EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 16, No. 4, 2023, 2156-2168
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Generalized Reflexive Structures Properties of Crossed Products Type

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Abstract. Let R be a ring and M be a monoid with a twisting map $f : M \times M \to U(R)$ and an action map $\omega : M \to Aut(R)$. The objective of our work is to extend the reflexive properties of rings by focusing on the crossed product $R * M$ over R. In order to achieve this, we introduce and examine the concept of strongly CM-reflexive. Although a monoid M and any ring R with an idempotent are not strongly CM-reflexive in general, we prove that R is strongly CM-reflexive under some additional conditions. Moreover, we prove that if R is a left $p.q.$-Baer (semiprime, left APP-ring, respectively), then R is strongly CM-reflexive. Additionally, for a right Ore ring R with a classical right quotient ring Q, we prove R is strongly CM-reflexive if and only if Q is strongly CM-reflexive. Finally, we discuss some relevant results on crossed products.

2020 Mathematics Subject Classifications: 16S36, 16N60, 16U99.

Key Words and Phrases: Left $p.q.$-Baer-ring, crossed product monoid $R * M$, CM-quasi Armendariz ring, strongly CM-reflexive ring.

1. Introduction

Unless otherwise stated, we assume that R is an associative ring with identity and M is a monoid. The concept of reflexive properties of rings was first studied by Mason [1]. In particular, a right ideal I of R is said to be reflexive if $xRy \subseteq I$ implies $yRx \subseteq I$ for any $x, y \in R$. This concept is also specialized to the zero ideal of a ring, where a ring R is said to be reflexive if its zero ideal is reflexive. Moreover, a ring R is called completely reflexive if $xy = 0$ implies $yx = 0$ for any $x, y \in R$. It is worth noting that reduced rings are completely reflexive, and every completely reflexive ring is semicommutative, as shown in the literature [1].

Several authors have discussed extensions of reflexive rings, including strongly reflexive rings, strongly M-reflexive rings, Armendariz rings, reversible rings, and reflexive on skew monoid rings, in numerous publications (see, for example, [2], [3], [4] and [5]). According to [6], a ring R is said to be an M-Armendariz ring of crossed product type relative to the given

DOI: https://doi.org/10.29020/nybg.ejpam.v16i4.4918

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twisting f and action ω, or an M-quasi Armendariz ring (or simply a CM-Armendariz ring or CM-quasi Armendariz ring, respectively), if for any $\phi = \sum_{i=1}^{n} a_i g_i, \psi = \sum_{j=1}^{m} b_j h_j \in R * M$ such that $\phi\psi = 0$ (resp., $\phi(R * M)\psi = 0$), it follows that $a_i \omega_{g_i}(b_j) = 0$ (resp., $a_i R \omega_{g_i l}(b_j) = 0$) for all i, j and all $g_i, h_j, l \in M$.

The focus of this paper is on investigating strongly CM-reflexive rings, which are a reflexive-like property defined for the monoid crossed product $R * M$ with respect to the given twisting map f and action map ω. This concept is a generalization of several other reflexive properties, including reflexive rings, strongly reflexive rings, strongly M-reflexive rings, and skew monoid rings.

The paper is devoted to presents several results, including, if R is a semiprime, then R is strongly CM-reflexive for a $u.p.$-monoid M. Also, if R is a left $p.q.$-Baer (semiprime, left APP-ring, respectively), then R is strongly CM-reflexive for a strictly totally ordered monoid. Additionally, if R is an M-compatible ring and M is a monoid with twisting f and action ω as above, then for any reduced ideal I of R such that R/I is strongly CM-reflexive, then R is strongly CM-reflexive. Moreover, for a right Ore ring R with classical right quotient ring Q, we show that R is strongly CM-reflexive if and only if Q is strongly CM-reflexive. Finally, we discuss example and some results in the subject.

To begin with, we introduce some notions and notations relevant to this paper. Let $\omega : M \to Aut(R)$ be a monoid homomorphism. For $h \in M$, we denote by ω_h the automorphism $\omega(h)$. The crossed product $R * M$ over R is defined as the set of all finite sums $R * M = \{x_h h | x_h \in R, h \in M\}$, where addition is defined component-wise and multiplication is defined using the distributive law and two rules known as action and twisting. Specifically, for $l, h \in M$ and $x \in R$, we have $hx = \omega_h(x)h$ and $l\,h = f(l, h)l\,h$, where $f : M \times M \to U(R)$ is a twisted function and $U(R)$ denotes the set of units of R. Here, the twisted function f and the action ω of M on R satisfy the following conditions: $\omega_l(\omega_h(x)) = f(l, h)\omega_l(\omega_h(x)f(l, h)^{-1})$, $\omega_l(f(h, k))f(l, hk) = f(l, h)f(l\,h, k)$, $f(1, l) = f(l, 1) = 1$ for all $l, h, k \in M$. It is worth noting that the monoid crossed product is a general ring construction.

