^{n-j} \binom{m}{i} \binom{n}{j} j^i..$

Proof. From Proposition 4,

$$\begin{aligned} \left|\dot{\mathfrak{I}}_{i,m}(n)\right| &= \binom{m}{i} n!S(i,n) \\ &= \binom{m}{i} n! \left\{ \frac{1}{n!} \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} j^i \right\} \\ &= \sum_{j=0}^{n} (-1)^{n-j} \binom{m}{i} \binom{n}{j} j^i. \ \square \end{aligned}$$

Consequently, using Corollary 1, the formula in Proposition 5 can also be written as follows:

Corollary 2. $\left|\dot{\mathfrak{I}}_m(n)\right| = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} (j+1)^m = S_{1,1}(m,n)$.

Proof. From Proposition 5,

$$\begin{aligned} \left|\dot{\mathfrak{I}}_m(n)\right| &= \sum_{i=0}^{m} \binom{m}{i} n!S(i,n) \\ &= \sum_{i=0}^{m} \sum_{j=0}^{n} \binom{m}{i} (-1)^{n-j} \binom{m}{i} \binom{n}{j} j^i \\ &= \sum_{j=0}^{n} (-1)^{n-j} \binom{m}{i} \left\{ \sum_{i=0}^{m} \binom{m}{i} j^i \right\} \\ &= \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} (j+1)^m \\ &= S_{1,1}(m,n) \ \square \end{aligned}$$

$S_{1,1}(m,n)$ is the $(r,\beta)-$ Stirling numbers with $r = 1$ and $\beta = 1$.

Remark 2. The formulas in Corollary 1 and 2 compute the values of $\left|\dot{\mathfrak{I}}_{i,m}(n)\right|$ and $\left|\dot{\mathfrak{I}}_{m}(n)\right|$, respectively, without using the values of the Stirling numbers of the second kind. In Example 4, $\left|\dot{\mathfrak{I}}_{3,4}(2)\right| = 24$. Using Corollary 1, we have.

$$\begin{aligned} \left|\dot{\mathfrak{I}}_{i,m}(n)\right| &= \sum_{j=0}^{2}(-1)^{2-j}\binom{4}{3}\binom{2}{j}j^3 \Big\} \\ &= 4(1)(0) - (4)(2)(1) + 4(1)(2^3) \\ &= 0 - 8 + 32 = 24. \end{aligned}$$

Also, in Example 5,

$$\begin{aligned} \left|\dot{\mathfrak{I}}_{4}(2)\right| &= \sum_{j=0}^{2}(-1)^{2-j}\binom{2}{j}(j+1)^4 \\ &= \binom{2}{0}1^4 - \binom{2}{1}2^4 + \binom{2}{2}3^4 \\ &= 1 - 32 + 81 = 50 = S_{1,1}(4,2). \end{aligned}$$

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References

[1] Bogart, Kenneth P. *Introductory Combinatorics.* Hardcourt Brace Company, 1990

[2] Caumeran, J.M.A., *Counting of Functions on a Formation of Points Which Have Integral Coordinates*, Masters Thesis, Mindanao State University-Marawi Campus, September 2000.

[3] Chen, C.C and Kho, K.M.: *Principles and Techniques in Combinatorics*, World Scientific Publishing Co. Pte. Ltd., 1992

[4] Comtet, Louie. *Advanced Combinatorics.* Reidel Publishing Company, Dordrecht. Holland, 1974

[5] Corcino, C.B., Corcino, R.B., Mező, I., Ramirez, J.L. *Some polynomials associated with the $r-$ Whitney numbers,* Proceedings-Mathematical Sciences 128 (3), Article 27.

[6] Corcino, R.B. (1999). The (r, β)-*Stirling numbers.* Mindanao Forum. **Vol. 14(2).**

[7] Corcino, R., Alquiza, M., Caumeran,J.M. C, *Counting Restricted Functions on the Formation of Points Having Integral Coordinates*, Mindanao Forum **19**(1) (2005), 9-18.

[8] Corcino, R.B. and Corcino, C.B. , *The Hankel Transform of Generalized Bell Numbers and Its q-Analogue*, Utilitas Mathematica, **89** (2012), 297-309.

[9] Corcino, R.B., Corcino, C.B., and Aldema, R. ,*Asymptotic Normality of the* (r, β)*-Stirling Numbers*, Ars Combinatorics, **81** (2006), 81-96.

[10] Mező, I. (2010). *A New Formula for the Bernoulli Polynomials.* Result Math. 58(3), 329-335.

[11] Rucinski, A. and B. Voigt (1990). *A Local Limit Theorem for Generalized Stirling Numbers.* Rev. Roumaine De Math. Pures Appl. **Vol. 35(2).** pp.161-172.

[12] Zehna, W.P and Johnson, R.L. *Elements of Set Theory*, Allyn and Bacon, Inc., United States of America, 1972.