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# $J^2$- Hop Domination in Graphs: Properties and Connections with other Parameters

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**Abstract.** A subset $T = \{v_1, v_2, \cdots, v_m\}$ of vertices of a graph $G$ is called a $J^2$-set if $N_G^2[v_i] \setminus N_G^2[v_j] \neq \varnothing$ for every $i \neq j$, where $i, j \in \{1, 2, \ldots, m\}$. A $J^2$-set $T$ is called a $J^2$-hop dominating in $G$ if for every $a \in V(G) \setminus T$, there exists $b \in T$ such that $d_G(a, b) = 2$. The $J^2$-hop domination number of $G$, denoted by $\gamma_{J^2h}(G)$, is the maximum cardinality among all $J^2$-hop dominating sets in $G$. In this paper, we initiate the study on $J^2$-hop domination and we establish its properties and connections with other known parameters in graph theory. We show that every maximum hop independent set is a $J^2$-hop dominating, hence, this parameter is greater than compare to the hop independence parameter on any graph. Moreover, we derive some lower and upper bounds of the parameter for a generalized graph, join and corona of two graphs, respectively. Finally, we obtain exact values of the parameter for some special graphs and shadow graph using the characterization results that are formulated in this study.

**2020 Mathematics Subject Classifications**: 05C69
**Key Words and Phrases**: $J^2$-set, $J^2$-hop dominating set, $J^2$-hop domination number

## 1. Introduction

Hop domination was introduced by Natarajan et al. in [9]. A subset $S$ of a vertices of a graph $G$ is called a hop dominating if for every $a \in V(G) \backslash S$, there exists $b \in S$ such that $d_G(a, b) = 2$. The minimum cardinality among all hop dominating sets of $G$, denoted by $\gamma_h(G)$, is called the hop domination number of $G$. This parameter had studied on some families of graphs and graphs obtained from some operations in [1, 2, 9]. Researchers in the field had further investigated this concept, and introduced new variants and obtained some significant results that contributed a lot to the hop domination theory (see [3–8, 10–12]).

In this paper, new parameter called $J^2$-hop domination in a graph will be introduced and investigated. We will establish its relationships with other known parameters in graph

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theory. Moreover, we will determine its bounds or exact values on some special graphs, shadow graph and join of two graphs. We believe that this parameter and its results would give additional insights to researchers in the field and would help them for more research directions in the future.

## 2. Terminology and Notation

A *path graph* is a non-empty graph with vertex-set $\{x_1, x_2, \dots, x_n\}$ and edge-set $\{x_1x_2, x_2x_3, \dots, x_{n-1}x_n\}$, where the $x_i^{'}s$ are all distinct. The path of order $n$ is denoted by $P_n$. If $G$ is a graph and $u$ and $v$ are vertices of $G$, then a path from vertex $u$ to vertex $v$ is sometimes called a *u-v path.* The *cycle graph* $C_n$ is the graph of order $n \geq 3$ with vertex-set $\{x_1, x_2, \dots, x_n\}$ and edge-set $\{x_1x_2, x_2x_3, \dots, x_{n-1}x_n, x_nx_1\}$.

Let $G = (V(G), E(G))$ be a simple and undirected graph. The *distance* $d_G(u, v)$ in $G$ of two vertices $u, v$ is the length of a shortest *u-v path* in $G$. The greatest distance between any two vertices in $G$, denoted by $diam(G)$, is called the *diameter* of $G$.

Two vertices $x, y$ of $G$ are *adjacent*, or *neighbors*, if $xy$ is an edge of $G$. The *open neighborhood* of $x$ in $G$ is the set $N_G(x) = \{y \in V(G) : xy \in E(G)\}$. The *closed neighborhood* of $x$ in $G$ is the set $N_G[x] = N_G(x) \cup \{x\}$. If $X \subseteq V(G)$, the *open neighborhood* of $X$ in $G$ is the set $N_G(X) = \bigcup_{x \in X} N_G(x)$. The *closed neighborhood* of $X$ in $G$ is the set $N_G[X] = N_G(X) \cup X$.

A vertex $a$ in $G$ is a *hop neighbor* of a vertex $b$ in $G$ if $d_G(a, b) = 2$. The set $N_G^2(a) = \{b \in V(G) : d_G(a, b) = 2\}$ is called the *open hop neighborhood* of $a$. The *closed hop neighborhood* of $a$ in $G$ is given by $N_G^2[a] = N_G^2(a) \cup \{a\}$. The *open hop neighborhood* of $S \subseteq V(G)$ is the set $N_G^2(S) = \bigcup_{a \in S} N_G^2(a)$. The *closed hop neighborhood* of $S$ in $G$ is the set $N_G^2[S] = N_G^2(S) \cup S$.

A subset $S$ of $V(G)$ is a *hop dominating* of $G$ if for every $a \in V(G) \backslash S$, there exists $b \in S$ such that $d_G(a, b) = 2$. The minimum cardinality among all hop dominating sets of $G$, denoted by $\gamma_h(G)$, is called the *hop domination number* of $G$. Any hop dominating set with cardinality equal to $\gamma_h(G)$ is called a *$\gamma_h$-set* of $G$.

A subset $S$ of $V(G)$ is called a *hop independent* if for every pair of distinct vertices $x, y \in S$, $d_G(x, y) \neq 2$. The maximum cardinality of a hop independent set in $G$, denoted by $\alpha_h(G)$, is called the *hop independence* number of $G$. Any hop independent set $S$ with cardinality equal to $\alpha_h(G)$ is called an $\alpha_h$-set of $G$.

Let $G$ and $H$ be any two graphs. The *join* of $G$ and $H$, denoted by $G + H$ is the graph with vertex set $V(G + H) = V(G) \cup V(H)$ and edge set

$$E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}.$$

The *corona* $G$ and $H$, denoted by $G \circ H$, the graph obtained by taking one copy of $G$ and $|V(G)|$ copies of $H$, and then joining the *ith* vertex of $G$ to every vertex of the *ith* copy of $H$. We denote by $H^v$ the copy of $H$ in $G \circ H$ corresponding to the vertex $v \in G$

and write $v + H^v$ for $\langle \{v\} + H^v \rangle$.

The *shadow graph* $S(G)$ of graph $G$ is constructed by taking two copies of $G$, say $G_1$ and $G_2$, and then joining each vertex $u \in V(G_1)$ to the neighbors of its corresponding vertex $u' \in V(G_2)$.

## 3. Results

We begin this section by introducing the concept of $J^2$-hop domination in a graph.

