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Characterizations of J-Total Dominating Sets of Some Graphs

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Abstract. Let G be a graph with no isolated vertex. A subset $M \subseteq V(G)$ is called a J-open set if $N_G(a)\backslash N_G(b) \neq \emptyset$ and $N_G(b)\backslash N_G(a) \neq \emptyset \ \forall \ a, b \in M$, where $a \neq b$. If in addition, M is a total dominating in G, then we call M a J-total dominating set in G. The maximum cardinality among all J-total dominating set in G, denoted by $\gamma_{Jt}(G)$, is called the J-total domination number of G. In this paper, we characterize J-total dominating sets in some special graphs and join of two graphs, and we use these results to obtain formulas for the parameters of these graphs. Moreover, we determine its relationships with other known parameters in graph theory. Finally, we derive the lower bound of the parameter for the corona of two graphs.

2020 Mathematics Subject Classifications: 05C69

Key Words and Phrases: J-open set, J-total dominating set, J-total domination number

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1. Introduction

The study of domination in graphs came about partially as a result of the study of games and recreational mathematics. In particular, mathematicians studied how chess pieces of a particular type could be placed on a chessboard in such a way that they would attack, or dominate, every square on the board.

Domination in a graph was introduced by Oystein Ore in 1962 in his book on graph theory [10]. A subset D of vertices of a graph G is called a *dominating* of G if for every $x \in V(G) \setminus D$, there exists $y \in D$ such that $xy \in E(G)$, that is, $N_G[D] = V(G)$. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set

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in G. A decade later, Cockayne and Hedetniemi [1] published a survey paper, in which the notation $\gamma(G)$ was first used for the domination number of a graph G. Since then, several mathematicians had studied and introduced new domination parameters in graphs. Some variants of domination were defined and further studied by researchers in [2–9, 11, 12].

In this paper, new variant of domination called J-total domination in a graph will be introduced and investigated. We will characterize J-total dominating sets in some classes of graphs and the join of two graphs, and we will use these results to determine the exact value of each of these graphs. Moreover, we will determine the bound of the parameter for the corona of two graphs. We believe that this new parameter would give additional insights to researchers in the field and would help them for more research directions in the future.

2. Terminology and Notation

Let $G = (V(G), E(G))$ be a graph. Two vertices a, b of G are said to be *adjacent*, or *neighbors*, if ab is an edge of G. The *open neighborhood* of x in G is the set defined by $N_G(x) = \{y \in V(G) : xy \in E(G)\}$. The *closed neighborhood* of x in G is the set $N_G[x] = N_G(x) \cup \{x\}$. If $X \subseteq V(G)$, then *open neighborhood* of X in G is the set $N_G(X) = \bigcup_{x\in X} N_G(x)$. The *closed neighborhood* of X in G is the set $N_G[X] = N_G(X) \cup X$.

A *path graph* is a non-empty graph with vertex-set $\{x_1, x_2, \ldots, x_n\}$ and edge-set $\{x_1x_2, x_2x_3, \ldots, x_{n-1}x_n\}$, where the $x_i^{'}s$ are all distinct. The path of order n is denoted by P_n. If G is a graph and u and v are vertices of G, then a path from vertex u to vertex v is sometimes called a *u-v path*. The *cycle graph* is the graph of order $n \geq 3$ with vertex-set $\{x_1, x_2, \ldots, x_n\}$ and edge-set $\{x_1x_2, x_2x_3, \ldots, x_{n-1}x_n, x_nx_1\}$. The cycle graph of order n is denoted by P_n.

A graph G is *connected* if every pair of its vertices can be joined by a path. Otherwise, G is *disconnected*. A maximal connected subgraph (not a subgraph of any connected subgraph) of G is called a *component* of G.

The *distance* $d_G(u, v)$ in G of two vertices u, v is the length of a shortest *u-v path* in G. The greatest distance between any two vertices in G, denoted by $diam(G)$, is called the *diameter* of G.

A subset S of $V(G)$ is called a *dominating* of G if for every $x \in V(G) \setminus S$, there exists $y \in S$ such that $xy \in E(G)$, that is, $N_G[S] = V(G)$. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set in G. Any dominating set S with cardinality equal to $\gamma(G)$ is called a γ-set of G.

A subset T of $V(G)$ is called a *total dominating* of G if for every $x \in V(G)$, there exists $y \in T$ such that $xy \in E(G)$, that is, $N_G(T) = V(G)$. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set in G. Any total dominating set T with cardinality equal to $\gamma_t(G)$ is called a γ_t-set of G.

A graph G is *complete* if every pair of distinct vertices of G are adjacent. A complete graph of order n is denoted by K_n.

