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Hop Italian domination in graphs

Sergio R. Canoy,Jr.[1,2], Ferdinand P. Jamil[1,2] and Sheila M. Menchavez[1,2,*]

[1] *Department of Mathematics and Statistics, College of Science and Mathematics, Mindanao State University-Iligan Institute of Technology, 9200 Iligan City, Philippines*
[2] *Center for Graph Theory, Algebra and Analysis, Premier Research Institute of Science and Mathematics (PRISM), Mindanao State University - Iligan Institute of Technology, 9200 Iligan City, Philippines*

Abstract. Given a simple graph $G = (V(G), E(G))$, a function $f : V(G) \to \{0, 1, 2\}$ is a hop Italian dominating function if for every vertex v with $f(v) = 0$ there exists a vertex u with $f(u) = 2$ for which u and v are of distance 2 from each other or there exist two vertices w and z for which $f(w) = 1 = f(z)$ and each of w and z is of distance 2 from v. The minimum weight $\sum_{v\in V(G)} f(v)$ of a hop Italian dominating function is the hop Italian domination number of G, and is denoted by $\gamma_{hI}(G)$. In this paper, we initiate the study of the hop Italian domination. First, we establish some properties of the the hop Italian dominating function and characterize graphs G with smaller values for $\gamma_{hI}(G)$. Next, we explore the relationships of the hop Italian domination number with closely related concepts, particularly the hop Roman domination number and the 2-hop domination number. Finally, we investigate the hop Italian domination in the complementary prism, join, corona and lexicographic product of graphs.

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1. Introduction

The history of the Roman domination in graphs can be traced back to the military strategy adapted by Constantine the Great (Emperor of Rome) during the fourth century AD (see [23, 26]). In order to defend his cities Constantine issued a decree that any city without a legion stationed to secure it must neighbor another city having two stationed legions. If the first were attacked, then the second could deploy a legion to protect it without becoming vulnerable itself. It is called defense-in-depth strategy, which used only four Field Armies (FA) available for deployment to defend a total of eight regions.

Roman domination as a mathematical concept was introduced by Cockayne, Dreyer, S.M. Hedetniemi and S.T. Hedetniemi [9] in 2004. Thereafter, it has become an active

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Email addresses: `sergio.canoy@g.msuiit.edu.ph` (S.R.Jr. Canoy), `ferdinand.jamil@g.msuiit.edu.ph` (F. Jamil), `sheila.menchavez@g.msuiit.edu.ph` (S. Menchavez)

research area (see [1, 6, 12, 18, 19, 21, 22, 25, 27]). It models many facility location problems (see [7]), where $f(v)$ is viewed as cost function. Units with cost 2 may be able to serve neighboring locations, while units with costs 1 can serve only their own location. In a communication network, $f(v) = 2$ is assigned to locations where we install wireless hubs which are more expensive but can serve neighboring locations, while $f(v) = 1$ is assigned to locations where we install wired hubs which function at low-range but are cheaper.

In 2016, the Roman 2-domination was introduced by Chellali, Haynes, Hedetniemi and McRae [8]. It is also called Italian domination. A function $f : V(G) \to \{0, 1, 2\}$ is an Italian dominating function provided for every vertex v with $f(v) = 0$ we have $\sum_{x \in N_G(v)} f(x) \geq 2$, where $N(v)$ is the set of all vertices adjacent to v. Apparently, a Roman dominating function is an Italian dominating function. The Italian domination number is the minimum weight of an Italian dominating function. Excellent references for Italian domination include [8, 20].

In 2017, Shabani [24] introduced the hop Roman domination. A hop Roman dominating function on G is a function $f : V(G) \to \{0, 1, 2\}$ satisfying the property that for every vertex v of G with $f(v) = 0$ there is a vertex u with $f(u) = 2$ for which the distance $d_G(u, v)$ between u and v is 2. It was largely motivated by the concept of hop domination which is relatively well-known to have a wide range of applications in social network. Hop Roman domination in graphs was further studied in [21, 22].

This present paper intends to introduce and initiate the study of the hop Italian domination. We will establish some of its properties and make characterizations for some special graphs. We will explore its relationships with the hop Roman domination and other related hop domination concepts. Finally, we will investigate the hop Italian domination in graphs under some binary operations.

All throughout this paper, we consider only graphs which are simple, finite and undirected.

Given a graph $G = (V(G), E(G))$, we call $V(G)$ the *vertex set* of G and $E(G)$ its *edge set*. The cardinality $|V(G)|$ of $V(G)$ is the *order* of G. All terminologies used here which are not being defined are adapted from [3].

Let G and H be disjoint graphs. The *complementary prism* $G\overline{G}$ is formed from G and its complement $\overline{G}$ by adding a perfect matching between corresponding vertices of G and $\overline{G}$. If for each $v \in V(G)$, $\overline{v}$ is the vertex in $\overline{G}$ corresponding to v, then $G\overline{G}$ is formed by adding the edge $v\overline{v}$ for every $v \in V(G)$. The *corona* of G and H is the graph $G \circ H$ obtained by taking one copy of G and $|V(G)|$ copies of H, and then joining the i^{th} vertex of G to every vertex in the i^{th} copy of H. In particular, we call $G \circ K_1$ the corona of G, and write $cor(G) = G \circ K_1$. The *composition* (or *lexicographic product*) of G and H is the graph $G[H]$ with $V(G[H]) = V(G) \times V(H)$ and $(u, v)(u', v') \in E(G[H])$ if and only if either $uu' \in E(G)$ or $u = u'$ and $vv' \in E(H)$. In any of these graphs, G and H are referred to as their basic component graphs.

For vertices u and v of a graph G, a *u-v geodesic* is any shortest path in G joining u and v. The length of a u-v geodesic is the *distance* between u and v, and is denoted by $d_G(u, v)$.

The *eccentricity* of v refers to the quantity $e(v) = \max\{d_G(u, v) : v \in V(G)\}$. Customarily, $diam(G) = \max\{e(v) : v \in V(G)\}$. In this paper, we write $e(G) = \min\{e(v) : v \in V(G)\}$.

Vertices u and v of a graph G are *neighbors* if $uv \in E(G)$. The *open neighborhood* of v refers to the set $N_G(v)$ consisting of all neighbors of v. The *degree* of v refers to the cardinality $|N_G(v)|$ of the open neighborhood of v, and $\delta(G)$ is the minimum degree of a vertex of G. The *closed neighborhood* of v is the set $N_G[v] = N_G(v) \cup \{v\}$. Customarily, for $S \subseteq V(G)$, $N_G(S) = \cup_{v \in S} N_G(v)$ and $N_G[S] = \cup_{v \in S} N_G[v]$. A subset $S \subseteq V(G)$ is a *dominating set* of G if $N_G[S] = V(G)$. The minimum cardinality $\gamma(G)$ of a dominating set of G is the *domination number* of G. A dominating set of cardinality $\gamma(G)$ is called a γ-*set* of G. The reader is referred to [2, 10, 11, 13, 16, 17] for the history, fundamental concepts and recent developments of domination in graphs as well as its various applications.

A set $S \subseteq V(G)$ is a *pointwise nondominating set of* G (or *PND-set of* G) if for each $v \in V(G) \setminus S$, there exists $u \in S$ such that $uv \notin E(G)$. The smallest cardinality of a pointwise nondominating set of G, denoted by $pnd(G)$, is called the *pointwise nondomination number* of G. Any point-wise nondominating (resp. dominating pointwise nondominating) set S of G of cardinality $|S| = pnd(G)$ (resp. $|S| = \gamma_{pnd}(G)$), is called a *pnd-set* (resp. γ_{pnd}-*set*) of G. PND-sets are introduced and discussed in [5].

A subset S of $V(G)$ is a *hop dominating set* of G if for each $v \in V(G) \setminus S$, there exists $u \in S$ for which $d_G(u, v) = 2$. The minimum cardinality of a hop dominating set is called the *hop domination number* of G, and is denoted by $\gamma_h(G)$. Any hop dominating set of cardinality $\gamma_h(G)$ is called γ_h-*set* of G. Good references on hop domination include [4, 5, 15]. At times we write $S \in HD(G)$ to mean that S is a hop dominating set of G.

For a vertex v of a connected graph G, $N_G(v, 2) = \{u \in V(G) : d_G(u, v) = 2\}$. Each element of $N_G(v, 2)$ is called a *hop-neighbor* of v. For $S \subseteq V(G)$, $N_G(S, 2) = \cup_{v \in S} N_G(v, 2)$ and $N_G[S, 2] = N_G(S, 2) \cup S$. Precisely, S is a hop dominating set if and only if $N_G[S, 2] = V(G)$.