**Definition 1.** Let $G$ be an undirected graph and $m \in N$. A subset $T = \{v_1, v_2, \cdots, v_m\}$ of vertices of $G$ is called a $J^2$-set if $N^2_G[v_i] \setminus N^2_G[v_j] \neq \varnothing$ for every $i \neq j$, where $i, j \in \{1, 2, \ldots, m\}$. A $J^2$-set $T$ is called a $J^2$*-hop dominating* in $G$, if $T$ is a hop dominating set in $G$. The $J^2$*-hop domination number* of $G$, denoted by $\gamma_{J^2h}(G)$, is the maximum cardinality among all $J^2$-hop dominating sets in $G$. Any $J^2$-hop dominating set $T$ with $|T| = \gamma_{J^2h}(G)$ (resp. $|T| = \gamma_h(G)$), is called a $\gamma_{J^2h}$-set or the maximum (resp. minimum) $J^2$-hop dominating set of $G$.

**Example 1.** Consider the graph $G$ in Figure 1 and let $T = \{u_1, u_2, \ldots, u_6\}$. Notice that $u_i \in N^2_G[u_i] \setminus N^2_G[u_j]$ $\forall$ $i \neq j$ where $i, j \in \{1, 2, \ldots, 6\}$. Thus, $T$ is a $J^2$-set of $G$. Since $N^2_G[T] = V(G)$, it follows that $T$ is a $J^2$-hop dominating set of $G$. Observe that $N^2_G[u_7] \subseteq N^2_G[u_3]$, $N^2_G[u_8] \subseteq N^2_G[u_3]$, $N^2_G[u_9] \subseteq N^2_G[u_1]$, and $N^2_G[u_{10}] \subseteq N^2_G[u_3]$. Thus, $T$ is a maximum $J^2$-hop dominating set of $G$. Hence, $\gamma_{J^2h}(G) = 6$.

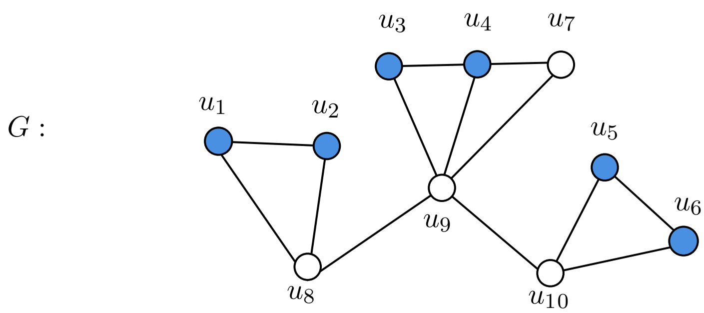


Figure 1: Graph $G$ with $\gamma_{J^2h}(G) = 6$

**Theorem 1.** *Let $G$ be any graph of order $m \geq 1$. Then each of the following holds:*

*(i)* *$N \subseteq V(G)$ is a $\gamma_h$-set in $G$ if and only if $N$ is a minimum $J^2$- hop dominating set in $G$.*

*(ii)* $\gamma_h(G) \leq \gamma_{J^2h}(G)$.

*(iii)* $1 \leq \gamma_{J^2h}(G) \leq m$.

*Proof.* (i) Suppose that $N \subseteq V(G)$ is a $\gamma_h$-set in $G$. Then $N$ is a minimum hop dominating set in $G$. It suffices to show that $N$ is a $J^2$-set in $G$. Suppose on the contrary that $N$ is not a $J^2$-set in $G$. Then there exist $x, y \in N$ such that either $N_G^2[x] \setminus N_G^2[y] = \varnothing$ or $N_G^2[y] \setminus N_G^2[x] = \varnothing$. This means that either $N_G^2[x] \subseteq N_G^2[y]$ or $N_G^2[y] \subseteq N_G^2[x]$. If $N_G^2[x] \subseteq N_G^2[y]$, then $D' = N \setminus \{x\}$ is a hop dominating set in $G$, contradicting the minimality of $N$. Similarly, when $N_G^2[y] \subseteq N_G^2[x]$. Consequently, $N$ is a minimum $J^2$-hop dominating set of $G$.

Conversely, suppose that $N$ is a minimum $J^2$-hop dominating set of $G$. Then $|N| = \gamma_h(G)$ (by definition). It follows that $N$ is a $\gamma_h$-set in $G$.

(ii) Let $S$ be a $\gamma_h$-set of $G$. Then by (i), $S$ is a minimum $J^2$-hop dominating set in $G$. Since $\gamma_{J^2h}(G)$ is the maximum cardinality among all $J^2$-hop dominating sets in $G$, it follows that $\gamma_h(G) = |S| \leq \gamma_{J^2h}(G)$.

(iii) Since $\gamma_h(G) \geq 1$ for any graph $G$ of order $m \geq 1$, we have $\gamma_{J^2h}(G) \geq 1$ by (ii). Since any $J^2$-hop dominating set $N$ of $G$ is always a subset of $V(G)$, it follows that $\gamma_{J^2h}(G) \leq |V(G)| = m$. Therefore, $1 \leq \gamma_{J^2h}(G) \leq m$. □

**Theorem 2.** *Let $G$ be any graph. Then $N \subseteq V(G)$ is a maximum $J^2$-set if and only if $N$ is a $\gamma_{J^2h}$-set of $G$.*

*Proof.* Let $N$ be a maximum $J^2$-set of $G$. Assume that $N$ is not a hop dominating set in $G$. Then there exists $u \in V(G) \setminus N$ such that $u \notin N_G^2[N]$. This implies that $u \notin N_G^2[v]$ for every $v \in N$. Let $N' = \{u\} \cup N$. Since $N$ is a $J^2$-set in $G$ and $u \in N_G^2[u]$, it follows that $N_G^2[a] \setminus N_G^2[b] \neq \varnothing$ and $N_G^2[b] \setminus N_G^2[a] \neq \varnothing$ for every $a \neq b$, where $a, b \in N'$. This means that $N'$ is a $J^2$-set in $G$, contradicting the maximality of $N$. Hence, $N$ is a hop dominating set of $G$. Since $N$ is a maximum $J^2$-set of $G$, $N$ is a maximum $J^2$-hop dominating set of $G$, that is, $N$ is a $\gamma_{J^2h}$-set of $G$.

Conversely, suppose that $N$ is a $\gamma_{J^2h}$-set of $G$. Then $N$ is a maximum $J^2$-hop dominating set of $G$. Hence, the assertion follows. □

The following result follows from Theorem 2.

