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A Comparative Study of Numerical Solution of Second-order Singular Differential Equations Using Bernoulli Wavelet Techniques

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Abstract. The main objective of this article is to discuss a numerical method for solving singular differential equations based on wavelets. Singular differential equations are first transformed into a system of linear algebraic equations, and then the linear system's solution produces the unknown coefficients. Along with its estimated error, the convergence of the approximative solution is also determined. Some numerical examples are thought to show that Bernoulli wavelet is better than Chebyshev and Legendre wavelet and other existing techniques.

2020 Mathematics Subject Classifications: Boundary value problem, Bernoulli wavelets, Hermite wavelets, collocation point, grid point

Key Words and Phrases: 65L10, 65T60

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1. Introduction

In this paper, we consider a general second-order singular linear boundary value problem (BVP) is the form as [[1], [13]]

$$y''(x) + \frac{\alpha}{x}y'(x) + A(x)y(x) = B(x), x \in [0, 1], \tag{1}$$

with boundary conditions

$$y(0) = a, y(1) = b, \tag{2}$$

where $A(x)$ and $B(x)$ are continuous functions defined in the interval $xin[0, 1]$ and a, b, and α are finite real constants. At the starting point $x = 0$ of (1), there is a singularity. The singularity at the point $x = 0$ is where the main problem resides. Numerical handling

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of singular boundary value problems has always been a complex and challenging utilize due to the unique behavior that happens at [9].

Singular boundary value problems (SBVP) have attracted a lot of attention in recent years, and numerous approaches have been put forth to examine them. There are issues with singular boundary values in many areas of applied mathematics, such as mechanics, nuclear physics, atomic theory, and chemical sciences. Singular boundary value issues have gained a lot of interest as a result, and many individuals have investigated them. The Chebyshev polynomial and the B-spline were utilized by Kadalbajoo and Agarwal to solve the homogeneous problem. Numerous writers have provided the equation $(1)'$s numerical solution using different numerical approaches, including the finite difference method and the cubic spline method[[4], [5]], cubic spline method [7]. variational iteration method [15], Galerkin and Collocation methods [8], Sinc-Galerkin [12], parametric spline method [10].

Currently under investigation in mathematics is the field of wavelet theory. Numerous engineering disciplines have successfully used wavelets, most notably signal analysis for waveform representation and segmentation, time-frequency analysis, and quick algorithms for straightforward implementation. Wavelets enable the accurate depiction of a large variety of operators and functions. In addition, wavelets are connected to quick numerical methods [[1],[3]]. A recently created mathematical technique called wavelet analysis has found extensive use in numerous technical applications. This has generated a lot of interest due to wavelets' robust mathematical capability and promising application potential in challenges related to science and engineering. Special focus has been given to the creation of compactly supported, smooth wavelet bases. As previously mentioned, spectral bases provide global support yet are endlessly differentiable. On the other hand, basis functions utilized in finite-element methods have poor continuity qualities while having a small, compact support. On the other hand, basis functions utilized in finite-element approaches have small, compact support but poor continuity qualities. We already know that finite element approaches perform well in terms of spatial localization but badly in terms of spectral localization, whereas spectral localization is a strong suit of spectral methods and a weak suit of spatial localization for spectral methods. Finite element and spectral base advantages are combined in wavelet bases.

We can anticipate numerical approaches based on wavelet bases to attain good spatial and spectral resolutions, [14]. One strategy for investigating differential equations in finite element-type methods is to use wavelet function bases instead of other traditional piecewise polynomial trial functions. The suggested approach transforms the provided SBVP into an unsolved system of algebraic equations that can be quickly solved with a mathematical program like MATLAB.

This paper's primary goal is to offer an algorithm for computing differentiation for the class of SBVP based on the BW.

The paper's outline is provided below. The fundamental characteristics of BW, Chebyshev wavelet(CW), and Legendre Wavelet(LW) are covered in Section 2. Singular boundary value issues were implemented and their solutions were reported in Section 3. Numerical examples are provided in Section 4 to illustrate the effectiveness and applicability of the suggested algorithm. The conclusion can be found in Section 5.