A graph G is called a *bipartite* graph if its vertex-set $V(G)$ can be partitioned into two nonempty subsets V_1 and V_2 such that every edge of G has one end in V_1 and one end in V_2. The sets V_1 and V_2 are called the *partite* sets of G. If each vertex in V_1 is adjacent to every vertex in V_2, then G is called a *complete bipartite* graph. If $|V_1| = m$ and $|V_2| = n$, then the complete bipartite graph is denoted by $K_{m,n}$. A *star* graph of order $n+1$ is the complete bipartite graph $K_{1,n}$.

Let G and H be any two graphs. The *join* of G and H, denoted by $G+H$ is the graph with vertex set $V(G+H) = V(G) \cup V(H)$ and edge set

$$E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}.$$

The *fan* F_n of order $n+1$, where $n \geq 1$, is given by $F_n = K_1 + P_n$.

The *corona* G and H, denoted by $G \circ H$, the graph obtained by taking one copy of G and $|V(G)|$ copies of H, and then Joining the *ith* vertex of G to every vertex of the *ith* copy of H. We denote by H^v the copy of H in $G \circ H$ corresponding to the vertex $v \in G$ and write $v + H^v$ for $\langle \{v\} + H^v \rangle$.

3. Results

We begin this section by introducing the concept of J-total domination in a graph.

Definition 1. Let G be a graph with no isolated vertex. A subset $M \subseteq V(G)$ is called a *J-open set* in G if $N_G(a) \setminus N_G(b) \neq \emptyset$ and $N_G(b) \setminus N_G(a) \neq \emptyset \ \forall \ a, b \in M, a \neq b$. If in addition, M is a total dominating in G, then we call M a *J-total dominating set* in G. The maximun cardinality among all J-total dominating sets in G, denoted by $\gamma_{Jt}(G)$, is called the *J-total domination number* of G. Any J-total dominating set M with $| M |= \gamma_{Jt}(G)$ (resp. $|M| = \gamma_t(G)$), is called a *γ_{Jt}-set* or the *maximum* (resp. *minimum)* *J-total dominating set* in G.

Remark 1. *Let G be a graph with no isolated vertex. Then each of the following is true.*

(i) *A total dominating set T of G may not be a J-open set in G (hence not a J-total dominating set).*

(ii) *A J-open set Q in G may not be a total dominating set in G (hence not a J-total dominating set).*

(iii) *A vertex set $V(G)$ of G may not be a J-total dominating set in G.*

Proposition 1. *Let G be a graph with no isolated vertex. Then*

(i) $\gamma_t(G) \leq \gamma_{Jt}(G)$.

(ii) $2 \leq \gamma_{Jt}(G) \leq | V(G) |$.

Proof. (i) Let G be a graph with no isolated vertex and let M be a maximum J-total dominating set of G. Then M is a total dominating set of G (by defintion). Since

$\gamma_t(G)$ is the minimum cardinality among all total dominating sets in G, it follows that $\gamma_{Jt}(G) =\mid M \mid \geq \gamma_t(G)$.

(ii) Since $\gamma_t(G) \geq 2$ for any graph G with no isolated vertex, and so $\gamma_{Jt}(G) \geq 2$ by (i). Since any J-total dominating set M is always a subset of a vertices $V(G)$ of G, it follows that $\gamma_{Jt}(G) \leq\mid V(G) \mid$. Consequently, $2 \leq \gamma_{Jt}(G) \leq\mid V(G) \mid$. □

Remark 2. *The bound given in Proposition 1 is tight. Moreover, strict inequality is attainable.*

For tightness, consider the graph G given in Figure 1 below.

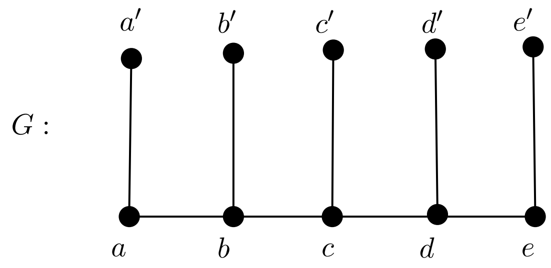

Figure 1: A graph G with $\gamma_t(G) = 5 = \gamma_{Jt}(G)$

Let $S = \{a, b, c, d, e\}$. Clearly, S is the minimum total dominating set of G. Thus, $\gamma_t(G) = 5$. Observe that $x' \in N_G(x) \setminus N_G(y)$ and $y' \in N_G(y) \setminus N_G(x)$ for every $x, y \in S$, where $x \neq y$. It follows that $N_G(x) \setminus N_G(y) \neq \varnothing$ and $N_G(y) \setminus N_G(x) \neq \varnothing$ for every $x, y \in S, x \neq y$. Hence, S is a J-open set in G, showing that S is a J-total dominating set of G. Notice that $N_G(a'), N_G(c') \subseteq N_G(b)$, $N_G(b'), N_G(d') \subseteq N_G(c)$ and $N_G(e') \subseteq N_G(d)$ and a, b, c, d, e must be in any total dominating set of G. Consequently, S is the maximum J-total dominating set of G, and so $\gamma_{Jt}(G) = 5$.