**Corollary 1.** *Let $G$ be a graph and let $N = \{x_1, x_2, \ldots, x_k\}$ be a $J^2$-set of $G$. Then*

$$|N| = k \leq \gamma_{J^2h}(G).$$

**Proposition 1.** *Given any positive integer $k \geq 1$, we have*

$$\gamma_{J^2h}(P_k) = \begin{cases} 1 \textit{ if } k = 1 \\ 2 \textit{ if } k = 2,3,4 \\ 3 \textit{ if } k = 5 \\ 4 \textit{ if } k = 6,7 \\ k-4 \textit{ if } k \geq 8 \end{cases}$$

*Proof.* Clearly, $\gamma_{J^2h}(P_1) = 1, \gamma_{J^2h}(P_k) = 2$ for $k = 2,3,4$, $\gamma_{J^2h}(P_5) = 3$ and $\gamma_{J^2h}(P_k) = 4$ for $k = 6,7$. Suppose that $k \geq 8$. Let $P_k = [a_1, a_2, \ldots, a_k]$ and let $S = \{a_3, a_4, \ldots, a_{k-3}, a_{k-2}\}$. Then $N^2_{P_k}[S] = V(P_k)$, showing that $S$ is a hop dominating set in $P_k$. Observe that $a_{i-2} \in N^2_{P_k}[a_i] \setminus N^2_{P_k}[a_j]$ and $a_{j+2} \in N^2_{P_k}[a_j] \setminus N^2_{P_k}[a_i]$ for all $j > i$, where $i,j \in \{3,4,\ldots,k-3,k-2\}$. Thus, $N^2_{P_k}[a_i] \setminus N^2_{P_k}[a_j] \neq \varnothing$ for all $i \neq j$, where $i,j \in \{3,4,\ldots,k-3,k-2\}$, that is, $S$ is a $J^2$-set in $P_k$. Therefore, $S$ is a $J^2$-hop dominating set in $P_k$. Since $N^2_{P_k}[a_1] \subseteq N^2_{P_k}[a_3]$, $N^2_{P_k}[a_2] \subseteq N^2_{P_k}[a_4]$, $N^2_{P_k}[a_k] \subseteq N^2_{P_k}[a_{k-2}]$, and $N^2_{P_k}[a_{k-1}] \subseteq N^2_{P_k}[a_{k-3}]$, it follows that $S$ is a maximum $J^2$-hop dominating set of $P_k$. Hence, $\gamma_{J^2h}(P_k) = k-4$ for all $k \geq 8$. □

**Theorem 3.** *Let $G$ be any graph of order $n$ and $N$ be any $J^2$-hop dominating set of $G$. Then each of the following holds:*

*(i)* $a \in N$ *if and only if* $N^2_G[a] \nsubseteq N^2_G[b]$ *and* $N^2_G[b] \nsubseteq N^2_G[a]$ $\forall$ $b \in N \setminus \{a\}$.

*(ii)* $\gamma_{J^2h}(G) = |V(G)| = n$ *if and only if* $N^2_G[v_i] \nsubseteq N^2_G[v_j]$ $\forall$ $i \neq j$ *where* $i,j \in \{1,2,\ldots,n\}$.

*(iii) If $G$ is $K_n$ or $\overline{K}_n$, then $\gamma_{J^2h}(G) = n$ for all $n \geq 1$.*

*Proof.* (i) Let $G$ be a graph and $N$ be a $J^2$-hop dominating set of $G$. Suppose that $a \in N$. Then $N^2_G[a] \setminus N^2_G[b] \neq \varnothing$ and $N^2_G[b] \setminus N^2_G[a] \neq \varnothing$ $\forall$ $b \in N \setminus \{a\}$. It follows that $N^2_G[a] \nsubseteq N^2_G[b]$ and $N^2_G[b] \nsubseteq N^2_G[a]$ $\forall$ $b \in N \setminus \{a\}$.

Conversely, suppose that $N^2_G[a] \nsubseteq N^2_G[b]$ and $N^2_G[b] \nsubseteq N^2_G[a]$ $\forall$ $b \in N \setminus \{a\}$. This means that $N^2_G[a] \setminus N^2_G[b] \neq \varnothing$ and $N^2_G[b] \setminus N^2_G[a] \neq \varnothing$ $\forall$ $b \in N \setminus \{a\}$. Hence, $a \in N$.

(ii) Suppose that $\gamma_{J^2h}(G) = |V(G)| = n$. Then $N = V(G) = \{v_1, v_2, \ldots, v_n\}$ is the $\gamma_{J^2h}$-set of $G$. Thus, $N^2_G[v_i] \setminus N^2_G[v_j] \neq \varnothing$ $\forall$ $i \neq j$, where $i,j \in \{1,2,\ldots,n\}$. It follows that $N^2_G[v_i] \nsubseteq N^2_G[v_j]$ $\forall$ $i \neq j$, where $i,j \in \{1,2,\ldots,n\}$.

Conversely, suppose that $N^2_G[v_i] \nsubseteq N^2_G[v_j]$ $\forall$ $i \neq j$, where $i,j \in \{1,2,\ldots,n\}$. Then $N^2_G[v_i] \setminus N^2_G[v_j] \neq \varnothing$ $\forall$ $i \neq j$, $i,j \in \{1,2,\ldots,n\}$. It follows that $v_i, v_j$ are in $J^2$-set $S$ of $G$ $\forall$ $i \neq j$, where $i,j \in \{1,2,\ldots,n\}$. Thus, $S = V(G)$. Consequently,

$$\gamma_{J^2h}(G) = |S| = |V(G)| = n.$$

(iii) Let $G = K_n$ and $V(G) = \{a_1, a_2, \ldots, a_n\}$. Then $\{a_i\} = N_G^2[a_i] \nsubseteq N_G^2[a_j] = \{a_j\}$ $\forall\ i \neq j$, where $i, j \in \{1, 2, \ldots, n\}$. Thus, by (ii), $\gamma_{J^2h}(G) = |V(G)| = n$. Similarly, if $G = \overline{K}_n$, then $\gamma_{J^2h}(G) = |V(G)| = n$.

□

**Theorem 4.** *Let $a, b$ be positive integers with $2 \leq a \leq b$. Then there exists a connected graph $G$ such that $\gamma_h(G) = a$ and $\gamma_{J^2h}(G) = b$. In other words, $\gamma_{J^2h}(G) - \gamma_h(G)$ can be made arbitrarily large.*

*Proof.* For $a = b$, consider $\overline{K}_a$. Then by Theorem 3, $\gamma_{J^2h}(\overline{K}_a) = a = \gamma_h(\overline{K}_a)$.

Suppose that $a < b$. Consider the following two cases:

**Case 1**: $a$ is odd.

Consider the graph $G$ in Figure 2. Let $m = b - a$ and let $S = \{x_1, x_2, \ldots, x_a\}$ and $S' = \{x_1, x_2, \ldots, x_{a-3}, u, v, x_a, c_1, c_2, \ldots, c_m\}$. Then $S$ and $S'$ are $\gamma_h$-set and $\gamma_{J^2h}$-set in $G$, respectively. Hence, $\gamma_h(G) = a$ and $\gamma_{J^2h}(G) = a + m = b$. Consequently, $\gamma_h(G) < \gamma_{J^2h}(G)$.