A subset S of $V(G)$ is a 2-*hop dominating set* of G if for each $v \in V(G) \setminus S$, there exist distinct vertices $u, w \in S$ for which $d_G(u, v) = 2 = d_G(w, v)$. The minimum cardinality of a 2-hop dominating set of G is the 2-*hop domination number* of G, denoted by $\gamma_{2h}(G)$. A comprehensive study on 2-hop domination is given in [14], where 2-hop domination is referred to as double hop domination. Here we also write $S \in 2\text{-}HD(G)$ to mean that S is a 2-hop dominating set of G

A set $S \subseteq V(G)$ is a $(1, 2)^*$-*dominating set of* G (resp. $(1, 2)^*$-*total dominating set*) if it is both a dominating (resp. a total dominating) set and a hop dominating set of G. The smallest cardinality of a $(1, 2)^*$ -dominating (resp. $(1, 2)^*$-total dominating) set of G, denoted by $\gamma^*_{1,2}(G)$ (resp.$\gamma^{*t}_{1,2}(G)$) is called the $(1, 2)^*$ -*domination number* (resp. $(1, 2)^*$-*total domination number*) of G. A $(1, 2)^*$ -dominating (resp. $(1, 2)^*$ -total dominating) set S with $|S| = \gamma^*_{1,2}(G)$ (resp. $|S| = \gamma^{*t}_{1,2}(G)$) is called a $\gamma^*_{1,2}$-set (resp. $\gamma^{*t}_{1,2}$-set) of G. The concept of $(1, 2)^*$-domination (a variation of $(1, 2)$-domination) is introduced in [4].

A function $f : V(G) \to \{0, 1, 2\}$ is a *hop Roman dominating function of* G if for each $v \in V(G)$ with $f(v) = 0$ there exists $u \in V(G)$ for which $d_G(u, v) = 2$ and $f(u) = 2$. The sum $\omega_G(f) = \sum_{v \in V(G)} f(v)$ is the *weight* of f in G. The minimum weight of a hop Roman dominating function of G is the *hop Roman domination number of* G, and is denoted by $\gamma_{hR}(G)$.

1.1. Some known results

Theorem 1.1. [24] *For any graph* G, $\gamma_{hR}(G) \leq 2\gamma_h(G)$.

Observation 1.2. (i) $\gamma_{hR}(P_n) = \begin{cases} 4k + r, & \text{if } n = 6k + r; 0 \leq r \leq 3; k \geq 0 \\ 4k + 4, & \text{if } n = 6k + r; 4 \leq r \leq 5; k \geq 0, \end{cases}$ *and*

$$(ii)\ \gamma_{hR}(C_n) = \begin{cases} 3, & \text{if } n = 3; \\ 4, & \text{if } n = 4, 5; \\ 4k + r, & \text{if } n = 6k + r; 0 \leq r \leq 3; k \geq 1 \\ 4k + 4, & \text{if } n = 6k + r; 4 \leq r \leq 5; k \geq 1. \end{cases}$$

2. Hop Italian domination

A function $f : V(G) \to \{0, 1, 2\}$ is a *hop Italian dominating function* (or hID*-function*) of G if for each $v \in V(G)$ with $f(v) = 0$, $\sum_{x \in N_G(v;2)} f(x) \geq 2$. More precisely, f is an hID-function of G if and only if at least one of the following holds for each $v \in V(G)$ with $f(v) = 0$:

(i) There exists $u \in V(G)$ for which $f(u) = 2$ and $d_G(u, v) = 2$;

(ii) There exist distinct $u, w \in V(G)$ for which $f(u) = 1 = f(w)$ and $d_G(u, v) = 2 = d_G(w, v)$.

The minimum weight $\sum_{v \in V(G)} f(v)$ of an hID-function of G is the *hop Italian domination number* of G, and is denoted by $\gamma_{hI}(G)$. If $hID(G)$ denotes the collection of all hID-functions of G, then

$$\gamma_{hI}(G) = \min\{\omega_G(f) : f \in hID(G)\}.$$

A hop Italian dominating function f of G with $\omega_G(f) = \gamma_{hI}(G)$ is called γ_{hI}*-function* of G.

As usual, for $f : V(G) \to \{0, 1, 2\}$ we write $f = (V_0, V_1, V_2)$, where $V_k = \{v \in V(G) : f(v) = k\}$ for each $k \in \{0, 1, 2\}$. Thus, $f = (V_0, V_1, V_2) \in hID(G)$ if and only if for each $v \in V_0$, $V_2 \cap N_G(v, 2) \neq \varnothing$ or $|V_1 \cap N_G(v)| \geq 2$.

If $f = (V_0, V_1, V_2)$ is a γ_{hI}-function of G, then $V_1 \cup V_2$ is a hop dominating set of G so that $\gamma_h(G) \leq |V_1 \cup V_2| \leq \omega_G(f) = \gamma_{hI}(G)$.

Now observe that if $S \subseteq V(G)$ is a γ_{2h}-set of G, then $f = (V(G) \setminus S, S, \varnothing) \in hID(G)$. Thus, $\gamma_{hI}(G) \leq |S| = \gamma_{2h}(G)$. Moreover, since a hop Roman dominating function is a hop Italian dominating function,

$$\gamma_{hI}(G) \leq \min\{\gamma_{hR}(G), \gamma_{2h}(G)\}. \tag{1}$$

Let G be the graph in Figure 1 obtained by joining two copies of P_5, say $[x_1, x_2, x_3, x_4, x_5]$ and $[y_1, y_2, y_3, y_4, y_5]$, using the edges x_1y_1, x_3y_3 and x_5, y_5. Then $\gamma_{hI}(G) = \gamma_{hR}(G) =$

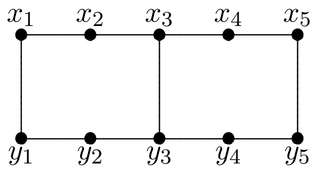

Figure 1: A graph G with $\gamma_{hI}(G) = \gamma_{hR}(G)$

$4 < 6 = \gamma_{2h}(G)$. In this case, $\gamma_{hR}(G) = \gamma_{hI}(G)$ is determined by the function $f = (V(G) \setminus \{x_3, y_3\}, \varnothing, \{x_3, y_3\})$. On the other hand, if $G = C_5$, then $\gamma_{hI}(G) = \gamma_{2h}(G) = 3 < 4 = \gamma_{hR}(G)$. However, if $G = K_p$ (the complete graph on p vertices), then $\gamma_{hI}(G) = \gamma_{hR}(G) = \gamma_{2h}(G) = p$.

Observation 2.1. *Let G be any graph. Then*

(i) $\gamma_{hI}(G) = \gamma_{hR}(G)$ *if and only if G has a γ_{hI}-function that is a hop Roman dominating function of G;*

(ii) $\gamma_{hI}(G) = \gamma_{2h}(G)$ *if and only if G has a γ_{hI}-function (V_0, V_1, V_0) for which $V_2 = \varnothing$.*

Observation 2.2. *On paths, cycles and complete bipartite graphs:*

$$(i)\ \gamma_{hI}(P_n) = \begin{cases} 2, & \text{if } n = 2; \\ 3, & \text{if } n = 3; \\ 2k+2, & \text{if } n = 4k + r \text{ with } 0 \le r \le 2; k \ge 1; \\ 2k+3, & \text{if } n = 4k + 3; k \ge 1. \end{cases}$$

$$(ii)\ \gamma_{hI}(C_n) = \begin{cases} 3, & \text{if } n = 3, 5; \\ 4, & \text{if } n = 4; \\ 2k+2, & \text{if } n = 4k + 2 + r \text{ with } 0 \le r \le 2; k \ge 1; \\ 2k+3, & \text{if } n = 4k + 5; k \ge 1. \end{cases}$$

$$(iii)\ \gamma_{hI}(K_{m,n}) = \begin{cases} 2, & \text{if } m = n = 1; \\ 3, & \text{if } m = 1 \text{ (resp. } n = 1) \text{ and } n \ge 2 \text{ (resp } m \ge 2); \\ 4, & \text{if } m \ge 2 \text{ and } n \ge 2 \end{cases}$$

2.1. Some properties and graphs with small values of γ_{hI}

Let $f = (V_0, V_1, V_2)$ be a γ_{hI}-function of G. A vertex $w \in V_0$ is an *Italian private hop-neighbor* of $v \in V_1 \cup V_2$ under f provided $\sum_{u \in N_G(w,2) \setminus \{v\}} f(u) < 2$. If no confusion arises, instead of saying *Itaian private hop-neighbor of v under f*, we simply say *Italian private hop-neighbor of v*.

Observe that the function given by $f(x) = 1$ for all $x \in V(G)$ is a γ_{hI}-function of $G = K_p$. In this case, $V_2 = \varnothing$, and such is a particular case of the following proposition.