For strict inequality, consider the graph G' given in Figure 2 below.

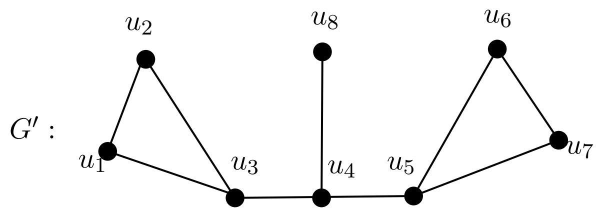

Figure 2: A graph G' with $\gamma_t(G') = 3 < 7 = \gamma_{Jt}(G')$

Let $T_1 = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$ and $T_2 = \{u_3, u_4, u_5\}$. Then T_2 is a minimum total dominating set of G'. Hence, $\gamma_t(G') = 3$. Since $T_2 \subseteq T_1$, it follows that T_1 is also a total dominating set of G'. Observe that $u_2 \in N_{G'}(u_1) \setminus N_{G'}(u_i)\ \forall\ i \neq 3$, $u_3 \in N_{G'}(u_1) \setminus N_{G'}(u_3)$, $u_1 \in N_{G'}(u_2) \setminus N_{G'}(u_j)\ \forall\ j \neq 3$, $u_3 \in N_{G'}(u_2) \setminus N_{G'}(u_3)$, $u_1 \in N_{G'}(u_3) \setminus N_{G'}(u_r)\ \forall\ r \neq 2$, $u_2 \in N_{G'}(u_3) \setminus N_{G'}(u_2)$, $u_8 \in N_{G'}(u_4) \setminus N_{G'}(u_q)\ \forall\ q \neq 4$, $u_7 \in N_{G'}(u_5) \setminus N_{G'}(u_s)\ \forall\ s \neq 6$, $u_6 \in N_{G'}(u_5) \setminus N_{G'}(u_6)$, $u_7 \in N_{G'}(u_6) \setminus N_{G'}(u_t)\ \forall\ t \neq 5$, $u_5 \in N_{G'}(u_6) \setminus N_{G'}(u_5)$ and $u_6 \in N_{G'}(u_7) \setminus N_{G'}(u_m)\ \forall\ m \neq 5$, $u_5 \in N_{G'}(u_7) \setminus N_{G'}(u_5)$. Thus, T_1 is a J-open set of G', and so T_1 is a J-total dominating set of G'. Hence, $\gamma_{Jt}(G') = 7$. Consequently, $\gamma_{Jt}(G') > \gamma_t(G')$.

Theorem 1. *Let K_n be a complete graph of order $n \geq 2$. Then $M \subseteq V(K_n)$ is a J-total dominating in K_n if and only if $\mid M \mid \geq 2$.*

Proof. Let $M \subseteq V(K_n)$ be a J-total dominating set in K_n. Then M is a total dominating set in K_n. Since $\gamma_t(K_n) = 2$ for all $n \geq 2$, it follows that $\mid M \mid \geq \gamma_t(K_n) = 2$.

Conversely, suppose that $M \subseteq V(K_n)$ with $\mid M \mid \geq 2$. Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ and $M = \{v_1, v_2, \ldots, v_s\} \subseteq V(K_n)$ where $s \in \{2, 3, \ldots, n\}$. Observe that $v_i \in N_{K_n}(v_j) \setminus N_{K_n}(v_i)$ for all $i \neq j, i, j \in \{1, 2, \ldots, s\}$. Thus,

$$N_{K_n}(v_j) \setminus N_{K_n}(v_i) \neq \emptyset \text{ for all } i \neq j, \text{where } i, j \in \{1, 2, \ldots, s\},$$

showing that M is a J-open set in $K_n\ \forall\ n \geq 2$. Since any set $\{v_i, v_j\}$, $i \neq j$, is a total dominating in K_n, it follows that M is J-total dominating set in $K_n\ \forall\ n \geq 2$. □

Corollary 1. *Let $n \geq 2$ be any positive integer. Then*

$$\gamma_{Jt}(K_n) = n.$$

Proof. Let $M = V(K_n) = \{a_1, a_2, ..., a_n\}$. Then by Theorem 1, M is J-total dominating set in K_n. Thus, $\gamma_{Jt}(K_n) \geq n$. By Proposition 1, $\gamma_{Jt}(K_n) = n$. □

Theorem 2. *Let m and n be positive integers with $2 \leq m \leq n$. Then there exists a connected graph H such that $\gamma_t(H) = m$ and $\gamma_{Jt}(H) = n$. That is, $\gamma_{Jt}(H) - \gamma_t(H)$ can be made arbitrarily large.*

Proof. For $m = n$, consider the graph H in Figure 3 below.