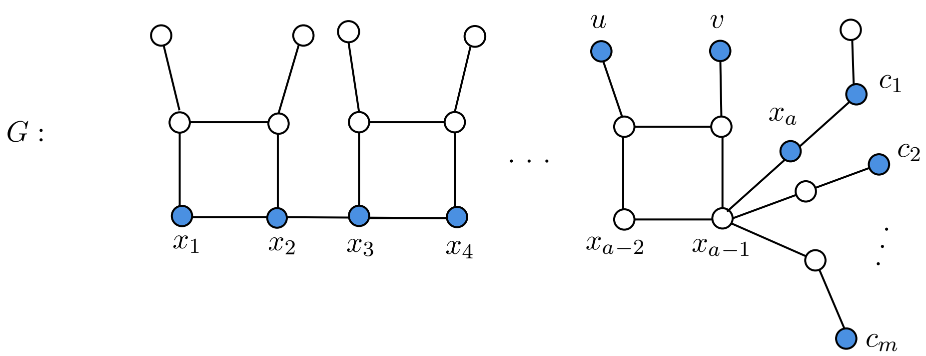


Figure 2: Graph $G$ with $\gamma_h(G) < \gamma_{J^2h}(G)$

**Case 2**: $a$ is even.

Consider the graph $H$ in Figure 3. Let $t = b - a$ and let $C = \{x_1, x_2, \ldots, x_a\}$ and $C' = \{x_1, x_2, \ldots, x_{a-2}, v, w, c_1, c_2, \ldots, c_t\}$. Then $C$ and $C'$ are $\gamma_h$-set and $\gamma_{J^2h}$-set in $H$, respectively. Therefore, $\gamma_h(H) = a$ and $\gamma_{J^2h}(H) = a + t = b$, showing that $\gamma_h(H) < \gamma_{J^2h}(H)$. □

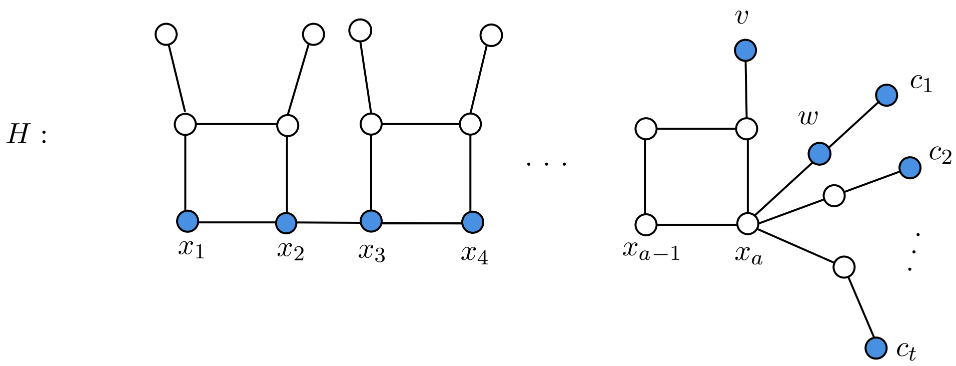


Figure 3: Graph $H$ with $\gamma_h(H) < \gamma_{J^2h}(H)$

**Theorem 5.** *Let $G$ be any graph and let $S \subseteq V(G)$. Then every hop independent set $S$ is a $J^2$-set in $G$. In particular, every $\alpha_h$-set is a $J^2$-hop dominating set. Moreover, $\alpha_h(G) \leq \gamma_{J^2h}(G)$.*

*Proof.* Let $S$ be a hop independent set in $G$. Then $d_G(a,b) \neq 2$ for every $a, b \in S$. Suppose on the contrary that $S$ is not a $J^2$-set in $G$. Then there exist $x, y \in S$ such that $N_G^2[x] \setminus N_G^2[y] = \varnothing$ or $N_G^2[y] \setminus N_G^2[x] = \varnothing$. It follows that $N_G^2[x] \subseteq N_G^2[y]$ or $N_G^2[y] \subseteq N_G^2[x]$. In either case, we have $d_G(x,y) = 2$, a contradiction to the fact that $S$ is a hop independent set in $G$. Therefore, $S$ is a $J^2$-set in $G$. Next, let $S'$ be an $\alpha_h$-set of $G$. Then $S'$ is a maximum hop independent set of $G$ (by definition). Thus, $S'$ is a $J^2$-set in $G$ by the first part. Now, suppose on the contrary that $S'$ is not a hop dominating set of $G$. Then there exists $x \in V(G) \setminus S'$ such that $x \notin N_G^2[y]$ $\forall$ $y \in S'$. This means that $d_G(x,y) \neq 2$ for all $y \in S'$. Thus, $S^* = \{x\} \cup S'$ is a hop independent set in $G$, contradicting the maximality of $S'$. Hence, $S'$ is a hop dominating set of $G$, showing that $S'$ is a $J^2$-hop dominating in $G$. Consequently, $\alpha_h(G) \leq \gamma_{J^2h}(G)$. □

**Theorem 6.** *Let $G$ and $H$ be two connected graphs. If $N = N_G \cup N_H \subseteq V(G+H)$, where $N_G$ and $N_H$ are $J^2$-sets in $G$ and $H$, respectively, then $N$ is a $J^2$-set in $G+H$.*

*Proof.* Let $a, b \in N$. Suppose that $a, b \in N_G$. If $d_G(a,b) = 1$, then $a \in N_{G+H}^2[a] \setminus N_{G+H}^2[b]$ and $b \in N_{G+H}^2[b] \setminus N_{G+H}^2[a]$. Since $a, b$ are arbitrary, the assertion follows. Assume that $d_G(a,b) = 2$. Since $N_G$ is a $J^2$- set in $G$, there exist $w, z \in V(G)$ such that $w \in N_G^2[a] \setminus N_G^2[b]$ and $z \in N_G^2[b] \setminus N_G^2[a]$. Let $s \in N_G(w) \cap N_G(a)$ and $t \in N_G(z) \cap N_G(b)$. Then $s \in N_{G+H}^2[b] \setminus N_{G+H}^2[a]$ and $t \in N_{G+H}^2[a] \setminus N_{G+H}^2[b]$. Since $a, b$ are arbitrary, $N$ is a $J^2$-set of $G+H$. Next, suppose that $d_G(a,b) \geq 3$. Let $u \in N_G(a)$ and $v \in N_G(b)$, then $u \in N_{G+H}^2[b] \setminus N_{G+H}^2[a]$ and $v \in N_{G+H}^2[a] \setminus N_{G+H}^2[b]$. Since $a, b$ are arbitrary, $N$ is a $J^2$-set of $G+H$. Similarly, if $a, b \in N_H$, then $N$ is a $J^2$-set of $G+H$. Next, suppose that $a \in N_G$ and $b \in N_H$. Then $a \in N_{G+H}^2[a] \setminus N_{G+H}^2[b]$ and $b \in N_{G+H}^2[b] \setminus N_{G+H}^2[a]$. Since $a, b$ are arbitrary, it follows that $N$ is a $J^2$-set of $G+H$. □

**Theorem 7.** *Let $G$ and $H$ be two connected graphs. If $N = N_G \cup N_H \subseteq V(G+H)$, where $N_G$ and $N_H$ are $J^2$-hop dominating sets in $G$ and $H$, respectively, then $N$ is a $J^2$-hop dominating set in $G+H$. Moreover, $\gamma_{J^2h}(G+H) \geq \gamma_{J^2h}(G) + \gamma_{J^2h}(H)$.*