Proposition 2.3. *For every graph G, there exists a γ_{hI}-function $f = (V_0, V_1, V_2)$ such that either $V_2 = \varnothing$ or $V_2 \neq \varnothing$ and v has at least three Italian private hop-neighbors for each $v \in V_2$.*

Proof: Let $f = (V_0, V_1, V_2)$ be a γ_{hI}-function of G with a minimum $|V_2|$. If $V_2 = \varnothing$, then the proposition holds. Suppose that $V_2 \neq \varnothing$, and let $v \in V_2$. We claim that v has at least three Italian private hop-neighbors. First, note that if v has no Italian private hop-neighbor in V_0, then $g = (V_0, V_1 \cup \{v\}, V_2 \setminus \{v\}) \in hID(G)$ with $\omega_G(g) < \omega_G(f)$, a contradiction. Next, suppose that v has exactly one Italian private hop-neighbor $w \in V_0$. If $\sum_{u \in N_G(w,2)\setminus\{v\}} f(u) = 0$, then $g = (V_0^*, V_1^*, V_2^*) \in hID(G)$ with $\omega_G(g) = \omega_G(f)$, where $V_0^* = V_0 \setminus \{w\}$, $V_1^* = V_1 \cup \{w, v\}$ and $V_2^* = V_2 \setminus \{v\}$. Since $|V_2^*| < |V_2|$, this is a contradiction to the choice of f. On the other hand, if $\sum_{u \in N_G(w,2)\setminus\{v\}} f(u) = 1$, then $g = (V_0, V_1 \cup \{v\}, V_2 \setminus \{v\}) \in hID(G)$ with $\omega_G(g) < \omega_G(f)$, a contradiction. Finally, suppose that v has exactly two neighbors w and z in V_0. Exactly one of the following holds:

(a) $\sum_{u \in N_G(w,2)\setminus\{v\}} f(u) = 0$ and $\sum_{u \in N_G(z,2)\setminus\{v\}} f(u) = 0$;

(b) $\sum_{u \in N_G(w,2)\setminus\{v\}} f(u) = 1$ and $\sum_{u \in N_G(z,2)\setminus\{v\}} f(u) = 1$;

(c) $\sum_{u \in N_G(w,2)\setminus\{v\}} f(u) = 0$ and $\sum_{u \in N_G(z,2)\setminus\{v\}} f(u) = 1$; and

(d) $\sum_{u \in N_G(w,2)\setminus\{v\}} f(u) = 1$ and $\sum_{u \in N_G(z,2)\setminus\{v\}} f(u) = 0$.

Suppose that (a) holds for f. Put $V_0^* = \{v\} \cup (V_0 \setminus \{w, z\})$, $V_1^* = V_1 \cup \{w, z\}$ and $V_2^* = V_2 \setminus \{v\}$. Then $g = (V_0^*, V_1^*, V_2^*) \in hID(G)$ with $w_G(g) = w_G(f)$. Since $|V_2^*| < |V_2|$, this is a contradiction to the assumption of f. Next, suppose that (b) holds for f. In this case, define $V_0^* = V_0$, $V_1^* = V_1 \cup \{v\}$ and $V_2^* = V_2 \setminus \{v\}$. Then $g = (V_0^*, V_1^*, V_2^*) \in hID(G)$ with $w_G(g) < w_G(f)$, a contradiction. Next, suppose that (c) holds for f. Define $V_0^* = V_0 \setminus \{w\}$, $V_1^* = V_1 \cup \{w, v\}$ and $V_2^* = V_2 \setminus \{v\}$. Then $g = (V_0^*, V_1^*, V_2^*) \in hID(G)$ with $w_G(g) = w_G(f)$. Since $|V_2^*| < |V_2|$, this is a contradiction. Similar contradiction is attained if (d) holds for f.

The above contradictions imply that v has at least three Italian private hop-neighbors . ■

Proposition 2.4. *Let G be a connected graph of order n. Then*

(i) *$\gamma_{hI}(G) = 1$ if and only if $G = K_1$;*

(ii) *$\gamma_{hI}(G) = 2$ if and only if $G = K_2$;*

(iii) *$\gamma_{hI}(G) = 3$ if and only if $\gamma_{2h}(G) = 3$ or $G = K_1 + (K_1 \cup H)$ for some graph H of order ≥ 3.*

Proof: **For** (i)**:** If $G = K_1$, then $\gamma_{hI}(G) = 1$. Conversely, if $\gamma_{hI}(G) = 1$, then $\gamma_h(G) = 1$ and so $G = K_1$.

For (ii)**:** If $G = K_2$, then $\gamma_{hI}(G) = 2$. Assume that $\gamma_{hI}(G) = 2$. By Proposition 2.3, G has a γ_{hI}-function $f = (V_0, V_1, V_2)$ for which either $V_2 = \varnothing$ or $V_2 \neq \varnothing$ and each $v \in V_2$ has at least 3 private hop-neighbors in V_0. Suppose that $V_2 \neq \varnothing$. Let $v \in V_2$ and let $u \in V_0$ be a private hop-neighbor of v. Then there exists a u-v geodesic $[u, w, v]$ in G. If $w \in V_1 \cup V_2$, then $w_G(f) \geq f(v) + f(w) \geq 3$. If $w \in V_0$ and $a \in V_2 \setminus \{v\}$ for which $d_G(a, w) = 2$, then $w_G(f) \geq f(v) + f(a) = 3$. Either case is a contradiction. Thus, $V_2 = \varnothing$ and $|V_1| = 2$. It follows that $V_0 = \varnothing$ and $|V(G)| = |V_1| = 2$. Therefore, $G = K_2$.

For (iii)**:** If $G = K_1 + (K_1 \cup H)$ for some graph H of order ≥ 3, then clearly $\gamma_{hI}(G) = 3$. Suppose that $\gamma_{2h}(G) = 3$. Then $G \notin \{K_1, K_2\}$. By (ii) and Equation 1, $\gamma_{hI}(G) = 3$. Conversely, suppose that $\gamma_{hI}(G) = 3$, and let $f = (V_0, V_1, V_2)$ be a γ_{hI}-function of G such that either $V_2 = \varnothing$ or $V_2 \neq \varnothing$ and each $v \in V_2$ has at least 3 private hop-neighbors in V_0. If $V_2 = \varnothing$, then by $\gamma_{2h}(G) = \gamma_{hI}(G) = 3$ by Observation 2.1(ii). Suppose that $V_2 \neq \varnothing$. Then $|V_2| = 1 = |V_1|$, say $V_2 = \{v\}$ and $V_1 = \{u\}$. By Proposition 2.3, v has at least 3 Italian private hop-neighbors in V_0. Thus, $G = \langle\{u\}\rangle + (\langle\{v\}\rangle \cup H) = K_1 + (K_1 \cup H)$, where $H = \langle V_0 \rangle$ of order ≥ 3. ■

It is worth noting that the family of graphs G for which $\gamma_{2h}(G) = \gamma_{hI}(G) = 3$ includes P_3, K_3, C_5, K_1-gluing of C_5 and K_2, K_2-gluing of C_5 and K_2; graph G containing $H = K_3$ such that each $v \in V(G) \setminus V(H)$ is adjacent to (exactly) one vertex of H (may be viewed as one generate by a triangle K_3; the graph G_1 in Figure 2 (may be viewed as one generated by mutually nonadjacent x, y and z); and the graph G_2 in Figure 2 (may be viewed as one generated by path $[x, y, z]$).

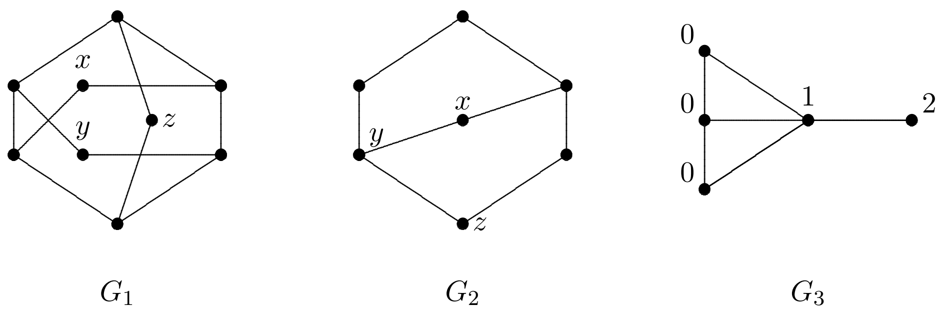

Figure 2: Examples of graphs G described in Proposition 2.4(iii) with $\gamma_{hI}(G) = 3$

Graph G_3 in Figure 2 shows an example of a graph G with $\gamma_{2h}(G) \neq 3 = \gamma_{hI}(G)$.