Given a monoid crossed product $R * M$ with twisting f and action ω, if the twisting f is trivial, (i.e., $f(a, b) = 1$) for all $a, b \in M$, then $R * M$ is the skew monoid ring $R * M$. If both the twisting f and the action ω are trivial, then $R * M$ is a monoid ring denoted by $R[M]$ (see [7] and [8]). A monoid M is said to be a $u.p.$-monoid (unique product monoid) if, for any two nonempty finite subsets X and Y of M, there exists a unique element $h \in M$ that can be written in the form $h = uv$ with $u \in X$ and $v \in Y$. An ordered monoid $(M, \preceq)$ is said to be strictly ordered if the following condition holds: whenever $g, k, h \in M$ with $g \prec k$, it follows that $gh \prec kh$ and $hg \prec hk$.

2. Generalized Reflexive rings of crossed product type

In this section, we will discuss the concept of strongly reflexive properties in the context of a monoid of crossed product $R*M$, where R is a ring and M is a monoid with a twisting map $f : M \times M \to U(R)$ and an action map $\omega : M \to Aut(R)$.

Definition 1. *A ring R is said to be strongly M-reflexive of crossed product type with respect to the given twisting map f and action map ω (or simply, strongly CM-reflexive) if for any $\phi = c_1l_1 + c_2l_2 + \cdots + c_nl_n$ and $\psi = a_1h_1 + a_2h_2 + \cdots + a_mh_m \in R * M$ satisfying that $\phi(R * M)\psi = 0$ implies that $c_i\omega_{l_i}(\omega_g(Ra_j)) = 0$, then $\psi(R * M)\phi = 0$ for each i, j and for all $g, l_i, h_j \in M$.*

Remark 1. (1) *If a ring R is strongly CM-reflexive with a trivial twisting map f, then we refer to the monoid M as a skew strongly M-reflexive ring. If R is strongly CM-reflexive with a trivial action map ω, then we call R a strongly TM-reflexive (i.e., twisted strongly M-reflexive) ring. Note that when both f and ω are trivial, R is simply strongly M-reflexive. In particular, if $M = (\mathbb{N} \cup \{0\}, +)$ and both f and ω are trivial, then R is strongly CM-reflexive if and only if R is strongly reflexive.*

(2) *If R is a strongly CM-reflexive ring with a trivial twisting map f, then any M-invariant subring S (i.e., $\omega_g(S) \subseteq S$ for all $g \in M$) of R is strongly CM-reflexive.*

An ideal I of a ring R is considered to be right s-unital if there exists an element $e \in I$ for every $t \in I$ such that $te = t$. A ring is referred to as a left APP-ring if the left annihilator $l_R(Rt)$ is right s-unital as an ideal of R for any element $t \in R$.

In their work [9], Nasr-Isfahani and Moussavi introduced a ring R with an endomorphism ω and defined it as ω-weakly rigid if the condition $cRt = 0$ holds if and only if $c\omega(Rt) = 0$ for any $c, t \in R$. It is worth noting that the category of ω-rigid rings and ω-compatible rings is a limited one, and it is evident that every ω-compatible ring falls under the category of ω-weakly rigid rings. However, there exist several classes of ω-weakly rigid rings that do not belong to the category of ω-compatible rings. By [10], R is α-rigid if and only if R is α-compatible and reduced. According to [9], any prime ring that has an automorphism ω is considered to be ω-weakly rigid. If a monoid homomorphism $\omega : M \to Aut(R)$ is weakly-rigid (compatible), it means that the ring R is also weakly rigid (compatible) with respect to each $g \in M$ under the automorphism ω_g.