*Proof.* Let $N = N_G \cup N_H$, where $N_G$ and $N_H$ are $J^2$-hop dominating sets in $G$ and $H$, respectively. Since $N_G$ and $N_H$ are $J^2$-sets in $G$ and $H$, respectively, it follows that $N$ is a $J^2$-set in $G+H$ by Theorem 6. Since $N_G$ and $N_H$ are hop dominating sets in $G$ and $H$, respectively, we have $N^2_G[N_G] = V(G)$ and $N^2_G[N_H] = V(H)$. Observe that $N^2_G[N_G] \subseteq N^2_{G+H}[N_G]$ and $N^2_H[N_H] \subseteq N^2_{G+H}[N_H]$. Thus,

$$N^2_{G+H}[N] = N^2_{G+H}[N_G \cup N_H] = V(G+H),$$

showing that $N$ is a hop dominating set in $G+H$. Therefore, $N$ is a $J^2$-hop dominating set in $G+H$.

Next, let $N' = N'_G \cup N'_H$, where $N'_G$ and $N'_H$ are $\gamma_{J^2h}$-sets in $G$ and $H$, respectively. Then by the first part, $N'$ is a $J^2$-hop dominating set in $G+H$. Consequently,

$$\gamma_{J^2h}(G+H) \geq |N'| = |N'_G| + |N'_H| = \gamma_{J^2h}(G) + \gamma_{J^2h}(H).$$

□

**Remark 1.** *The bound given in Theorem 7 is sharp. Moreover, strict inequality is attainable.*

For the sharpness, consider the join graph $P_3 + P_4$ in Figure 4. Let $S = \{a, b, e, f\}$. Then $N^2_{P_3+P_4}[S] = V(P_3 + P_4)$, showing that $S$ is a hop dominating set in $P_3 + P_4$. Observe that $x \in N^2_{P_3+P_4}[x] \setminus N^2_{P_3+P_4}[y]$ and $y \in N^2_{P_3+P_4}[y] \setminus N^2_{P_3+P_4}[x]$ for every $x \neq y$ where $x, y \in S$. This means that $N^2_{P_3+P_4}[x] \setminus N^2_{P_3+P_4}[y] \neq \varnothing$ and $N^2_{P_3+P_4}[y] \setminus N^2_{P_3+P_4}[x] \neq \varnothing$ for every $x \neq y$ where $x, y \in S$. Thus, $S$ is a $J^2$-hop dominating set in $P_3 + P_4$. Since $N^2_{P_3+P_4}[c] \subseteq N^2_{P_3+P_4}[a]$, $N^2_{P_3+P_4}[f] \subseteq N^2_{P_3+P_4}[d]$, and $N^2_{P_3+P_4}[e] \subseteq N^2_{P_3+P_4}[g]$, it follows that $S$ is a maximum $J^2$-hop dominating set of $P_3 + P_4$. Hence, $\gamma_{J^2h}(P_3 + P_5) = 4$ . By Proposition 1, $\gamma_{J^2h}(P_3) = 2$ and $\gamma_{J^2h}(P_4) = 2$. Consequently,

$$\gamma_{J^2h}(P_3 + P_4) = 4 = \gamma_{J^2h}(P_3) + \gamma_{J^2h}(P_4).$$

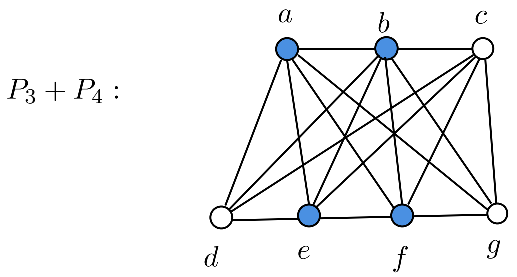


Figure 4: Graph $P_3 + P_4$ with $\gamma_{J^2h}(P_3 + P_4) = 4 = \gamma_{J^2h}(P_3) + \gamma_{J^2h}(P_4)$

For strict inequality, consider the graph $P_2+P_8$ in Figure 5. Let $S' = \{a, b, d, e, f, g, h, i\}$. Then $S'$ is a $\gamma_{J^2h}$-set in $P_2+P_8$. Thus, $\gamma_{J^2h}(P_2+P_8) = 8$. By Proposition 1, $\gamma_{J^2h}(P_2) = 2$ and $\gamma_{J^2h}(P_8) = 4$. Hence,

$$\gamma_{J^2h}(P_2 + P_8) = 8 > 6 = \gamma_{J^2h}(P_2) + \gamma_{J^2h}(P_8).$$

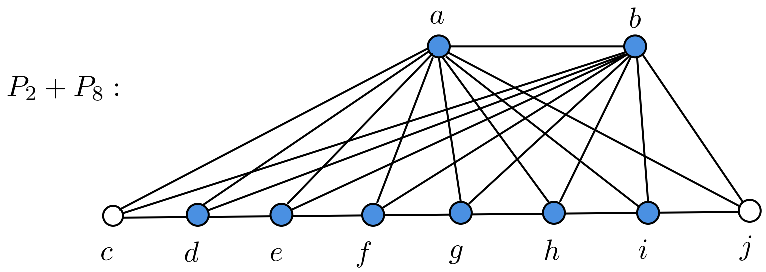


Figure 5: Graph $P_2 + P_8$ with $\gamma_{J^2h}(P_2 + P_8) > \gamma_{J^2h}(P_2) + \gamma_{J^2h}(P_8)$

**Theorem 8.** *Let $G$ be any non-trivial connected graph and $H$ be any connected graph. If $T = \bigcup_{v\in V(G)} T_v$, where $T_v$ is a maximum $J^2$-set in $H^v$ for each $v \in V(G)$, then $T$ is a $J^2$-hop dominating set in $G \circ H$. Moreover, $\gamma_{J^2h}(G \circ H) \geq |V(G)| \cdot \gamma_{J^2h}(H)$.*

*Proof.* Suppose that $T = \bigcup_{v\in V(G)} T_v$, where $T_v$ is a maximum $J^2$-set in $H^v$ for each $v \in V(G)$. Let $a, b \in T$. Suppose that $a, b \in T_u$ for some $u \in V(G)$. If $d_H(a, b) = 1$, then $a \in N^2_{G\circ H}[a] \setminus N^2_{G\circ H}[b]$ and $b \in N^2_{G\circ H}[b] \setminus N^2_{G\circ H}[a]$. It follows that $T$ is a $J^2$-set in $G \circ H$. Assume that $d_H(a, b) = 2$. Since $T_u$ is a $J^2$- set in $H^u$, there exist $w, z \in V(H^u)$ such that $w \in N^2_{H^u}[a]\setminus N^2_{H^u}[b]$ and $z \in N^2_{H^u}[b]\setminus N^2_{H^u}[a]$. Let $s \in N_{H^u}(w) \cap N_{H^u}(a)$ and $t \in N_{H^u}(z) \cap N_{H^u}(b)$. Then $s \in N^2_{G\circ H}[b]\setminus N^2_{G\circ H}[a]$ and $t \in N^2_{G\circ H}[a]\setminus N^2_{G\circ H}[b]$. Since $a, b$ are arbitrary, $T$ is a $J^2$-set of $G \circ H$. Next, suppose that $d_H(a, b) \geq 3$. Let $u \in N_H(a)$ and