Proposition 2.5. (i) *For every nonnegative integer k, there exists a connected graph G for which $\gamma_{hR}(G) = \gamma_{hI}(G) + k$.*

(ii) *For every pair of positive integers a and b with $4 \leq a \leq b$, there exists a connected graph G for which $\gamma_{hI}(G) = a$ and $\gamma_{2h}(G) = b$. Consequently, for each nonnegative integer k, there exists a connected graph G with $\gamma_{2h}(G) = \gamma_{hI}(G) + k$.*

Proof: **For** (i)**:** If $k = 0$, then we take a complete graph G. Suppose that $k \geq 1$. First, suppose that k is even, say $k = 2j$ for some integer $j \geq 1$. Choose G to be the path P_n, where $n = 12j + 3$. Writing $n = 6(2j) + 3$, Observation 1.2 yields $\gamma_{hR}(G) = \gamma_{hR}(P_n) = 4(2j) + 3 = 8j + 3$. Similarly, by Observation 2.2, $\gamma_{hI}(G) = 2(3j) + 3$. Thus, $\gamma_{hR}(G) = (6j + 3) + 2j = \gamma_{hI}(G) + k$.

Next, suppose that $k = 2j + 1$ for some integer $j \geq 0$. If $j = 0$, then we take $G = C_7$. Assume that $j \geq 1$. Consider the graph $G = C_n$, a cycle on n vertices, where $n = 12j + 3$. By Observation 1.2 and Observation 2.2, $\gamma_{hR}(G) = 4(2j) + 3 = (6j + 2) + (2j + 1) = [(2(3j) + 2] + (2j + 1) = \gamma_{hI}(G) + k$.

For (ii): If $a = b$, then we take $G = K_a$, the complete graph on a vertices. Suppose that $b = a + k$ with $k \geq 1$. We consider the following cases:

Case 1: Suppose that $a = 2n+2$ for some $n \geq 1$. Let $t = 4n$, and put $P_t = [x_1, x_2, \ldots, x_t]$, a path on t vertices. If $n = 1$, then we take $G = G_1$, where G_1 is the graph in Figure 3 obtained from P_4 by adding $k + 1$ distinct paths $[x_3, y_j, z_j]$, $j = 1, 2, \ldots, k + 1$. Define

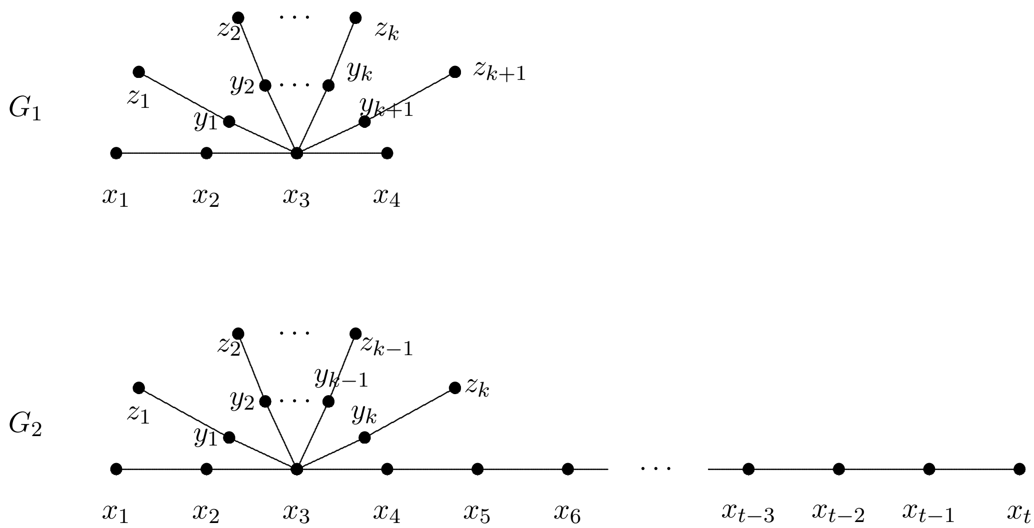

Figure 3: Examples of graphs G with $\gamma_{hI}(G) = a$ and $\gamma_{2h}(G) = b$

$V_2 = \{x_3, x_4\}$, $V_1 = \varnothing$ and $V_0 = V(G) \setminus \{x_3, x_4\}$. Then $f = (V_0, V_1, V_2)$ is a γ_{hI}-function of G. Thus, $\gamma_{hI}(G) = 4 = a$. On the other hand, the set $\{x_1, x_2, x_4\} \cup \{z_j : j = 1, 2, \ldots, k+1\}$ is a γ_{2h}-set of G, implying that $\gamma_{2h}(G) = 3+k+1 = 4+k = b$. Suppose that $n \geq 2$. Obtain G as the graph G_2 in Figure 3 from P_t by adding k distinct paths $[x_3, y_j, z_j]$, $j = 1, 2, \ldots, k$. Define $V_2 = \{x_3, x_4\}$, $V_1 = \{x_7, x_8, x_{11}, x_{12}, \ldots, x_{t-1}, x_t\}$ and $V_0 = V(G) \setminus (V_1 \cup V_2)$. Then $f = (V_0, V_1, V_2)$ is a γ_{hI}-function of G, implying that $\gamma_{hI}(G) = \gamma_{hI}(P_t) = 2n + 2 = a$. On the other hand, necessarily, $S = \{z_j : j = 1, 2, \ldots, k\}$ is contained in any 2-hop dominating set of G. Observe that $S \cup \{x_1, x_2, x_4, x_5, x_8, x_9, \ldots, x_{t-4}, x_{t-3}, x_{t-1}, x_t\}$ is a γ_{2h}-set of G. Thus, $\gamma_{2h}(G) = (2n + 2) + k = a + k = b$.

Case 2: Suppose that $a = 2n + 3$ for some $n \geq 1$. Put $t = 4n$ and let $P_t = [x_1, x_2, \ldots, x_t]$. If $n = 1$, then obtain G as the graph G_1 in Figure 4by adding to P_6 k geodesics, namely

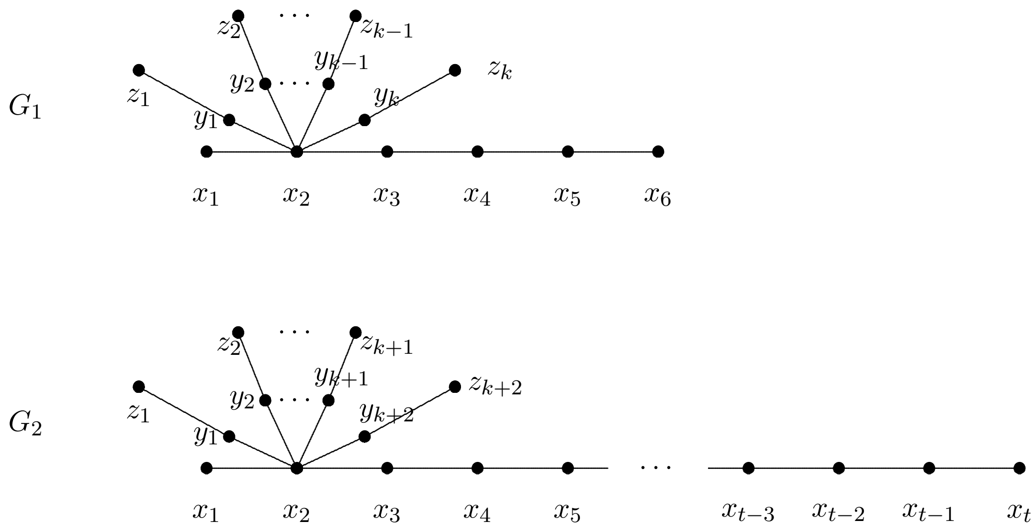

Figure 4: Examples of graphs G with $\gamma_{hI}(G) = a$ and $\gamma_{2h}(G) = b$

$[x_2, y_j, z_j]$, $j = 1, 2, \ldots, k$. Then $\gamma_{hI}(G) = 5 = a$, which is determined by the γ_{hI}-function $f = (V_0, V_1, V_2)$ with $V_1 = \{x_6\}$, $V_2 = \{x_2, x_3\}$ and $V_0 = V(G) \setminus \{x_2, x_3, x_6\}$. On the other hand, $S = \{z_j : j = 1, 2, \ldots, k\}$ is always contained in a 2-hop dominating set of G so that $S \cup (V(P_6) \setminus \{x_2\})$ is a γ_{2h}-set of G. Thus, $\gamma_{2h}(G) = 5 + k = b$. Now, suppose that $n \geq 2$. Obtain G as the graph G_2 in Figure 4 from P_t by adding k distinct paths $[x_3, y_j, z_j]$, $j = 1, 2, \ldots, k + 2$. Define $V_2 = \{x_2\}$, $V_1 = \{x_1, x_3, x_4, x_7, x_8, \ldots, x_{t-1}, x_t\}$ and $V_0 = V(G) \setminus (V_1 \cup V_2)$. Then $f = (V_0, V_1, V_2)$ is a γ_{hI}-function of G, implying that $\gamma_{hI}(G) = \gamma_{hI}(P_t) = 2n + 3 = a$. On the other hand, if $S = \{z_j : j = 1, 2, \ldots, k\}$, then $S \cup \{x_1, x_3, x_4, x_7, x_8, x_{11}, x_{12}, \ldots, x_{t-1}, x_t\}$ is a γ_{2h}-set of G. Thus, $\gamma_{2h}(G) = (2n + 1) + (k + 2) = a + k = b$. ■

2.2. PND_I-functions

A function $f = (V_0, V_1, V_2)$ on $V(G)$ is a *PND_I-function* of G if for each $v \in V_0$ one of the following holds:

(i) there exists $u \in V_2$ for which $v \notin N_G(u)$;

(ii) there exist vertices u and w in V_1 for which $v \notin N_G(u) \cup N_G(w)$.