2. *Preliminaries:*

2.1. *Definition of wavelet*

In [2] describes the four kinds of CW, LW and [[2], [11], [6]] are mentioned the BW and we mention some properties related to BW.

2.2. *Formulation for the Solution of Singular Differential Equations*

A function $y(x)$ defined over $[0,1)$ can be expanded by CW, LW and BW as [2]

$$y(x) = \sum_{q=1}^{2^{l-1}} \sum_{p=0}^{P-1} g_{qp}\psi_{qp}(x) = G^T\psi(x), \tag{3}$$

where G and ψ are $(2^{l-1}P \times 1)$ matrices given by

$$G = [g_{10}, g_{11}, \ldots, g_{1(P-1)}, g_{20} \ldots, g_{2(P-1)}, \ldots, g_{2^{l-1}0}, \ldots, g_{2^{l-1}(P-1)}]^T,$$

$$\psi(x) = [\psi(x)_{10}, ..., \psi(x)_{1(P-1)}, ..., \psi(x)_{2^{l-1}0}, ..., \psi(x)_{2^{l-1}(P-1)}]^T.$$

The boundary conditions (2) leads, respectively, to the following equations:

$$G^T\psi(0) = a, \qquad G^T\psi(1) = b. \tag{4}$$

Now total conditions will be reduced to $2^{l-1}P - 2$ to recover the unknown coefficients g_{qp}, which can be obtained by substituting equation (3) in the expression (1) as follows:

$$\frac{d^2}{dx^2}\sum_{q=1}^{2^{l-1}} \sum_{p=0}^{P-1} g_{qp}\psi_{qp}(x) + \frac{\alpha}{x}\frac{d}{dx}\sum_{q=1}^{2^{l-1}} \sum_{p=0}^{P-1} g_{qp}\psi_{qp}(x) + A(x)\sum_{q=1}^{2^{l-1}} \sum_{p=0}^{P-1} g_{qp}\psi_{qp}(x) = B(x), \tag{5}$$

In the equation (5) replacing x by x_j, we get

$$\frac{d^2}{dx^2}\sum_{q=1}^{2^{l-1}} \sum_{p=0}^{P-1} g_{qp}\psi_{qp}(x_j) + \frac{\alpha}{x_j}\frac{d}{dx}\sum_{q=1}^{2^{l-1}} \sum_{p=0}^{P-1} g_{qp}\psi_{qp}(x_j) + A(x_j)\sum_{q=1}^{2^{l-1}} \sum_{p=0}^{P-1} g_{qp}\psi_{qp}(x_j) = B(x_j), \tag{6}$$

where $x_j's$ are collocation points of following as

$$x_j = \frac{T(1 + cos\frac{(j-1)\pi}{2^{l-1}P-1})}{2}, \quad j = 2, 3, ..., 2^{l-1}P - 1. \tag{7}$$

On combining equations (4) and (6), we get $2^{l-1}P$ linear equations from which we can obtain the unknown coefficients g_{qp}. Similarly, we can proceed with higher orders as well. Kailash Yadav and J. P. Jaiswal [2] have shown that convergence analysis and error estimation.

3. Numerical Examples

Example 4.1: Take into consideration the following [1] second-order singular boundary value problem equation:

$$y''(x) + \left(\frac{2}{x}\right) y'(x) - \left(\frac{2}{x^2}\right) y(x) = 4, \ 0 \leq x \leq 1, \tag{8}$$

with conditions

$$y(0) = 0, \quad y(1) = 0. \tag{9}$$

This is the precise answer:

$$y(x) = x^2 - x. \tag{10}$$

Table 1 compares the absolute error, for example, 4.1 with a few other techniques that are currently in use and covered in the article [1]. Here, LWGM stands for the Laguerre wavelet-based Galerkin method, HWGM for the Hermite wavelet-based Galerkin method, and BW for the Bernoulli wavelet approach. FMD is for the finite difference method. The absolute errors obtained using the Legendre wavelet (LW), Chebyshev wavelet (CHW) (all four kinds), and Bernoulli wavelet (BW) methods are shown in Table 2. The tables show that, for the most part, the Bernoulli wavelet technique yields findings that are more accurate. Figure 1 depicts, for $l = 1, P = 3$, respectively, the physical behavior of the numerical solution and precise solution at grid locations.