$v \in N_H(b)$, then $u \in N^2_{G\circ H}[b] \setminus N^2_{G\circ H}[a]$ and $v \in N^2_{G\circ H}[a] \setminus N^2_{G\circ H}[b]$. Since $a, b$ are arbitrary, $T$ is a $J^2$-set of $G \circ H$. Next, assume that $a \in T_x$ and $b \in T_y$ for some $x, y \in V(G), x \neq y$. Then $a \in N^2_{G\circ H}[a] \setminus N^2_{G\circ H}[b]$ and $b \in N^2_{G\circ H}[b] \setminus N^2_{G\circ H}[a]$. Thus, $N^2_{G\circ H}[a] \setminus N^2_{G\circ H}[b] \neq \varnothing$ and $N^2_{G\circ H}[b] \setminus N^2_{G\circ H}[a] \neq \varnothing$. Since $a$ and $b$ are arbitrary, $T$ is a $J^2$-set in $G \circ H$. Now, since $T_v$ is a maximum $J^2$-set in $H^v$ for each $v \in V(G)$, it follows that $T_v$ is a maximum $J^2$-hop dominating set in $H^v$ for every $v \in V(G)$ by Theorem 2. Thus,

$$\bigcup_{v\in V(G)} V(H^v) \subseteq N^2_{G\circ H}[T].$$

Now, let $r \in V(G \circ H) \setminus \bigcup_{v\in V(G)} V(H^v)$. Then $r \in V(G)$. Since $G$ is a non-trivial connected graph, there exists $q \in T_s$ such that $d_{G\circ H}(r, q) = 2$ for some $s \in V(G)$. Hence, $N^2_{G\circ H}[T] = V(G \circ H)$, and so $T$ is a $J^2$-hop dominating set in $G \circ H$. Consequently, $\gamma_{J^2h}(G \circ H) \geq |V(G)| \cdot \gamma_{J^2h}(H)$. □

**Lemma 1.** [7] *Let $G$ be a non-trivial connected graph and let $G_1$ and $G_2$ be two copies of $G$ in the graph $S(G)$. If $w \in V(G_1)$ and $w' \in V(G_2)$ is the corresponding vertex of $w$, then*

$$N^2_{S(G)}[w] = N^2_{G_1}[w] \cup N^2_{G_2}[w'] = N^2_{S(G)}[w'].$$

**Lemma 2.** *Let $G$ be a non-trivial connected graph and let $G_1$ and $G_2$ be two copies of $G$ in the graph $S(G)$. If $N^2_{G_1}[a] \subseteq N^2_{G_1}[b]$ or $N^2_{G_2}[a] \subseteq N^2_{G_2}[b]$, then $N^2_{S(G)}[a] \subseteq N^2_{S(G)}[b]$.*

*Proof.* Let $a, b \in V(G_1)$ and suppose that $N^2_{G_1}[a] \subseteq N^2_{G_1}[b]$. Let $x \in N^2_{S(G)}[a]$. Then $d_{S(G)}(a, x) = 2$. If $x \in V(G_1)$, then $d_{G_1}(a, x) = 2$. So, $x \in N^2_{G_1}[a]$. Thus, by assumption, $x \in N^2_{G_1}[b]$. By Lemma 1, $x \in N^2_{S(G)}[b]$, and we are done. Suppose that $x \in V(G_2)$. Then $x \in N^2_{G_2}[a']$ for some $a' \in V(G_2)$. Since $N^2_{G_2}[a'] \subseteq N^2_{G_2}[b'] \subseteq N^2_{S(G)}[b]$, it follows that $x \in N^2_{S(G)}[b]$, and so $N^2_{S(G)}[a] \subseteq N^2_{S(G)}[b]$. Similarly, if $N^2_{G_2}[a] \subseteq N^2_{G_2}[b]$, then $N^2_{S(G)}[a] \subseteq N^2_{S(G)}[b]$. □

**Theorem 9.** *Let $G$ be a connected non-trivial graph. Then $T \subseteq V(S(G))$ is a $J^2$-set in $S(G)$ if and only if $T$ satisfies one of the following conditions:*

(*i*) *$T$ is a $J^2$-set in $G_1$.*

(*ii*) *$T$ is a $J^2$-set in $G_2$.*

(*iii*) *$T = T_{G_1} \cup T_{G_2}$, where $T_{G_1} \cup T'_{G_2}$ and $T'_{G_1} \cup T_{G_2}$ are $J^2$-sets in $G_1$ and $G_2$, respectively, where $T'_{G_2} = \{x \in V(G_1) : x' \in T_{G_2}\}$ and $T'_{G_1} = \{y \in V(G_2) : y' \in T_{G_1}\}$.*

*Proof.* Suppose that $T$ is a $J^2$-set in $S(G)$. Let $T_{G_1} = T \cap V(G_1)$ and $T_{G_2} = T \cap V(G_2)$. If $T_{G_2} = \varnothing$, then $T = T_{G_1}$ is a $J^2$-set in $G_1$. If $T_{G_1} = \varnothing$, then $T = T_{G_2}$ is a $J^2$-set in $G_2$, showing that (*i*) or (*ii*) holds. Assume that $T_{G_1} \neq \varnothing$ and $T_{G_2} \neq \varnothing$. Suppose on the contrary that $S = T_{G_1} \cup T'_{G_2}$ is not a $J^2$-set in $G_1$. Then there exist $a, b \in S$ such that $N^2_{G_1}[a] \setminus N^2_{G_1}[b] = \varnothing$ or $N^2_{G_1}[b] \setminus N^2_{G_1}[a] = \varnothing$. It follows that $N^2_{G_1}[a] \subseteq N^2_{G_1}[b]$ or $N^2_{G_1}[b] \subseteq N^2_{G_1}[a]$. If $a, b \in T_{G_1}$, then $a, b \in T$. Since $N^2_{G_1}[a] \subseteq N^2_{G_1}[b]$ or $N^2_{G_1}[b] \subseteq N^2_{G_1}[a]$, we have $N^2_{S(G)}[a] \subseteq N^2_{S(G)}[b]$ or $N^2_{S(G)}[b] \subseteq N^2_{S(G)}[a]$ by Lemma 2. Thus, $N^2_{S(G)}[a] \setminus N^2_{S(G)}[b] = \varnothing$ or $N^2_{S(G)}[b] \setminus N^2_{S(G)}[a] = \varnothing$, a contradiction to the fact that $T$ is a $J^2$-set in $S(G)$. Suppose that $a, b \in T'_{G_2}$. Then $a', b' \in T_{G_2} \subseteq T$. Since $N^2_{G_1}[a] \setminus N^2_{G_1}[b] = \varnothing$ or $N^2_{G_1}[b] \setminus N^2_{G_1}[a] = \varnothing$, it follows that $N^2_{G_2}[a'] \setminus N^2_{G_2}[b'] = \varnothing$ or $N^2_{G_2}[b'] \setminus N^2_{G_2}[a'] = \varnothing$. Thus, $N^2_{G_2}[a'] \subseteq N^2_{G_2}[b']$ or $N^2_{G_2}[b'] \subseteq N^2_{G_2}[a']$, and so