The minimum weight of an PND_I-function of G is the *PND_I number* of G, denoted by $pnd_I(G)$. Any PND_I-function of G with weight equal to $pnd_I(G)$ is a *pnd_I-function.*

Example 2.6. (1) $pnd_I(P_n) = \begin{cases} 1, & \text{if } n = 1; \\ 2, & \text{if } n = 2; \\ 3, & \text{if } n \geq 3. \end{cases}$

(2) $pnd_I(C_n) = \begin{cases} 4, & \text{if } n = 4; \\ 3, & \text{otherwise} \end{cases}$.

(3) $pnd_I(K_p) = p$ *for* $p \geq 1$ *and* $pnd_I(K_{m,n}) = 4$ *for* $m, n \geq 2$.

If $f = (V_0, V_1, V_2)$ is a PND_I-function of G, then $V_1 \cup V_2$ is a PND-set of G. Thus, $pnd(G) \leq |V_1| + |V_2| \leq \omega_G(f)$ for all PND_I-functions $f = (V_0, V_1, V_2)$ of G. On the other hand, if $S \subseteq V(G)$ is a PND-set of G, then $f = (V(G) \setminus S, \varnothing, S)$ is a PND_I-function of G. Also, every hop Italian dominating function is a PND_I-function. Thus, $pnd(G) \leq pnd_I(G) \leq \min\{2\, pnd(G), \gamma_{hI}(G)\}$.

Observation 2.7. *Let* G *be any graph. Then*

(*i*) $pnd_I(G) = 1$ *if and only if* $G = K_1$*;*

(*ii*) $pnd_I(G) = 2$ *if and only if either* $G = K_2$ *or* G *is a nontrivial graph with an isolated vertex;*

(*iii*) $pnd_I(G) = 3$ *if and only if one of the following holds:*

  (*a*) G *has an endvertex;*

  (*b*) G *has a set of vertices* $S = \{x, y, z\}$ *for which every* $v \in V(G) \setminus S$ *is adjacent to at most one vertex in* S.

Lemma 2.8. *Let* G *be a noncomplete graph. Then* G *admits a* pnd_I*-function* $f = (V_0, V_1, V_2)$ *for which* $V_2 \neq \varnothing$.

Proof: Let $f = (V_0, V_1, V_2)$ be a pnd_I-function of G with $V_2 = \varnothing$. Then $V(G) = V_1$. Since G is noncomplete, there exist $u, v \in V(G)$ such that$d_G(u, v) = 2$. Observe that $g = (\{u\}, V_1 \setminus \{u, v\}, \{v\})$ is a PND_I-function of G with $\omega_G(g) = \omega_G(f)$. ■

3. Graphs under binary operations

In view of Proposition 2.4, $\gamma_{hI}(G\overline{G}) \geq 2$ for any graph G.

Proposition 3.1. (complementary prism of graphs) *Let* G *be any graph. Then*

(*i*) $\gamma_{hI}(G\overline{G}) = 2$ *if and only if* $G = K_1$.

(*ii*) $\gamma_{hI}(G\overline{G}) = 4$ *for all nontrivial graphs* G.

Proof: If $G = K_1$, then $\gamma_{hI}(G\overline{G}) = \gamma_{hI}(P_2) = 2$. Conversely, if $\gamma_{hI}(G\overline{G}) = 2$, then $G\overline{G} = K_2$ by Proposition 2.4(ii). This means that $G = K_1$.

Suppose that $G \neq K_1$. First, we claim that $\gamma_{hI}(G\overline{G}) \geq 4$. By ($i$), $\gamma_{hI}(G\overline{G}) \geq 3$. Suppose that $\gamma_{hI}(G\overline{G}) = 3$. In view of Proposition 2.4(iii), $\gamma(G\overline{G}) = 1$. This is possible only when $G\overline{G} = K_2$ so that $\gamma_{hI}(G\overline{G}) = 2$, a contradiction. Thus, $\gamma_{hI}(G\overline{G}) \geq 4$. Let $v \in V(G)$ and define $V_2 = \{v, \overline{v}\}$, $V_1 = \varnothing$ and $V_0 = V(G\overline{G}) \setminus \{v, \overline{v}\}$. Let $z \in V_0$. Assume that $z \in V(G)$ (the case where $z \in V(\overline{G})$ is done similarly). If $zv \in E(G)$, then $[z, v, \overline{v}]$ is a geodesic in $G\overline{G}$ so that $\overline{v} \in V_2 \cap N_{G\overline{G}}(z, 2)$. On the other hand, if $zv \notin E(G)$, then $[z, \overline{z}, \overline{v}]$ is a geodesic in $G\overline{G}$ so that $\overline{v} \in V_2 \cap N_{G\overline{G}}(z, 2)$. This shows that $f = (V_0, V_1, V_2) \in hI(G\overline{G})$, and consequently, $\gamma_{hI}(G\overline{G}) \leq \omega_{G\overline{G}}(f) = 4$. ■

From Proposition 3.1(ii), for $n \geq 5$, $\gamma_{hI}(K_n\overline{K_n}) = 4$ while $\max\{\gamma_{hI}(K_n), \gamma_{hI}(\overline{K_n})\} = n$. Thus, contrary to the case of Italian domination in complementary prisms, it is not always true that $\gamma_{hI}(G\overline{G}) \geq \max\{\gamma_{hI}(G), \gamma_{hI}(\overline{G})\}$.

Corollary 3.2. *For all graphs G,*

$$\gamma_{hI}(G\overline{G}) \leq \gamma_{hI}(G) + \gamma_{hI}(\overline{G}),$$

and this bound is sharp.

Proof: If $G = K_1$, then by Proposition 3.1(i) and Proposition 2.4(i), $\gamma_{hI}(G\overline{G}) = 2 = \gamma_{hI}(G) + \gamma_{hI}(\overline{G})$. Suppose that $G \neq K_1$. Since $2 \leq \gamma_{hI}(G)$ and $2 \leq \gamma_{hI}(\overline{G})$, $4 \leq \gamma_{hI}(G) + \gamma_{hI}(\overline{G})$. The conclusion follows immediately from Proposition 3.1.

To show sharpness of the bound, consider $G = P_2$. By Observation 2.1(ii), $\gamma_{hI}(G\overline{G}) = \gamma_{hI}(P_4) = 4 = \gamma_{hI}(G) + \gamma_{hI}(\overline{G})$. ■

Strict inequality can be obtained in Theorem 3.2. Consider $G = K_1 \cup K_3$. The graph $G\overline{G}$ is as shown in Figure 5. For this graph, $\gamma_{hI}(G\overline{G}) = 4$, $\gamma_{hI}(G) = 4$ and $\gamma_{hI}(\overline{G}) = 3$.

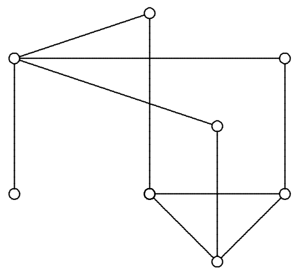

Figure 5: The graph of $G\overline{G}$ where $G = K_1 \cup K_3$

Theorem 3.3. (join of graphs) *Let G and H be any graphs, and* $f = (V_0, V_1, V_0)$ *be a function on* $V(G + H)$*. Then* $f \in hID(G + H)$ *if and only if* $f|_G \in PND_I(G)$ *and* $f|_H \in PND_I(H)$*, where* $f|_G$ *and* $f|_H$ *are the restrictions of f to G and H, respectively.*

Proof: Let $f = (V_0, V_1, V_2) \in hID(G+H)$. Let $v \in V_0 \cap V(G)$. Then $|V_2 \cap N_{G+H}(v,2)| \geq 1$ or $|V_1 \cap N_{G+H}(v,2)| \geq 2$. Suppose that $|V_2 \cap N_{G+H}(v,2)| \geq 1$, and let $u \in V_2 \cap N_{G+H}(v,2)$. Since $d_{G+H}(u,v) = 2$, $u \in V_2 \cap V(G)$ and $u \notin N_G(v)$. Suppose, on the other hand, that $|V_1 \cap N_{G+H}(v,2)| \geq 2$, say $u, w \in V_1 \cap N_{G+H}(v,2)$. Then $u, w \in V_1 \cap V(G)$ and $v \notin N_G(u) \cup N_G(w)$. This shows that $f|_G = (V_0 \cap V(G), V_1 \cap V(G), V_2 \cap V(G)) \in PND_I(G)$. Similarly, $f|_H \in PND_I(H)$.