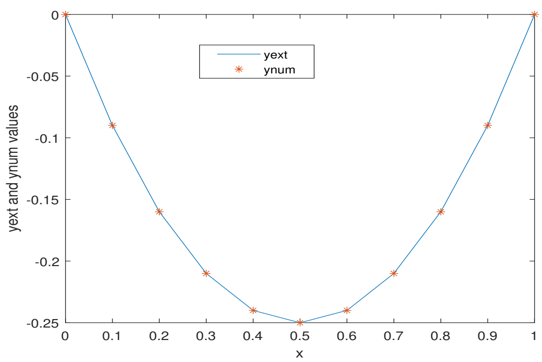
Table 1: Comparison of Example 4.1's absolute in accuracy

x	FDM	LWGM	HWGM	BW
		k=1, M=3	k=1, M=3	l=1, P=3
0.1	7.88E-02	3.32E-03	1.90E-04	4.88E-17
0.2	1.33E-02	3.18E-03	2.00E-03	6.94E-17
0.3	1.66E-02	1.58E-03	1.40E-03	5.55E-17
0.4	1.79E-02	1.30E-04	4.60E-04	8.33E-17
0.5	1.75E-02	1.19E-03	2.60E-04	5.55E-17
0.6	1.55E-02	1.33E-03	5.60E-04	2.78E-17
0.7	1.22E-02	7.00E-04	4.10E-04	5.55E-17
0.8	8.08E-02	2.30E-04	1.00E-05	1.11E-16
0.9	3.69E-02	7.60E-04	3.40E-04	0.00E-00

Table 2: Comparison of the Example 4.1 absolute error (where $l = 1$ and $P = 6$)

x	LW	FSTCHW	SNDCHW	THDCHW	FTHCHW	BW
0.1	2.78E-17	2.78E-17	0.00E-00	2.78E-17	2.78E-17	0.00E-00
0.2	2.78E-17	2.78E-17	2.78E-17	2.78E-17	2.78E-17	8.94E-18
0.3	5.55E-17	5.55E-17	0.00E-00	2.78E-17	0.00E-00	0.00E-00
0.4	2.78E-17	2.78E-17	0.00E-00	5.55E-17	2.78E-17	0.00E-00
0.5	5.55E-17	0.00E-17	5.55E-17	5.55E-17	5.55E-17	0.00E-00
0.6	8.33E-17	8.33E-12	1.67E-16	8.33E-17	0.00E-00	0.00E-00
0.7	0.00E-00	5.55E-17	1.39E-16	1.11E-16	1.67E-16	1.08E-17
0.8	5.55E-17	1.11E-17	1.39E-16	2.22E-16	2.78E-17	0.00E-00
0.9	1.11E-17	1.39E-17	1.39E-16	2.08E-16	5.55E-17	8.34E-18

Figure 1: Physical characteristics of the precise and approximative solutions at the collocation points in Example 4.1 for $l = 1$ and $P = 3$.



Example 4.2: Think about the following [1] second-order singular boundary value problem equation:

$$y''(x) + \left(\frac{1}{x}\right) y'(x) + y(x) = x^2 - x^3 - 9x + 4, \ 0 \leq x \leq 1, \tag{11}$$

with conditions

$$y(0) = 0, \quad y(1) = 0. \tag{12}$$

The actual response is

$$y(x) = x^2 - x^3. \tag{13}$$

Tables 3 and 4 compare the absolute errors for Example 4.2 derived from the Bernoulli wavelets (BW) technique with some other methods discussed in the reference [[1], [13]], and respectively Legendre wavelet (LW), Chebyshev wavelet (CHW) (all four kinds), and Bernoulli wavelet (BW). The tables show that BW's method consistently yields more accurate results. For $l = 1, P = 3$, respectively, Figure 2 depicts the physical behavior of the numerical solution and the precise solution.