Lemma 1. *[11, Lemma 1.1]. If M is a u.p.-monoid, then M is cancellative (i.e., for $\ell, h, \lambda \in M$, if $\ell\lambda = h\lambda$ or $\lambda\ell = \lambda h$, then $\ell = h$).*

Lemma 2. *Suppose R is a ring and M is a u.p.-monoid with a twisting map $f : M \times M \to U(R)$ and an action map $\omega : M \to Aut(R)$. If R is an M-rigid ring, then the monoid ring $R * M$ is reduced.*

Proof. Assume that $\phi = c_1h_1 + \cdots + c_nh_n \in R * M$ satisfies $\phi^2 = 0$. According to Proposition 2.2 [6], R is CM-Armendariz, this implies $c_i\omega_{h_i}(b_j)f(l_i, h_j))(l_ih_j) = 0$ for all i and j, by Lemma 1, M is a cancellative so $c_i\omega_{h_i}(b_j) = 0$. As R is an M-rigid, then R is a reduced, we can conclude that $c_i = 0$ for all $1 \leq i \leq n$. Consequently, $\phi = 0$, and hence $R * M$ is a reduced. □

Theorem 1. *Let R be a semiprime ring and M be a u.p.-monoid with a twisting map $f : M \times M \to U(R)$ and an action map $\omega : M \to Aut(R)$. If R is an M-compatible ring, then R is strongly CM-reflexive.*

Proof. The evidence has been modified from the Theorem 1.1 of [12]. Let $\phi = c_1l_1 + c_2l_2 + \cdots + c_nl_n, \psi = a_1h_1 + a_2h_2 + \cdots + a_mh_m \in R * M$ satisfy $\phi(R * M)\psi = 0$. Then for any $r \in R$ and $g \in M$, we have

$$(c_1l_1 + c_2l_2 + \cdots + c_nl_n)gr(a_1h_1 + a_2h_2 + \cdots + a_mh_m) = 0. \tag{2.1}$$

We will employ mathematical induction on n to demonstrate that $c_iR\omega_{l_i}(\omega_g(a_j)) = 0$ for all $1 \leq i \leq n, 1 \leq j \leq m$, and for any $g \in M$. This can be achieved by utilizing the fact that M is a compatible monoid. If we take $n = 1$, then we have $(c_1l_1)gr(a_1h_1+a_2h_2+\cdots+a_mh_m) = 0$. Therefore, for each $1 \leq j \leq m$, we have $c_1R\omega_{l_1}(\omega_g(a_j))f(l_i, h_j))(l_ih_j) = 0$. By Lemma 1, M is a cancellative, this means $l_1h_i \neq l_1h_j$ for any i and j with $1 \leq i \neq j \leq m$. Thus, $c_1R\omega_{l_1}(\omega_g(a_j)) = 0$. For the case where $n \geq 2$, we can use the assumption that M is a uniquely presented monoid to find s and t with $1 \leq s \leq n$ and $1 \leq t \leq m$ such that l_sgh_t is uniquely represented by considering two subsets $K = \{l_1g, l_2g, \ldots, l_ng\}$ and $H = \{h_1, h_2, \ldots, h_m\}$ of the monoid M. Without loss of generality, we may assume that $s = 1$ and $t = 1$. From Eq. (2.1), we can deduce that $c_1\omega_{l_1}(\omega_g(Ra_1))f(l_1, h_1)(l_1h_1) = 0$, which implies that $c_1R\omega_{l_1}(\omega_g(a_1)) = 0$. Since ω_g and ω_{l_1} are automorphisms of R, we have $c_1R\omega_{l_1}(\omega_g(a_1)) = 0$. As a result, for every $z \in R$, we have $c_1R\omega_{l_1}(\omega_g(a_1za_1))f(l_1, h_1) = 0$, which implies that $0 = (c_1l_1 + c_2l_2 + \cdots + c_nl_n)gra_1z(gra_1z(a_1h_1 + a_2h_2 + \cdots + a_mh_m) = (c_2l_2 + \cdots + c_nl_n)gr(a_1za_1h_1 + a_1za_2h_2 + \cdots + a_1za_mh_m)$.

By applying the induction hypothesis, it follows that $c_i\omega_{l_i}(\omega_g(ra_1za_j)) = 0$ for all $2 \leq i \leq n$ and $1 \leq j \leq m$. Thus, we have $c_iR\omega_{l_i}(\omega_g(a_1))R\omega_{l_i}(\omega_g(a_1)) = 0$, which implies that $c_iR\omega_{l_i}(\omega_g(a_1)) = 0$ for all $2 \leq i \leq n$, as R is a semiprime ring. Therefore, we have $c_iR\omega_{l_i}(\omega_g(a_1)) = 0$ for all $1 \leq i \leq n$. As a result, the Eq. (2.1) becomes $(c_1l_1 + c_2l_2 + \cdots + c_nl_n)gr(a_2h_2 + \cdots + a_mh_m) = 0$. We can repeat this process to show that $c_i\omega_{l_i}(\omega_g(ra_j)) = 0$ for all $g \in M$ and all i, j. This shows that $c_iR\omega_{l_i}(\omega_g(a_j)) = 0$. Consequently, we can see that $a_jR\omega_{h_j}(\omega_g(c_i)) = 0$ for all $g \in M$, $1 \leq j \leq m$, and $1 \leq i \leq n$. Therefore, R is strongly CM-reflexive. □

The following example demonstrates the existence of a ring R over a field F that is not strongly CM-reflexive.