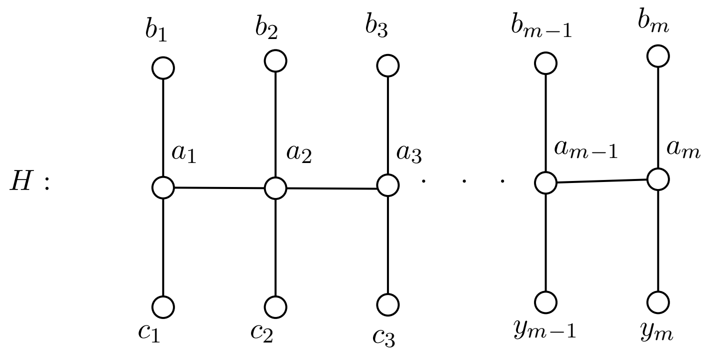

Figure 3: A graph H with $\gamma_t(H) = m = \gamma_{Jt}(H)$

Let $M = \{a_1, a_2, ..., a_{m-1}, a_m\}$. Then M is a minimum total dominating set of H, and so $\gamma_t(H) = m$. Since $b_i, c_i \in N_H(a_i) \setminus N_H(a_j)$ for every $i \neq j, i, j \in \{1, 2, \ldots, m\}$, it follows that M is a J-open set in H. Thus, M is a J-total dominating set of H. Now, observe that $N_H(b_i), N_H(c_i) \subseteq N_H(a_{i+1})$, $\forall\ i \in \{1, 2, \ldots, m-1\}$, $N_H(b_m), N_H(c_m) \subseteq N_H(a_{m-1})$ and a_i must be in any total dominating set of H for each $i \in \{1, 2, \ldots, m\}$. Therefore, M is the maximum J-total dominating set of H, and so $\gamma_{Jt}(H) = m$.

Suppose that $m < n$. Let $s = n - m$ and consider the graph H' in Figure 4 below, where $\langle\{a_m, b_1, b_2, \ldots, b_s\}\rangle$ induced a complete graph for all positive integer $s \geq 1$.

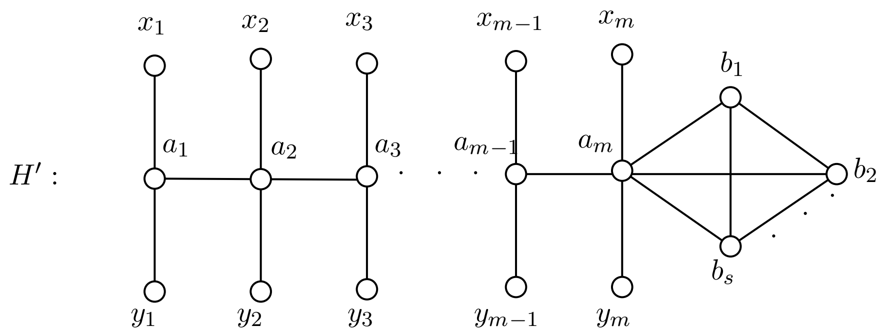

Figure 4: A graph H' with $\gamma_t(H') < \gamma_{Jt}(H')$

Let $M_1 = \{a_1, a_2, ..., a_m\}$ and $M_2 = \{a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_s\}$. Then M_1 is the minimum total dominating set of H'. Thus, $\gamma_t(H') = m$. Since $M_1 \subseteq M_2$, M_2 is also a total dominating set of H'. Observe that M_2 is a J-open set in H'. Hence,M_2 is a J-total dominating set of H'. Applying the same argument in the equality part and $V(K_s)$ is a J-open set in $K_s, s \geq 2$ by Theorem 1, it follows that M_2 is the maximum J-total dominating set of H'. Consequently, $\gamma_{Jt}(H') = s + m = n$. □

Theorem 3. *Let G be a graph with no isolated vertex. Then*

(i) *M is a γ_t-set in G if and only if M is a minimum J-total dominating set in G.*

(ii) *If every component of G is non-trivial complete graph, then $\gamma_{Jt}(G) = |V(G)|$. However, the converse is not true.*

Proof. (i) Let M be a γ_t-set in G. Then M is the minimum total dominating set in G. Suppose that M is not a J-open set in G. Then there exist $a, b \in M$ such that $N_G(a) \backslash N_G(b) = \emptyset$ or $N_G(b) \backslash N_G(a) = \emptyset$. It follows that $N_G(a) \subseteq N_G(b)$ or $N_G(b) \subseteq N_G(a)$. Assume that $N_G(a) \subseteq N_G(b)$, then $M \backslash \{a\}$ is a total dominating set in G. However, this is a contradiction to our assumption that M is the minimum total dominating set in G. Hence, M is a J-open set in G, and so M is a minimum J-total dominating set in G.