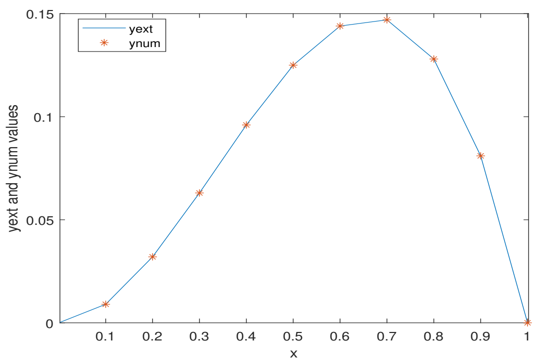
Table 3: The absolute inaccuracy for Example 4.2 is compared.

x	FDM [1]	LWGM [1]	HWGM [1]	LWGM [13]	BW	BW
		k=1, M=3	k=1, M=3	k=1, M=5	l=1, P=3	l=1, P=4
0.1	2.37E-02	1.67E-03	5.10E-05	4.82E-04	0.00E-00	1.73E-18
0.2	4.75E-02	1.16E-03	5.90E-05	6.70E-05	2.77E-17	1.39E-17
0.3	6.56E-02	2.90E-04	4.60E-05	4.90E-05	0.00E-00	4.16E-17
0.4	8.06E-02	1.19E-04	2.30E-05	2.80E-05	8.33E-17	2.78E-17
0.5	8.84E-02	3.40E-05	4.00E-06	1.80E-05	5.55E-17	1.39E-17
0.6	8.74E-02	4.29E-04	8.00E-06	6.10E-05	5.55E-17	2.78E-17
0.7	7.69E-02	6.23E-04	9.00E-06	3.90E-05	5.55E-17	0.00E-00
0.8	5.74E-02	3.50E-04	3.00E-06	3.70E-05	1.11E-16	0.00E-00
0.9	3.07E-02	1.84E-04	6.00E-06	7.50E-05	0.00E-00	8.33E-17

Table 4: Comparison of Example 4.2's absolute error (where $l = 1$ and $P = 6$)

x	LW	FSTCHW	SNDCHW	THDCHW	FTHCHW	BW
0.1	3.64E-17	2.08E-17	1.94E-16	4.16E-17	1.94E-16	3.47E-18
0.2	2.08E-17	2.08E-17	2.29E-16	4.88E-17	2.78E-17	6.94E-18
0.3	1.38E-17	5.55E-17	1.94E-16	2.78E-17	2.22E-16	0.00E-00
0.4	1.38E-17	2.78E-17	2.22E-16	4.16E-17	2.22E-16	0.00E-00
0.5	2.78E-17	2.78E-17	2.22E-16	5.55E-17	2.22E-16	0.00E-00
0.6	5.55E-17	5.55E-17	1.67E-16	8.33E-17	1.63E-16	2.78E-17
0.7	0.00E-00	0.00E-00	1.67E-16	2.78E-17	1.62E-16	5.56E-18
0.8	1.11E-16	5.55E-17	1.67E-16	2.28E-17	2.78E-16	1.95E-18
0.9	1.11E-16	8.33E-17	2.78E-17	1.39E-17	5.55E-17	8.67E-18

Figure 2: Physical characteristics of the precise and approximative solutions at the collocation locations of Example 4.2 for $l = 1$ and $P =$.



Example 4.3: Consider the equation for the second-order singular boundary value problem. [1]:

$$y''(x) + \left(\frac{1}{x}\right) y'(x) + y(x) = x^2 - x^3 - 9x + 4, \ 0 \leq x \leq 1, \tag{14}$$

with conditions

$$y(0) = 0, \quad y(1) = 0. \tag{15}$$

The actual result is

$$y(x) = x^2 - x^3. \tag{16}$$

The absolute error for Example 4.3 obtained using the technique under consideration and the method described in the article [1] and Bernoulli wavelet (BW), Legendre wavelet (LW), Chebyshev wavelet (CHW) (all four kinds), and Bernoulli wavelet (BW), respectively, are shown in Tables 5 and 6. It is concluded that the Bernoulli wavelet produces results that are comparably better. Figure 3 depicts, for $l = 1, P = 3$, respectively, the physical behavior of the numerical solution and precise solution at grid locations.