Conversely, let $v \in V_0$. Suppose that $v \in V(G)$. If $f|_G \in PND_I(G)$, then there exists $u \in V_2 \cap V(G)$ for which $v \notin N_G(u)$ or there exist $w, z \in V_1 \cap V(G)$ for which $v \notin N_G(w) \cup N_G(z)$. The former implies that $u \in V_2 \cap N_{G+H}(v,2)$, while the latter implies that $w, z \in V_1 \cap N_{G+H}(v,2)$. Similarly, if $v \in V(H)$ and $f|_H \in PND_I(H)$, then $|V_2 \cap N_{G+H}(v,2)| \geq 1$ or $|V_1 \cap N_{G+H}(v,2)| \geq 2$. Therefore, $f \in hID(G+H)$. ■

Corollary 3.4. *Let G and H be any graphs of orders m and n, respectively. Then*

$$\gamma_{hI}(G+H) = pnd_I(G) + pnd_I(H).$$

In particular,

(i) $\gamma_{hI}(G+H) = m+n$ *if G and H are complete graphs;*

(ii) $\gamma_{hI}(G+H) = 4$ *if both G and H have isolated vertices;*

(iii) $\gamma_{hI}(G+H) = 1 + pnd_I(H)$ *if $G = K_1$;*

Proposition 3.5. *Let G be a graph with no isolated vertices. Then $\gamma_{hI}(G \circ H) \leq \gamma^{*t}_{1,2}(G)$.*

Proof: Let $S \subseteq V(G)$ be a $\gamma^{*t}_{1,2}$-set of G, and define $f = (V_0, V_1, V_2)$, where $V_0 = V(G \circ H) \setminus S$, $V_1 = \varnothing$ and $V_2 = S$. Let $v \in V_0 \cap V(G)$. Since V_2 is a hop dominating set of G, there exists $u \in V_2$ for which $d_G(u,v) = 2$. Let $v \in V_0 \cap V(H^u)$, where $u \in V(G)$. Since V_2 is a total dominating set of G, there exists $w \in V_2 \cap N_G(u)$. Then $d_{G \circ H}(u,w) = 2$. Thus, $f \in hID(G \circ H)$. Consequently, $\gamma_{hI}(G \circ H) \leq \omega_{G \circ H}(f) = 2|S| = 2\gamma^{*t}_{1,2}(G)$. ■

Theorem 3.6. (corona of graphs) *Let G be a nontrivial connected graph and H any graph, and let $f = (V_0, V_1, V_2)$ be a function on $V(G \circ H)$. Then $f \in hID(G \circ H)$ if and only if each of the following holds:*

(i) One of the following holds for each $v \in V_0 \cap V(G)$:

- *(a)* $|V_2 \cap N_G(v,2)| \geq 1$ *or* $|V_1 \cap N_G(v,2)| \geq 2$*;*
- *(b) There exists $w \in N_G(v)$ for which $|V_2 \cap V(H^w)| \geq 1$;*
- *c) There exists $w \in N_G(v)$ for which $|V_1 \cap V(H^w)| \geq 2$;*
- *(d) There exist $u, w \in N_G(v)$ for which $|V_1 \cap V(H^w)| = 1 = |V_1 \cap V(H^u)|$;*
- *(e)* $|V_1 \cap N_G(v,2)| = 1$ *and there exists $w \in N_G(v)$ for which $|V_1 \cap V(H^w)| = 1$.*

(ii) Each of the following holds for every $v \in V(G)$ with $V_2 \cap N_G(v) = \varnothing$:

- *(a) $f|_{H^v}$ is a PND_I-function of H^v if $N_G(v) \subseteq V_0$;*

(b) $V(H^v) \setminus V_0$ *is a* PND*-set of* H^v *if* $|V_1 \cap N_G(v)| = 1$.

Proof: Suppose that $f \in hID(G \circ H)$. Then (i) is clear. Let $v \in V(G)$ with $V_2 \cap N_G(v) = \varnothing$. Suppose that $N_G(v) \subseteq V_0$, and let $u \in V_0 \cap V(H^v)$. Then $|V_2 \cap N_{G\circ H}(u, 2)| \geq 1$ of $|V_1 \cap N_{G\circ H}(u, 2)| \geq 2$. Since $N_G(v) \subseteq V_0$, the preceding statement implies that $|V_2 \cap N_{H^v}(u, 2)| \geq 1$ or $|V_1 \cap N_{H^v}(u, 2)| \geq 2$. It means that there exists $w \in V_2 \cap V(H^v)$ for which $u \notin N_{H^v}(w)$ or there exist w and z in $V_1 \cap V(H^v)$ for which $u \notin N_{H^v}(w) \cup N_{H^v}(z)$. Thus, $f|_{H^v} = (V_0 \cap V(H^v), V_1 \cap V(H^v), V_2 \cap V(H^v))$ is a PND_I-function of H^v and $(ii)(a)$ holds. Suppose that $|V_1 \cap N_G(v)| = 1$. Let $u \in V_0 \cap V(H^v)$. Following similar argument, since $|V_1 \cap N_G(v)| = 1$, we have $|V_2 \cap N_{H^v}(u, 2)| \geq 1$ or $|V_1 \cap N_{H^v}(u, 2)| \geq 1$. In any case, there exists $w \in V(H^v) \setminus V_0$ such that $u \notin N_{H^v}(w)$, showing that $V(H^v) \setminus V_0$ is a PND-set of H^v. This proves $(ii)(b)$.

Conversely, suppose that (i) and (ii) hold for f. Let $v \in V_0$. If $v \in V(G)$, then (i) implies the existence of $u \in V_2$ such that $d_{G\circ H}(u, v) = 2$ or of vertices u and w in V_1 such that $d_{G\circ H}(u, v) = 2 = d_{G\circ H}(w, v)$. Now, suppose that $v \in V(H^u)$ for some $u \in V(G)$. If $V_2 \cap N_G(u) \neq \varnothing$, and $w \in V_2 \cap N_G(u)$, then w is the desired vertex for which $w \in V_2$ and $d_{G\circ H}(v, w) = 2$. Suppose that $V_2 \cap N_G(u) = \varnothing$. We consider two cases:

Case 1: If $N_G(u) \subseteq V_0$, then by condition $(ii)(a)$, there exists there exists $w \in V_2 \cap V(H^u)$ for which $v \notin N_{H^u}(w)$ or there exist vertices z and w in $V_1 \cap V(H^u)$ for which $v \notin N_{H^u}(w) \cup N_{H^v}(z)$. The former implies that $d_{G\circ H}(w, v) = 2$, while latter implies that $d_{G\circ H}(w, v) = 2 = d_{G\circ H}(z, v)$.

Case 2: Suppose that $N_G(u) \cap V_1 \neq \varnothing$. If $|N_G(u) \cap V_1| \geq 2$, say $w, z \in N_G(u) \cap V_1$, then $d_{G\circ H}(w, v) = 2 = d_{G\circ H}(z, v)$. Suppose that $|N_G(u) \cap V_1| = 1$, say $x \in N_G(u) \cap V_1$. By $(ii)(b)$, $V(H^u) \setminus V_0$ is a PND-set of H^u so that there exists $w \in V(H^u) \setminus V_0$ such that $v \notin N_{H^u}(w)$. We either have $w \in V_2$ and $d_{G\circ H}(w, v) = 2$ or $w \in V_1$ and $d_{G\circ H}(w, v) = 2 = d_{G\circ H}(x, v)$.

Accordingly, $f \in hID(G \circ H)$. ■

Corollary 3.7. *Let* G *be a connected graph of order* n *and* H *be any graph.*

(i) *If* $\gamma(G) = 1$, *then* $4 \leq \gamma_{hI}(G \circ H) \leq 6$. *More precisely,*

  (a) $\gamma_{hI}(G \circ H) = 4$ *if* $\gamma_h(G) = 2$ *or* $H = K_2$ *or* H *has an isolated vertex;*

  (b) $\gamma_{hI}(G \circ H) = 5$ *if* $pnd_I(H) = 3$; *and*

  (c) $\gamma_{hI}(G \circ H) = 6$ *if* $pnd_I(H) \geq 4$.