The converse is clear.

(ii) Suppose that every component H of G is a non-trivial complete graph. Let $H_1, \ldots, H_k$, $k \geq 2$ be components of G. Then by Corollary 1, $\gamma_{Jt}(H_i) =| V(H_i) |$.
It follows that $\gamma_{Jt}(G) = \gamma_{Jt}(H_i) + \cdots + \gamma_{Jt}(H_k)$
$=| V(H_i) | + \cdots + | V(H_k)$
$=| V(G) |$.

To see that the converse is not true, consider $G = C_5$ below.

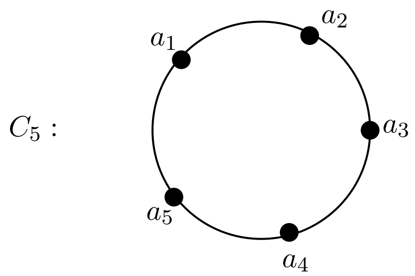

Figure 5: A graph C_5 with $\gamma_{Jt}(C_5) = 5$

Let $M = \{a_1, a_2, ..., a_5\} = V(G)$. Observe that

$$a_2 \in N_G(a_1) \setminus N_G(a_i) \ \forall \ i \neq 3,$$

$$a_5 \in N_G(a_1) \setminus N_G(a_j) \ \forall \ j \neq 4,$$

$$a_1 \in N_G(a_2) \setminus N_G(a_k) \ \forall \ k \neq 5,$$

$$a_3 \in N_G(a_2) \setminus N_G(a_e) \ \forall \ e \neq 4,$$

$$a_2 \in N_G(a_3) \setminus N_G(a_s) \ \forall \ s \neq 1,$$

$$a_4 \in N_G(a_3) \setminus N_G(a_t) \ \forall \ t \neq 5,$$

$$a_3 \in N_G(a_4) \setminus N_G(a_r) \ \forall \ r \neq 2,$$

$$a_5 \in N_G(a_4) \setminus N_G(a_q) \ \forall \ q \neq 1,$$

$$a_1 \in N_G(a_5) \setminus N_G(a_m) \ \forall \ m \neq 2, and$$

$$a_4 \in N_G(a_5) \setminus N_G(a_n) \ \forall \ n \neq 3.$$

Thus, $M = V(G)$ is a J-open set in G. Since $N_G(M) = V(G)$, it follows that M is a J-total dominating set in G. Hence, $\gamma_{Jt}(G) = 5 =\mid V(G) \mid$. □

Theorem 4. *Let $K_{m,n}$ be a complete bipartite graph where $m, n \geq 1$. Then $N \subseteq V(K_{m,n})$ is a J-total dominating in $K_{m,n}$ if and only if $N = \{a, b\}$ for some $a \in V(\overline{K}_m)$ and $b \in V(\overline{K}_n)$.*

Proof. Let $N \subseteq V(K_{m,n})$ be a J-total dominating in $K_{m,n}, m, n \geq 1$. Then N is a total dominating set in $K_{m,n}$. Thus, $|N| \geq 2$. Let $V(\overline{K}_m) = \{u_1, u_2, \dots, u_m\}$ and $V(\overline{K}_n) = \{v_1, v_2, \dots, v_n\}$. Observe that $N_{k_{m,n}}(u_i) = N_{K_{m,n}}(u_j) \ \forall \ i \neq j$, $i, j \in \{1, 2, \dots, m\}$ and $N_{K_{m,n}}(v_r) = N_{K_{m,n}}(v_q) \quad \forall r \neq q, \quad r, q \in \{1, 2, \dots, n\}$. This means that there are $m-1$ and $n-1$ vertices of $\overline{K}_m$ and $\overline{K}_n$, respectively, cannot be in any J-total dominating set of $K_{m,n}$. Hence, $\mid N \mid \leq 2$, and so $\mid N \mid = 2$. Thus, $N = \{a, b\}$ for some $a \in V(\overline{K}_m)$ and $b \in V(\overline{K}_n)$.

Conversely, let $N = \{a, b\}$ for some $a \in V(\overline{K}_m)$ and $b \in V(\overline{K}_n)$. Then $N_{K_{m,n}}(a) = V(\overline{K}_n)$ and $N_{K_{m,n}}(b) = V(\overline{K}_m)$. Hence, $N_{K_{m,n}}(N) = V(K_{m,n})$, and $N_{k_{m,n}}(a) \setminus N_{K_{m,n}}(b) = V(\overline{K}_n) \neq \emptyset$ and $N_{K_{m,n}}(b) \setminus N_{K_{m,n}}(a) = V(\overline{K}_m) \neq \emptyset$. Consequently, $N = \{a, b\}$ is a J-total dominating set in $K_{m,n}$. □

The following result follows immediately for Theorem 4.