$$N^2_{S(G)}[a'] \subseteq N^2_{S(G)}[b'] \text{ or } N^2_{S(G)}[b'] \subseteq N^2_{S(G)}[a']$$

by Lemma 2, which is a contradiction. Now, suppose that $a \in T_{G_1}$ and $b \in T'_{G_2}$. Then $b' \in T_{G_2}$. Since $N^2_{G_1}[a] \subseteq N^2_{G_1}[b]$ or $N^2_{G_1}[b] \subseteq N^2_{G_1}[a]$, it follows that $N^2_{S(G)}[a] \subseteq N^2_{S(G)}[b']$ or $N^2_{S(G)}[b'] \subseteq N^2_{S(G)}[a]$ by Lemma 1 and Lemma 2, a contradiction. Thus, $S = T_{G_1} \cup T'_{G_2}$ is a $J^2$-set in $G_1$. Similarly, $T'_{G_1} \cup T_{G_2}$ is a $J^2$-set in $G_2$. Thus, (iii) holds.

Conversely, if (i) or (ii) holds, then the assertion follows. Assume that (iii) holds. Let $x, y \in T = T_{G_1} \cup T_{G_2}$. If $x, y \in T_{G_1} \subseteq T_{G_1} \cup T'_{G_2}$, then $N^2_{G_1}[x] \setminus N^2_{G_1}[y] \neq \varnothing$ and $N^2_{G_1}[y] \setminus N^2_{G_1}[x] \neq \varnothing$ by assumption. This means that $N^2_{G_1}[x] \nsubseteq N^2_{G_1}[y]$ and $N^2_{G_1}[y] \nsubseteq N^2_{G_1}[x]$. Thus, $N^2_{S(G)}[x] \nsubseteq N^2_{S(G)}[y]$ and $N^2_{S(G)}[y] \nsubseteq N^2_{S(G)}[x]$, and we are done. If $x, y \in T_{G_2}$, then $x', y' \in T'_{G_2} \subseteq T_{G_1} \cup T'_{G_2}$. Since $T_{G_1} \cup T'_{G_2}$ is a $J^2$-set in $G_1$, we have $N^2_{G_1}[x'] \nsubseteq N^2_{G_1}[y']$ and $N^2_{G_1}[y'] \nsubseteq N^2_{G_1}[x']$. Thus, by Lemma 1, $N^2_{S(G)}[x] \nsubseteq N^2_{S(G)}[y]$ and $N^2_{S(G)}[y] \nsubseteq N^2_{S(G)}[x]$. Now, assume that $x \in T_{G_1}$ and $y \in T_{G_2}$. Then $y' \in T'_{G_2}$, and so $x, y' \in T_{G_1} \cup T'_{G_2}$. Since $T_{G_1} \cup T'_{G_2}$ is a $J^2$-set in $G_1$, we have $N^2_{G_1}[x] \nsubseteq N^2_{G_1}[y']$ and $N^2_{G_1}[y'] \nsubseteq N^2_{G_1}[x]$. Thus, by Lemma 1, $N^2_{S(G)}[x] \nsubseteq N^2_{S(G)}[y]$ and $N^2_{S(G)}[y] \nsubseteq N^2_{S(G)}[x]$. Since $x, y$ are arbitrary, it follows that $T$ is a $J^2$-set in $S(G)$. □

**Theorem 10.** *[5] Let $G$ be a non-trivial connected graph. Then $S$ is a hop dominating set in $S(G)$ if and only if one of the following conditions holds:*

*(i) $S$ is a hop dominating set in $G_1$.*

*(ii) $S$ is a hop dominating set in $G_2$.*

*(iii) $S = S_{G_1} \cup S_{G_2}$ such that $S_{G_1} \cup S'_{G_2}$ and $S'_{G_1} \cup S_{G_2}$ are hop dominating sets in $G_1$ and $G_2$, respectively, where*

$$S'_{G_2} = \{a \in V(G_1) : a' \in S_{G_2}\} \text{ and } S'_{G_1} = \{b \in V(G_2) : b' \in S_{G_1}\}.$$

**Theorem 11.** *Let $G$ be a connected non-trivial graph. Then $T \subseteq V(S(G))$ is a $J^2$-hop dominating set in $S(G)$ if and only if $T$ satisfies one of the following conditions:*

*(i) $T$ is a $J^2$-hop dominating set in $G_1$.*

*(ii) $T$ is a $J^2$-hop dominating set in $G_2$.*

(*iii*) $T = T_G \cup T_H$, *where* $T_{G_1} \cup T'_{G_2}$ *and* $T'_{G_1} \cup T_{G_2}$ *are* $J^2$*-hop dominating sets in* $G_1$ *and* $G_2$*, respectively, where*

$$T'_{G_2} = \{x \in V(G_1) : x' \in T_{G_2}\} \text{ and } T'_{G_1} = \{y \in V(G_2) : y' \in T_{G_1}\}.$$

*Proof.* Suppose that $T$ is a $J^2$-hop dominating set in $S(G)$. Let $T_{G_1} = T \cap V(G_1)$ and $T_{G_2} = T \cap V(G_2)$. If $T_{G_2} = \varnothing$, then $T = S_{G_1}$ is a $J^2$-hop dominating set of $G_1$. If $T_{G_1} = \varnothing$, then $T = T_{G_2}$ is a $J^2$-hop dominating set of $G_2$, showing that ($i$) or ($ii$) holds. Now, since $T$ is a $J^2$-set in $S(G)$, $T_{G_1} \cup T'_{G_2}$ and $T'_{G_1} \cup T_{G_2}$ are $J^2$-sets in $G_1$ and $G_2$, respectively, by Theorem 9. Also, since $T$ is a hop dominating set in $S(G)$, $T_{G_1} \cup T'_{G_2}$ and $T'_{G_1} \cup T_{G_2}$ are hop dominating sets in $G_1$ and $G_2$, respectively, by Theorem 10. Consequently, $T_{G_1} \cup T'_{G_2}$ and $T'_{G_1} \cup T_{G_2}$ are $J^2$-hop dominating sets in $G_1$ and $G_2$, respectively.