Example 1. *Let M be a monoid with at least two elements, and let $S = M_2(F)$ be the matrix ring over a field F with a twisting map $f : M \times M \to U(R)$, then S is not strongly CM-reflexive.*

Solution. Take $e \neq h \in M$, we define $\omega : M \to Aut(S)$ by

$$\omega_h\left(\begin{pmatrix} a & d \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & -d \\ 0 & c \end{pmatrix}.$$

If the twisting map f is trivial (i.e., $f(x, y) = 1$ for all $x, y \in M$), then the ring S is not strongly CM-reflexive. To see this, consider $\phi = E_{12}e+E_{11}h$ and $\psi = (E_{11}+E_{12})h \in S*M$. For $\varphi = (E_{11} + E_{22})h \in S * M$, we can easily verify that $\phi\varphi\psi = 0$. However, we have $\psi\varphi\phi \neq 0$, which implies that S is not strongly CM-reflexive. □

A ring R is categorized as a right PP-ring or left PP-ring if the right or left annihilator of an element in R, respectively, is generated by an idempotent. A (quasi-) Baer ring is one where the right annihilator of every nonempty subset or every right ideal of R is generated by an idempotent. Principally quasi-Baer rings, introduced by Birkenmeier et al. [13], extend the concept of quasi-Baer rings. A ring R is referred to as left principally quasi-Baer or simply left $p.q.$-Baer if the left annihilator of a principal left ideal in R is generated by an idempotent. It is important to note that biregular rings and quasi-Baer rings are examples of left $p.q.$-Baer rings. For more information and examples of left $p.q.$-Baer rings, see Birkenmeier et al. ([13], [14]) and Liu [15]. Since right PP-rings and left $p.q.$-Baer rings both fall under the category of left APP [16], the following results can be deduced.

Theorem 2. *Suppose R is a reduced ring, M is a strictly totally ordered monoid with a twisting map $f : M \times M \to U(R)$ and an action map $\omega : M \to Aut(R)$ that is compatible with the multiplication in M. If R is a left p.q.-Baer ring, then R is strongly CM-reflexive.*

Proof. The proof is a variant of the proof given in Proposition 2.9 [17]. Let $\phi = c_1l_1 + c_2l_2 + \cdots + c_nl_n, \psi = a_1h_1 + a_2h_2 + \cdots + a_mh_m \in R*M$ satisfy $\phi(R*M)\psi = 0$. Since M is a strictly totally ordered monoid, we can assume that $l_i \preceq l_j$ and $h_i \preceq h_j$ whenever $i < j$. Now, we claim $c_i\omega_{l_i}(\omega_g(Ra_j)) = 0$ for all i, j. Let r be an element of R. Then, we have $\phi(re)\psi = 0$ since $\phi(R * M)\psi = 0$. Thus, we have

$$\begin{aligned} 0 = \; & \phi(re)\psi = c_1rf(l_1, e)a_1f(l_1, h_1)l_1h_1 + \cdots + [c_nrf(l_n, e)a_{m-2}f(l_n, h_{m-2})l_nh_{m-2} \\ + \; & c_{n-1}rf(l_{n-1}, e)a_{m-1}f(l_{n-1}, h_{m-1})l_{n-1}h_{m-1} + c_{n-2}rf(l_{n-2}, e)l_mf(l_{n-2}, h_m)l_{n-2}h_m] \\ + \; & [c_nrf(l_n, e)a_{m-1}f(l_n, h_{m-1})l_nh_{m-1} + a_{n-1}rf(l_{n-1}, e)a_mf(l_{n-1}, h_m)l_{n-1}h_m] \\ + \; & c_nrf(l_n, e)a_mf(l_n, h_m)l_nh_m. \end{aligned} \quad (2.2)$$