Corollary 2. *Let $m, n \geq 1$ be positive integers. Then $\gamma_{Jt}(K_{m,n}) = 2$.*

Theorem 5. *Let $m \geq 2$ be positive integer. Then*

$$\gamma_{Jt}(P_m) = \begin{cases} 2 \quad if & m = 2, 3, 4 \\ 4 \quad if & m = 5 \\ m-2 \quad if & m \geq 6. \end{cases}$$

Proof. Clearly, $\gamma_{Jt}(P_2) = 2$. For $m = 3$, let $V(P_3) = \{v_1, v_2, v_3\}$ and let $S = \{v_1, v_2\}$. Then $v_2 \in N_{P_3}(v_1) \setminus N_{P_3}(v_2)$ and $v_1 \in N_{P_3}(v_2) \setminus N_{P_3}(v_1)$. Thus, S is a J-open set in P_3. Since $N_{P_3}(S) = V(P_3)$, it follows that S is a J-total dominating set of P_3. Notice that $N_{P_3}(v_1) = N_{P_3}(v_3)$. Hence, v_1 and v_3 cannot be both in any J-open set of P_3. Therefore, $S = \{v_1, v_2\}$ is a maximum J-total dominating set in P_3, showing that $\gamma_{Jt}(P_3) = 2$.

For $m = 4$, let $V(P_4) = \{a_1, a_2, a_3, a_4\}$ and $S' = \{a_2, a_3\}$. Then

$$a_1, a_3 \in N_{P4}(a_2) \setminus N_{P_4}(a_3) \text{ and } a_2, a_4 \in N_{P_4}(a_3) \setminus N_{P_4}(a_2).$$

Thus, S' is a J-open set in P_4. Observe that $N_{P_4}(S') = V(P_4)$. Therefore, S' is a J-total dominating set in P_4. Notice that $N_{P_4}(a_1) \subseteq N_{P_4}(a_3)$ and $N_{P_4}(a_4) \subseteq N_{P_4}(a_2)$. This means that a_1 and a_3 (resp. a_2 and a_4) cannot be both in any J-open set of P_4. Consequently, $S' = \{v_2, v_3\}$ is a maximum J-total dominating set of P_4, and so $\gamma_{Jt}(P_4) = 2$.

For $m = 5$, let $V(P_5) = \{u_1, u_2, u_3, u_4, u_5\}$ and consider $C = \{u_1, u_2, u_4, u_5\}$. Then $u_1 \in N_{P_5}(u_2) \setminus N_{P_5}(u_i) \quad \forall \quad i \neq 2, u_2 \in N_{P_5}(u_1) \setminus N_{P_5}(u_j) \quad \forall j \neq 1, u_4 \in N_{P_5}(u_5) \setminus N_{P_5}(u_r)$ $\forall\ r \neq 5$ and $u_5 \in N_{P_5}(u_4) \setminus N_{P_5}(u_q) \quad \forall \quad q \neq 4$. Hence, C is a J-open set in P_5. Since $N_{P_5}(C) = V(_5)$, it follows that C is a J-total dominating set of P_5. Notice that $N_{P_5}(u_1) \subseteq N_{P_5}(u_3)$. Thus, u_1 and u_3 cannot be both in any J-open set of P_5. Consequently, C is a maximum J-total domianting set of P_5, and so $\gamma_{Jt}(P_5) = 4$.

Next, suppose that $m \geq 6$. Let $V(P_m) = \{w_1, w_2, \ldots, w_m\}$ and consider $C' = \{w_2, w_3, ..., w_{m-2}, w_{m-1}\}$. Notice that $w_{i-1} \in N_{P_m}(w_i) \setminus N_{P_m}(w_j)$ and $w_{j+1} \in N_{P_m}(w_j) \setminus N_{P_m}(w_i) \quad \forall i < j, i, j \in \{2, 3, \ldots, m-1\}$. It follows that

$$N_{P_m}(w_i) \setminus N_{P_m}(w_j) \neq \emptyset \quad \forall\ i \neq j, i, j \in \{2, 3, ..., m-1\}.$$

Thus, C' is a J-open set in P_m for all $m \geq 6$. Since $N_{P_m}(C') = V(P_m)$, C' is a J-total dominating set of P_m. Now, observe that $N_{P_m}(w_1) \subseteq N_{P_m}(w_3)$ and $N_{P_m}(w_m) \subseteq N_{P_m}(w_{m-2})$. Hence, w_1 and w_3 (resp. w_{m-2} and w_m) cannot be both in any J-open set of P_m. Therefore, C' is a maximum J-total dominating set of P_m, and so $\gamma_{Jt}(P_m) = m - 2$ for all $m \geq 6$. □