(ii) *In general,*

$$4 \leq \gamma_{hI}(G \circ H) \leq \rho_H(G),$$

where $\rho_H(G) = \min\{2|S| + (n - |N_G(S)|)\, pnd_I(H) : S \in HD(G)\}$, *and this bound is tight.*

Proof: In any case $\gamma_{hI}(G \circ H) \geq 4$ by Proposition 2.4. Suppose that $\gamma(G) = 1$, and let $u \in V(G)$ for which $N_G[u] = V(G)$. Pick $v^u \in V(H^u)$ and $w \in V(G) \setminus \{u\}$. Put $S = \{u, v^u, w\}$, and define $V_2 = S$, $V_1 = \varnothing$ and $V_0 = V(G \circ H) \setminus S$. By Theorem 3.6, $f = (V_0, V_1, V_2) \in hID(G \circ H)$. Thus, $\gamma_{hI}(G \circ H) \leq \omega_{G \circ H}(f) = 6$.

Suppose that $\gamma_h(G) = 2$, and let S be a γ_h-set of G. Necessarily, $u \in S$. Put $S = \{u, v\}$, where $d_G(x, v) = 2$ for all $x \in V(G) \setminus \{u\}$. Define $V_2 = S$, $V_1 = \varnothing$ and $V_0 = (V(G) \setminus S) \cup \big(\cup_{x \in V(G)} V(H^x)\big)$. Since $V(H^u) \cup (V(G) \setminus S) \subseteq N_{G \circ H}(v, 2)$ and $\cup_{x \in V(G) \setminus \{u\}} V(H^x) \subseteq N_{G \circ H}(u, 2)$, $f \in hID(G \circ H)$. Thus, $\gamma_{hI}(G \circ H) \leq \omega_{G \circ H}(f) = 4$.

Suppose that H has an isolated vertex v. Put $S = \{u, v^u\}$, where v^u is the copy of vertex v in H^u. Define $V_2 = S$, $V_1 = \varnothing$ and $V_0 = (V(G) \setminus \{u\}) \cup \big(\big(\cup_{x \in V(G)} V(H^x)\big) \setminus \{v^u\}\big)$. Then $(V(G) \setminus \{u\}) \cup (V(H^u) \setminus \{v^u\}) \subseteq N_{G \circ H}(v^u, 2)$ and $\cup_{x \in V(G) \setminus \{u\}} V(H^x) \subseteq N_{G \circ H}(u, 2)$. Thus, $f \in hID(G \circ H)$ so that $\gamma_{hI}(G \circ H) \leq \omega_{G \circ H}(f) = 4$.

Suppose that $\gamma_h(G) \neq 2$ and $pnd_I(H) \geq 2$. Let $f_u = (V_0^u, V_1^u, V_2^u)$ be a pnd_I-function of H^u. Define $V_2 = \{u\} \cup V_2^u$, $V_1 = V_1^u$ and $V_0 = (V(G) \setminus \{u\}) \cup \big(\cup_{x \in V(G) \setminus \{u\}} V(H^x)\big) \cup V_0^u$. Then $f = (V_0, V_1, V_2) \in hID(G)$ with $\omega_{G \circ H}(f) = 2 + (|V_1^u| + 2|V_2^u|) = 2 + pnd_I(H)$. If $pnd_I(H) = 2$ (i.e., $H = K_2$), then $\gamma_{hI}(G \circ H) = 4$. If $pnd_I(H) = 3$, then the preceding result implies that $\gamma_{hI}(G \circ H) = 5$; and by a similar reason, if $pnd_I(H) \geq 4$, then $\gamma_{hI}(G \circ H) = 6$.

To prove (ii), let $S \subseteq V(G)$ be a hop dominating set of G. For each $v \in V(G) \setminus N_G(S)$, let $f_v = (V_0^v, V_1^v, V_2^v)$ be a pnd_I-function of $H = H^v$. Define the following

- $V_0 = [V(G) \setminus S] \cup \big[\cup_{v \in V(G) \cap N_G(S)} V(H^v)\big] \cup \big[\cup_{v \in V(G) \setminus N_G(S)} V_0^v\big]$;
- $V_1 = \cup_{v \in V(G) \setminus N_G(S)} V_1^v$;
- $V_2 = S \cup \big[\cup_{v \in V(G) \setminus N_G(S)} V_2^v\big]$.

Put $f = (V_0, V_1, V_2)$. Let $v \in V_0 \cap V(G)$. Since S is a hop dominating set of G and $v \in V(G) \setminus S$, there exists $u \in S \subseteq V_2 \cap V(G)$ for which $d_G(u, v) = 2$, showing that condition $(i)(b)$ of Theorem 3.6 is satisfied. Let $v \in V(G)$ for which $V_2 \cap N_G(v) = \varnothing$. Since $V_1 \cap V(G) = \varnothing$, $v \in V(G) \setminus N_G(S)$. Then $f|_{H^v} = f_v$, and therefore $f|_{H^v}$ is a PND_I-function of H^v. By Theorem 3.6, $f \in hID(G \circ H)$. Moreover,

$$\begin{aligned}
\gamma_{hI}(G \circ H) &\leq \omega_{G \circ H}(f) \\
&= |V_1| + 2|V_2| \\
&= \sum_{v \in V(G) \setminus N_G(S)} |V_1^v| + 2|S| + \sum_{v \in V(G) \setminus N_G(S)} |V_2^v| \\
&= 2|S| + \sum_{v \in V(G) \setminus N_G(S)} (|V_1^v| + 2|V_2^v|) \\
&= 2|S| + [n - |N_G(S)|]\, pnd_I(H).
\end{aligned}$$

Since S is arbitrary, $\gamma_{hI}(G \circ H) \leq \rho_H(G)$.

Consider $G = P_4$. For any graph H, $\gamma_{hI}(G \circ H) = 4 = \rho_H(G)$. This proves the tightness of the bound. ■

It is also worth noting that for graphs G with no isolated vertices, Corollary 3.7 is an improvement of Proposition 3.5 as $\rho_H(G) \leq 2\gamma^{*t}_{1,2}(G)$.

Theorem 3.8. (lexicographic product of graphs) *Let G and H be connected graphs and $f = (V_0, V_1, V_2)$ a function on $V(G[H])$. Let A, B and C be subsets of $V(G)$ and let A_x, B_x and C_x be subsets of $V(H)$ such that $V_0 = \cup_{x\in A}(\{x\} \times A_x)$, $V_1 = \cup_{x\in B}(\{x\} \times B_x)$ and $V_2 = \cup_{x\in C}(\{x\} \times C_x)$. Then $f \in hID(G[H])$ if and only if each of the following holds:*

- (i) *$B \cup C$ is a hop dominating set of G;*
- (ii) *For each $x \in A$ for which $C \cap N_G(x, 2) = \varnothing$, one of the following holds:*
  - (a) *$|B \cap N_G(x, 2)| \geq 2$;*
  - (b) *$B \cap N_G(x, 2) = \{w\}$ such that $|B_w| \geq 2$;*
  - (c) *$x \in B \cup C$, $B \cap N_G(x, 2) = \{w\}$ with $|B_w| = 1$ and $B_x \cup C_x$ is a PND-set of H;*
  - (d) *$x \in B \cup C$, $B \cap N_G(x, 2) = \varnothing$ and the restriction $f|_{\langle\{x\}\times V(H)\rangle}$ of f on $\langle\{x\} \times V(H)\rangle$ is a PND_I-function of $\langle\{x\} \times V(H)\rangle$.*

Proof: Assume that $f \in hID(G[H])$. Then $V_1 \cup V_2$ is a hop dominating set of $G[H]$. This implies that $B \cup C$ is a hop dominating set of G, and (i) holds. Next, to prove (ii), let $x \in A$ for which $C \cap N_G(x, 2) = \varnothing$. We consider the following cases:

Case 1: Suppose that $B \cap N_G(x, 2) = \varnothing$. Since $B \cup C$ hop-dominates A, $x \in B \cup C$. Put $T_x = \langle\{x\}\times V(H)\rangle$. Let $y \in A_x$. Then $|V_2 \cap N_{G[H]}((x, y), 2)| \geq 1$ or $|V_1 \cap N_{G[H]}((x, y), 2)| \geq 2$. If $(u, v) \in V_2 \cap N_{G[H]}((x, y), 2)$, then $x = u$ so that $(u, v) \in V_2^x = V_2 \cap V(T_x)$ and $(x, y)(u, v) \notin E(T_x)$. On the other hand, if $(u, v), (w, z) \in V_1 \cap N_{G[H]}((x, y), 2)$, then $u = w = x$ so that $(u, v), (w, z) \in V_1^x = V_1 \cap V(T_x)$ and $(x, y)(u, v), (x, y)(w, z) \notin E(T_x)$. Since y is arbitrary, $f|_{T_x} = (V_0^x, V_1^x, V_2^x)$ is a PND_I-function of T_x, where $V_0^x = V_0 \cap V(T_x)$. This proves $(ii)(d)$.