For the converse, suppose ($i$) holds. Then $T$ is both a $J^2$-set and a hop dominating in $G_1$. Thus, by Theorem 9 and by Theorem 10, $T$ is a $J^2$-hop dominating set in $S(G)$. Similarly, if ($ii$) holds, then $T$ is a $J^2$-hop dominating set of $S(G)$. Suppose that (iii) holds. Then by Theorem 9 and Theorem 10, $T$ is a $J^2$-hop dominating set of $S(G)$. □

**Corollary 2.** *Let* $G$ *be a connected non-trivial graph. Then*

$$\gamma_{J^2h}(S(G)) = \gamma_{J^2h}(G).$$

*Proof.* Let $T$ be a $\gamma_{J^2h}$-set of $G$. Then by Theorem 11, $T$ is a $J^2$-hop dominating set of $S(G)$. Thus, $\gamma_{J^2h}(S(G)) \geq |T| = \gamma_{J^2h}(G)$. On the other hand, suppose $T^*$ is a $\gamma_{J^2h}$-set of $S(G)$. If $T^*$ is of type ($i$) or ($ii$), then $T^*$ is a $J^2$-hop dominating set of $G$ by Theorem 11($i$) and ($ii$). Hence, $\gamma_{J^2h}(S(G)) = |T^*| \leq \gamma_{J^2h}(G)$. Next, suppose $T^*$ is of type ($iii$), say $T^* = T_{G_1} \cup T_{G_2}$. Then $T^*_G = T_{G_1} \cup T'_{G_2}$ is a $J^2$-hop dominating set of $G_1$ by Theorem 11($iii$). This implies that $\gamma_{J^2h}(S(G)) = |T^*| = |T^*_G| \leq \gamma_{J^2h}(G)$. Consequently, $\gamma_{J^2h}(S(G)) = \gamma_{J^2h}(G)$. □

**Example 2.** Consider the shadow graph $S(C_4)$ of $C_4$ in Figure 6. Let $V(C_4) = \{a, b, c, d\}$ and let $N = \{a, b\}$. Then $N^2_{C_4}[N] = V(C_4)$, $a \in N^2_{C_4}[a] \setminus N^2_{C_4}[b]$ and $b \in N^2_{C_4}[b] \setminus N^2_{C_4}[a]$. Thus, $N$ is a $J^2$-hop dominating set of $C_4$. Since $N^2_{C_4}[d] = N^2_{C_4}[a]$ and $N^2_{C_4}[c] = N^2_{C_4}[b]$, it follows that $N$ is a maximum $J^2$-hop dominating set of $C_4$. Hence, $\gamma_{J^2h}(C_4) = 2$. Observe that $a \in N^2_{S(C_4)}[a] \setminus N^2_{S(C_4)}[b]$ and $b \in N^2_{S(C_4)}[b] \setminus N^2_{S(C_4)}[a]$, showing that $N$ is a $J^2$-set in $S(C_4)$. Since $N^2_{S(C_4)}[N] = V(S(C_4))$, it follows that $N$ is a $J^2$-hop dominating set in $S(C_4)$. By Lemma 1, $N^2_{S(C_4)}[u] = N^2_{S(C_4)}[u']$ for every $u \in V(C_4)$. Since $N^2_{S(C_4)}[a] = N^2_{S(C_4)}[d]$ and $N^2_{S(C_4)}[b] = N^2_{S(C_4)}[c]$, it follows that $N$ is a maximum $J^2$-hop dominating set of $S(C_4)$. Thus, $\gamma_{J^2h}(C_4) = 2 = \gamma_{J^2h}(S(C_4))$.

$S(C_4)$ :

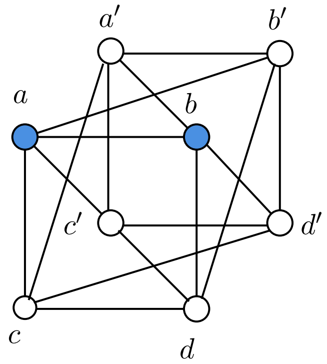


Figure 6: Graph $C_4$ with $\gamma_{J^2h}(C_4) = 2 = \gamma_{J^2h}(S(C_4))$

## 4. Conclusion

The concept of $J^2$-hop domination has been introduced and initially investigated in this study. Its bounds with respect to other known parameters in graph theory have been determined. In addition, characterizations of $J^2$-hop dominating sets in some graphs and shadow graph have been formulated and were used to solve exact value of the parameter of each of these graphs. Interested researchers may study further this parameter on graphs that were not considered in this study. Further, researchers may consider the investigation on the complexity of solving this parameter and provide application especially in real-life situation, network and other fields.

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## References

[1] S. Ayyaswamy, B. Krishnakumari, B. Natarjan, and Y. Venkatakrishnan. Bounds on the hop domination number of a tree. *Proceedings-Mathematical Sciences.*, 125(4):449–455, 2015.

[2] S. Ayyaswamy, C. Natarajan, and G. Sathiamoorphy. A note on hop domination

number of some special families of graphs. *International Journal of Pure and Applied Mathematics.*, 119(12):11465–14171, 2018.

[3] J. Hassan and S. Canoy Jr. Connected grundy hop dominating sequences in graphs. *Eur. J. Pure Appl. Math.*, 16(2):1212–1227, 2023.

[4] J. Hassan and S. Canoy Jr. Grundy dominating and grundy hop dominating sequences in graphs: Relationships and some structural properties. *Eur. J. Pure Appl. Math.*, 16(2):1154–1166, 2023.

[5] J. Hassan, S. Canoy Jr., and Chrisley Jade Saromines. Convex hop domination in graphs. *Eur. J. Pure Appl. Math.*, 16(1):319–335, 2023.

[6] J. Hassan, A. Lintasan, and N.H. Mohammad. Some properties and realization problems involving connected outer-hop independent hop domination in graphs. *Eur. J. Pure Appl. Math.*, 16(3):1848–1861, 2023.

[7] Canoy S. Jr. and G. Salasalan. Revisiting domination, hop domination, and global hop domination in graphs. *Eur. J. Pure Appl. Math.*, 14:1415–1428, 2021.

[8] S. Canoy Jr and J. Hassan. Weakly convex hop dominating sets in graphs. *Eur. J. Pure Appl. Math.*, 16(2):1196–1211, 2023.

[9] C. Natarajan and S. Ayyaswamy. Hop domination in graphs ii. *Versita,*, 23(2):187–199, 2015.

[10] Y. Pabilona and H. Rara. Total hop dominating sets in the join, corona and lexicographic product of graphs. *Jour. of Algebra and Appl. Math.*, 2:105–115, 2017.

[11] Y. Pabilona and H. Rara. Connected hop domination in graphs under some binary operations. *Asian-Eur. J. Math.*, 11(5):1850075–1–1850075–11, 2018.

[12] G. Salasalan, S. Canoy Jr., and A. Aradais. Global hop domination numbers of graphs. *Eur. J. Pure Appl. Math.*, 14(1):112–125, 2021.