Theorem 6. *Let n be any positive integer. Then*

$$\gamma_{Jt}(F_n) = \begin{cases} 2 \quad if & n = 1 \\ 3 \quad if & n = 2, 3, 4 \\ 5 \quad if & n = 5 \\ n - 1 \quad if & n \geq 6 \end{cases}$$

Proof. Since F_1 and F_2 are complete graphs, $\gamma_{Jt}(F_1) = 2$ and $\gamma_{Jt}(F_2) = 3$ by Corollary 1. For $n = 3$, let $V(F_3) = \{v_0, v_1, v_2, v_3\}$, where v_0 is the dominating vertex of F_3. Consider $M = \{v_0, v_1, v_2\}$. Then $v_i \in N_{F_3}(v_0) \setminus N_{F_3}(v_i)$ and $v_0 \in N_{F_3}(v_i) \setminus N_{F_3}(v_0)$ $\forall\ i \neq 0$, and $v_2 \in N_{F_3}(v_1) \setminus N_{F_3}(v_2)$ and $v_1 \in N_{F_3}(v_2) \setminus N_{F_3}(v_1)$. Thus, M is a J-open set in F_3. Since $N_{F_n}(M) = V(F_3)$, it follows that M is a J-total dominating set of F_3. Since $N_{F_3}(v_1) = N_{F_3}(v_3)$, v_1 and v_3 cannot be both in any J-open set of F_3. Therefore, M is a maximum J-total dominating set of F_3, and so $\gamma_{Jt}(F_3) = 3$. Similarly, if $n = 4$, then $\gamma_{Jt}(F_4) = 3$.

For $n = 5$, let $V(F_5) = \{v_0, v_1, v_2, V_3, v_4, v_5\}$, where v_0 is the dominating vertex of F_5. Let $M' = \{v_0, v_1, v_2, , v_4, v_5\}$. Then $v_j \in N_{F_5}(v_0) \setminus N_{F_5}(v_j)$ and $v_0 \in N_{F_5}(v_j) \setminus N_{F_5}(v_0)$ $\forall\ j \neq 0$. Since $\{v_1, v_2, v_4, v_5\}$ is a J-open set in P_5 by Theorem 5, it follows that M' is a J-open set in F_5. Notice that $N_{F_5}(M') = V(F_5)$. Thus, M' is a J-total dominating set of F_5. Since $N_{F_5}(v_1) \subseteq N_{F_5}(v_3)$, it follows that v_1 and v_3 cannot be both in any J-open set of F_5. Therefore, M' is a maximum J-total dominating set of F_5, and so $\gamma_{Jt}(F_5) = 5$.

Next, suppose that $n \geq 6$. Let $V(F_n) = \{u_0, u_1, \ldots, u_n\}$, where u_0 is the dominating vertex of F_n. Let $C = \{u_0, u_2, u_3, \ldots, u_{n-1}\}$. Observe that $u_r \in N_{F_n}(u_o) \setminus N_{F_n}(u_r)$ and $u_0 \in N_{F_n}(u_r) \setminus N_{F_n}(u_o)$ $\forall$ $r \neq 0$. Since $\{u_2, u_3, ..., u_{n-1}\}$ is a J-open set in P_n by Theorem 5, it follows that C is a J-open set in F_n. Observe further that $N_{F_n}(C) = V(F_n)$. Hence, C is a J-total dominating set of F_n. Since $N_{F_n}(u_1) \subseteq N_{F_n}(u_3)$ and $N_{F_n}(u_n) \subseteq N_{F_n}(u_{n-2})$, u_1 and u_3 (resp. u_{n-2} and u_n) cannot be both in any J-open set of F_n. Therefore, C is a maximum J-total dominating set of F_n, showing that $\gamma_{Jt}(F_n) = n - 1$ for all $n \geq 6$. □

Theorem 7. *Let G and H be two graphs with no isolated vertices. A subset M of vertices of $G+H$ is a J-total dominating set of $G+H$ if and only if one of the following conditions holds:*

(*i*) *M is a J-total dominating set of G.*

(*ii*) *M is a J-total dominating set of H.*

(*iii*) *$M = M_G \cup M_H$, where M_G and M_H are J-open sets in G and H, respectively.*

Proof. Let M be a J-total dominating set of $G + H$. If $M_H = \varnothing$, then $M = M_G$ is a J-total dominating set in G. Thus, (i) holds. If $M_G = \varnothing$, then $M = M_H$ is a J-total dominating set in H, and hence (ii) holds. Next, assume that M_G and M_H are both non-empty. Suppose on the contrary that M_G is not a J-open set in G. Then there exist $a, b \in M_G \subseteq M$ such that either $N_G(a) \setminus N_G(b) = \varnothing$ or $N_G(b) \setminus N_G(a) = \varnothing$. Thus, $N_{G+H}(a) \setminus N_{G+H}(b) = \varnothing$ or $N_{G+H}(a) \setminus N_{G+H}(b) = \varnothing$, a contradiction to the fact that M is a J-open set in $G + H$. Therefore, D_G is a J-open set in G. Similarly, M_H is a J-open set in H. Consequently, (iii) holds.