Case 2: Suppose that $B \cap N_G(x, 2) \neq \varnothing$. If $|B \cap N_G(x, 2)| \geq 2$, then $(ii)(a)$ is done. Note that such holds particularly when $x \notin B \cup C$, $y \in A_x$ and we have $(u, v), (w, z) \in V_1 \cap N_{G[H]}((x, y), 2)$ with $u \neq w$.

Assume that $|B \cap N_G(x, 2)| = 1$, say $B \cap N_G(x, 2) = \{w\}$. If $|B_w| \geq 2$, then $(ii)(b)$ holds. Note that this readily follows if $x \notin B \cup C$. Now suppose that $|B_w| = 1$. Since $C \cap N_G(x, 2) = \varnothing$, necessarily $x \in B \cup C$. We claim that $B_x \cup C_x$ is a PND-set of H. Let $y \in V(H) \setminus (B_x \cup C_x) = A_x \setminus (B_x \cup C_x)$. Then $(x, y) \in V_0 \setminus (V_1 \cup V_2)$. There exists $(a, b) \in V_2 \cap N_{G[H]}((x, y), 2)$ or there exist distinct $(a, b), (s, t) \in V_1 \cap N_{G[H]}((x, y), 2)$. The former implies that $b \in C_x$ and $by \notin E(H)$. Since $|B_w| = 1$, the latter implies that $a \in B_x$ and $by \notin E(H)$ or $s \in B_x$ and $ty \notin E(H)$. Accordingly, $B_x \cup C_x$ is a PND-set of H. This proves $(ii)(d)$.

Conversely, suppose that conditions (i) and (ii) all hold for f. Let $(x, y) \in V_0$. Then $x \in A$. If $u \in C \cap N_G(x, 2)$, then for any $v \in C_u$, $(u, v) \in V_2 \cap N_{G[H]}((x, y), 2)$. Now assume that $C \cap N_G(x, 2) = \varnothing$. It is straightforward to show that if $(ii)(a)$ or $(ii)(b)$ holds for x, then $|V_1 \cap N_{G[H]}((x, y), 2)| \geq 2$. Suppose that $(ii)(c)$ holds for x. Let $B \cap N_G(x, 2) = \{w\}$ and let $z \in B_w$. If $t \in C_x$ for which $ty \notin E(H)$, then $(x, t) \in V_2 \cap N_{G[H]}((x, y), 2)$. On the other hand, if $t \in B_x$ for which $ty \notin E(H)$, then (x, t) and (w, z) are distinct vertices in $V_1 \cap N_{G[H]}((x, y), 2)$. Finally, suppose that $(ii)(d)$ holds for x. Put $T_x = \langle \{x\} \times V(H) \rangle$ and define $V_i^x = V_i \cap V(T_x)$ for $i = 0, 1, 2$. Then $f|_{T_x} = (V_0^x, V_1^x, V_2^x)$. Since $(x, y) \in V_0^x$ and $f|_{T_x}$ is a PND_I-function of T_x, there exists $(x, v) \in V_2^x$ with $(x, y)(x, v) \notin E(T_x)$ or there exist distinct $(x, v), (x, z) \in V_1^x$ such that $(x, y)(x, v), (x, y)(x, z) \notin E(T_x)$. The former implies that $(x, v) \in V_2 \cap N_{G[H]}((x, y), 2)$, while the latter implies that $(x, v), (x, z) \in V_1 \cap N_{G[H]}((x, y), 2)$. Therefore, $f \in hID(G[H])$. ■

Corollary 3.9. *Let G and H be nontrivial connected graphs where H is noncomplete. Then*

$$\gamma_{hI}(G[H]) \leq \min\{2|S \cap N_G(S, 2)| + pnd_I(H)|S \setminus N_G(S, 2)| : S \in HD(G)\},$$

and this bound is sharp.

Proof: Put $\alpha_H(G) = \min\{2|S \cap N_G(S, 2)| + pnd_I(H)|S \setminus N_G(S, 2)| : S \in HD(G)\}$. Let $S \subseteq V(G)$ be a hop dominating set of G. For each $x \in S \setminus N_G(S, 2)$, let $f_x = (V_0^x, V_1^x, V_2^x)$ be a pnd_I-function of $\langle \{x\} \times V(H) \rangle$. By Lemma 2.8, since $\langle \{x\} \times V(H) \rangle$ is noncomplete, we assume that $V_2^x \neq \varnothing$ for each $x \in S \setminus N_G(S, 2)$. Pick $y \in V(H)$. Define the following sets:

- $V_2 = \left[\cup_{x \in S \cap N_G(S,2)} \{(x, y)\}\right] \cup \left[\cup_{x \in S \setminus N_G(S,2)} V_2^x\right]$;
- $V_1 = \cup_{x \in S \setminus N_G(S,2)} V_1^x$; and
- $V_0 = V(G[H]) \setminus (V_1 \cup V_2)$.

Let $f = (V_0, V_1, V_2)$. As in Theorem 3.8, write $V_0 = \cup_{x \in A} (\{x\} \times A_x)$, $V_1 = \cup_{x \in B} (\{x\} \times B_x)$ and $V_2 = \cup_{x \in C} (\{x\} \times C_x)$. Since $V_2^x \neq \varnothing$ for each $x \in S \setminus N_G(S, 2)$, $C = S$ and $B = S \setminus N_G(S, 2)$. Thus $B \cup C$ is a hop dominating set of G. Let $x \in A$ with $C \cap N_G(x, 2) = \varnothing$. Since C is a hop dominating set of G, $x \in C \setminus N_G(C, 2) = B$. Note that if $B \cap N_G(x, 2) \neq \varnothing$ and $u \in B \cap N_G(x, 2)$, then $u \in C \cap N_G(x, 2)$, a contradiction. Thus, $B \cap N_G(x, 2) = \varnothing$. Since $f|_{\langle \{x\} \times V(H) \rangle} = f_x$ for each $x \in C \setminus N_G(C, 2)$, $f \in hID(G[H])$ by Theorem 3.8. Therefore,

$$\begin{aligned} \gamma_{hI}(G[H]) \leq 2|V_2| + |V_1| &= 2|S \cap N_G(S, 2)| + \sum_{u \in S \setminus N_G(S,2)} [2|V_2^u| + |V_1^u|] \\ &= 2|S \cap N_G(S, 2)| + pnd_I(H)|S \setminus N_G(S, 2)|. \end{aligned}$$

Since S is arbitrary, $\gamma_{hI}(G[H]) \leq \alpha_H(G)$.

To show the sharpness of the upperbound, consider the graph G in Figure 6. Verify

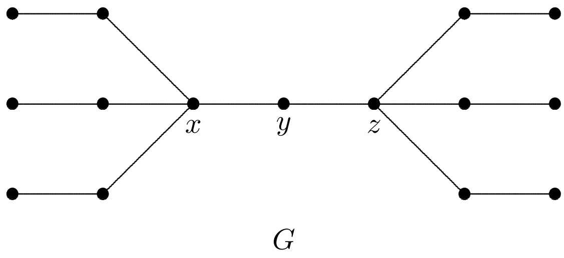

Figure 6: Graph G showing sharpness of the bound in Corollary 3.9

that for $n \geq 3$, $\gamma_{hI}(G[P_n]) = 7$. Note that $pnd_I(P_n) = 3$ while the set $S = \{x, y, z\}$ is a hop dominating set of G with $|S \cap N_G(S, 2)| = 2$ and $|S \setminus N_G(S, 2)| = 1$. In this case, $\alpha_H(G) = 2(2) + pnd_I(P_n)(1) = 7$. ■

Strict inequality in Corollary 3.9 can also be attained. Note that for $n \geq 3$, $\gamma_{hI}(C_5[P_n]) = 5$ while $\alpha_{P_n}(C_5) = 6$. The same example also shows that $\alpha_H(G)$ need not be determined by a γ_h-set S of G. If $C_5 = [x_1, x_2, x_3, x_4, x_5, x_1]$, then $S = \{x_1, x_2, x_4\}$ is a hop dominating set but not a γ_h-set of C_5. However, $\alpha_{P_n}(C_5) = 2|S \cap N_{C_5}(S, 2)| + pnd_I(P_n)|S \setminus N_{C_5}(S, 2)| = 6$.

4. Conclusion

It turned out that the hop Italian domination is directly related to both the hop Roman domination and the 2-hop domination. More precisely, $\gamma_{hI}(G) \leq \min\{\gamma_{hR}(G), \gamma_{2h}(G)\}$ for all graphs G. More interestingly, it is shown that, in fact, the difference $\gamma_{hR}(G) - \gamma_{hI}(G)$ can be made arbitrary large, and that any pair of positive integers a and b with $4 \leq a \leq b$ are realizable as the hop Italian domination number and the 2-hop domination number, respectively, of some connected graph. Finally, for graphs under the complementary prism, join, corona and lexicographic product of graphs, the hop Italian domination number is expressible in terms of the hop Italian domination numbers or of the pnd_I numbers of its factors.

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