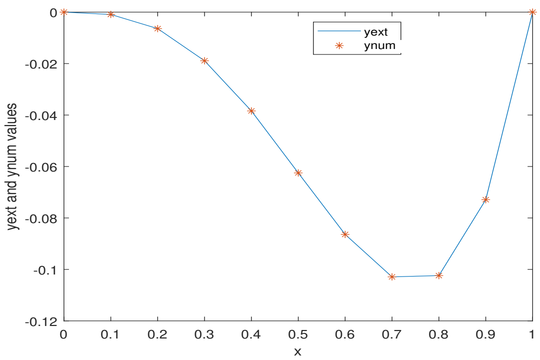
Table 5: The absolute error for Example 4.3 is compared

x	FDM [1]	LWGM [1]	HWGM [1]	BW
		k=1, M=3	k=1, M=3	l=1, P=3
0.1	2.55E-02	7.70E-05	0.00E-00	0.00E-00
0.2	3.09E-02	1.56E-03	1.00E-06	1.72E-16
0.3	3.40E-02	2.04E-03	4.00E-06	1.21E-16
0.4	3.83E-02	1.10E-03	7.00E-06	7.63E-17
0.5	4.05E-02	4.86E-04	1.20E-05	2.78E-17
0.6	4.05E-02	1.45E-03	1.70E-05	1.39E-17
0.7	3.74E-02	8.44E-04	2.00E-06	6.94E-17
0.8	3.02E-02	1.27E-03	2.00E-06	1.25E-17
0.9	1.81E-02	3.02E-03	6.00E-06	1.25E-17

Table 6: Comparison of the Example 4.3 absolute error (where $l = 1$ and $P = 6$)

x	LW	FSTCHW	SNDCHW	THDCHW	FTHCHW	BW
0.1	6.18E-18	2.26E-17	2.21E-16	6.84E-16	2.26E-16	2.27E-18
0.2	2.60E-18	3.03E-17	7.55E-17	2.15E-16	3.04E-17	1.81E-18
0.3	3.47E-18	4.86E-17	3.12E-17	5.90E-17	4.86E-17	19.97E-18
0.4	2.78E-17	2.08E-17	1.25E-16	3.19E-16	2.08E-17	0.00E-00
0.5	2.78E-17	1.04E-17	2.22E-16	6.31E-16	1.04E-16	3.45E-18
0.6	4.16E-17	8.33E-17	2.22E-16	6.66E-16	8.33E-17	2.78E-17
0.7	2.77E-17	5.55E-17	1.67E-16	4.16E-16	5.55E-17	5.56E-18
0.8	5.55E-17	2.22E-16	1.67E-16	5.55E-17	2.22E-16	2.31E-18
0.9	1.24E-16	1.11E-16	1.11E-16	2.22E-16	1.11E-16	1.23E-18

Figure 3: For $l = 1$ and $P = 3$, the physical behavior of the exact and approximative solutions at collocation locations in Example 4.3.



4. Conclusion

In this research work, A numerical method based on Bernoulli wavelets is developed to estimate the numerical result of time discretization. We transformed the proposed equation into a system of algebraic equations using the collocation point and the properties of the Bernoulli wavelet series, then solved these equations to obtain the required unknown Bernoulli wavelet coefficients. The proposed method is applied to various test cases of well-known singular differential equations, and the numerical results are compared to analytical and existing solutions. The correctness and consistency of the suggested method are shown by the numerical findings reported in the previous section. The proposed approach is unique in that it can quickly solve general singular differential equations while maintaining simplicity. We will suggest applying the method to numerical SBVP of additional sorts, such as singular fractional integro-differential equations, singular fractional boundary value differential equations, and singular partial differential equations, in the upcoming work.

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