Conversely, if (i) or (ii) holds, then the assertion follows. Next, suppose that (iii) holds. Since M_G and M_H are both non-empty, it follows that M is a total dominating set in $G+H$. Let $a, b \in M$. Suppose that $a, b \in M_G \subseteq M$. Since M_G is a J-open set in G, we have $N_G(a) \setminus N_G(b) \neq \varnothing$ and $N_G(b) \setminus N_G(a) \neq \varnothing$. It follows that $N_{G+H}(a) \setminus N_{G+H}(b) \neq \varnothing$ and $N_{G+H}(b) \setminus N_{G+H}(a) \neq \varnothing$. Hence M is a J-open set in $G+H$. Similarly, if $a, b \in M_H \subseteq M$, then M is a J-open set in $G + H$. Now, assume that $a \in M_G$ and $b \in M_H$. If a is a dominating vertex of G, then we are done. Similarly, if b a dominating vertex of H. Suppose that a and b are not dominating vertices of G and H, respectively. Let $x \in V(G)$ and $y \in V(H)$, where $x \notin N_G(a)$ and $y \notin N_H(b)$. Then $y \in N_{G+H}(a) \setminus N_{G+H}(b)$ and $x \in N_{G+H}(b) \setminus N_{G+H}(a)$. Thus, M is a J-open set in $G+H$. Consequently, D is a J-total dominating set in $G + H$. □

The following result follows immediately from Theorem 7.

Corollary 3. *Let G and H be two graphs with no isolated vertices. Then*

$$\gamma_{Jt}(G + H) = \gamma_{Jt}(G) + \gamma_{Jt}(H).$$

Theorem 8. *Let G be a connected non-trivial graph and H be a graph with no isolated vertex. If $M = V(G) \cup (\bigcup_{v \in V(G)} M_v)$, where M_v is a J-total dominating set in H^v for each $v \in V(G)$, then M is a J-total dominating set in $G \circ H$. Moreover,*

$$\gamma_{Jt}(G \circ H) \geq |V(G)| + \gamma_{Jt}(H) \cdot |V(G)|.$$

Proof. Let $M = V(G) \cup (\bigcup_{v \in V(G)} M_v)$, where M_v is a J-total dominating set in H^v for each $v \in V(G)$. Since G is connected, it follows that $V(G)$ is a total dominating set in G. Moreover, since M_v is a total dominating set in H^v for each $v \in V(G)$, M is a total dominating set in $G \circ H$. Now, let $a, b \in M$. If $a, b \in M_u$ for some $u \in V(G)$, then $N_H(a) \setminus N_H(b) \neq \varnothing$ and $N_H(b) \setminus N_H(a) \neq \varnothing$. It follows that $N_{G \circ H}(a) \setminus N_{G \circ H}(b) \neq \varnothing$ and $N_{G \circ H}(b) \setminus N_{G \circ H}(a) \neq \varnothing$. Thus, M is a J-open set in $G \circ H$. Similarly, if $a, b \in V(G)$, then M is a J-open set in $G \circ H$. Assume that $a \in M_s$ and $b \in M_t$ for some $s, t \in V(G), s \neq t$. Then $s \in N_{G \circ H}(a) \setminus N_{G \circ H}(b)$ and $t \in N_{G \circ H}(b) \setminus N_{G \circ H}(a)$, hence we are done. Suppose that $a \in M_w$ for some $w \in V(G)$ and $b \in V(G)$. If $w = b$, then $b \in N_{G \circ H}(a) \setminus N_{G \circ H}(b)$ and $a \in N_{G \circ H}(b) \setminus N_{G \circ H}(a)$, and we are done. Suppose $w \neq b$. Since H is graph with no isolated vertex, there exists $q \in H^w$ such that $q \in N_{G \circ H}(a) \setminus N_{G \circ H}(b)$. Clearly, $N_{G \circ H}(b) \setminus N_{G \circ H}(a) = M_b \neq \varnothing$. Therefore, M is a J-open set in $G \circ H$, showing that M is a J-total dominating set in $G \circ H$. Consequently,

$$\gamma_{Jt}(G \circ H) \geq |V(G)| + \gamma_{Jt}(H) \cdot |V(G)|.$$

□

4. Conclusion

The concept of J-total domination has been introduced and investigated in this study. Characterizations of J-total dominating sets in some graphs and join of two graphs are formulated and were used to solve exact values of the parameters of these graphs. Some bounds and relationships of this newly defined parameter have been established. Other graphs that were not considered in this study could be an interesting topic to consider by researchers for further investigation of the concept. They may also consider the bounds of the parameter with respect to other well known parameters in graph theory.

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