It follows that $c_nrf(l_n, e)a_mf(l_n, h_m) = 0$ since l_nh_m is of highest order in the $l_ih_j's$. Hence $c_nrf(l_n, e)a_m = 0$. This shows that $c_n \in \ell_R(Rf(l_n, e)a_m) = \ell_R(Ra_m)$. Hence, $\ell_R(Ra_m) = Re_m$ for some idempotent e_m by hypothesis. Replacing r by re_m in Eq. (2.2) we obtain $0 = c_1re_mf(l_1, e)a_1f(l_1, h_1)l_1h_1 + \cdots + [c_nre_mf(l_n, e)a_{m-2}f(l_n, h_{m-2})l_nh_{m-2}$ $+c_{n-1}re_mf(l_{n-1}, e)a_{m-1}f(l_{n-1}, h_{m-1})l_{n-1}h_{m-1}]+c_nre_mf(l_n, e)a_{m-1}f(l_n, h_{m-1})l_nh_{m-1}$(2.3)

So $c_nre_mf(l_n, e)a_{m-1}f(l_n, h_{m-1}) = 0$, because l_nh_{m-1} is of highest order in $\{l_ih_j|1 \leq i \leq n, 1 \leq j \leq m\}$ $\{l_{n-1}h_m, l_nh_m\}$. Hence $c_nre_mf(l_n, e)a_{m-1} = 0$. Since Re_m is an ideal of R and $e_m \in Re_m$, we have $e_mr \in Re_m$ and thus $e_mr = e_mre_m$ for all $r \in R$. On the other hand, we also have $c_n = c_ne_m$ since $c_n \in \ell_R(Ra_m) = Re_m$. Hence $c_nrf(l_n, e)a_{m-1} = c_ne_mrf(l_n, e)a_{m-1} = c_ne_mre_mf(l_n, e)a_{m-1} = c_nre_mf(l_n, e)a_{m-1} = 0$. This implies that $c_n \in \ell_R(Ra_m + Ra_{m-1})$, and hence $\ell_R(Ra_m + Ra_{m-1}) = Re_{m-1}$ for some idempotent $e_{m-1} \in R$ since R is a left $p.q.$-Baer ring. Replacing r by re_{m-1} in equation (2.3) we obtain $c_nre_{m-1}f(l_n, e)a_{m-2}f(l_n, h_{m-2}) = 0$ in the same way as above. This shows that $c_n \in \ell_R(Ra_m + Ra_{m-1} + Ra_{m-2})$. Continuing this process we obtain $c_nRa_t = 0$ for all $t = 1, 2, \ldots, m$. So, we have $(c_1l_1 + c_2l_2 + \cdots + c_{n-1}l_{n-1})(R*M)(a_1h_1 + a_2h_2 + \cdots + a_mh_m) = 0$. Using induction on $m+n$, we obtain $c_i\omega_{l_i}(\omega_g(Ra_j)) = 0$ for all i, j. So it is easy to see that $a_j\omega_{h_j}(\omega_g(Rc_i)) = 0$ by a reduced ness. Therefore, R is strongly CM-reflexive. □

If N is an ideal of the monoid M with twisting $f : M \times M \to U(R)$ and action

$\omega : M \to Aut(R)$, then the restrictions $f|_{N\times N} : N \times N \to U(R)$ and $\omega|_N : N \to Aut(R)$ are induced twisting and action.

Proposition 1. *Let R be an M-compatible ring and M be a commutative, cancellative monoid and N be an ideal of M with a center element λ. If R is strongly CN-reflexive, then R is strongly CM-reflexive.*

Proof. Let $\phi = \sum_{i=1}^{n} c_i l_i, \psi = \sum_{j=1}^{m} a_j h_j \in R * M$ satisfying $\phi\varphi\psi = 0$ for any $\varphi = \sum_{r=1}^{v} \ell_r g_r \in R * M$. Since $\lambda \in N$ is a center element, this implies that

$$\lambda l_1, \lambda l_2, \ldots, \lambda l_n, \lambda g_1\lambda, \lambda g_2\lambda, \ldots, \lambda g_v\lambda, h_1\lambda, h_2\lambda, \ldots, h_m\lambda \in N,$$

such that $\lambda l_i \neq \lambda l_j, \lambda g_i\lambda \neq \lambda g_j\lambda$ and $h_i\lambda \neq h_j\lambda$ for all $i \neq j$. Then, we have

$$\phi_1\varphi_1\psi_1 = \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{r=1}^{v}(c_i\omega_{l_i}(\ell_r\omega_\lambda(a_j)))f(l_i\lambda, h_j)(\lambda^2 l_i g_r h_j \lambda^2) = 0.$$

Since φ, ϕ and ψ are nonzero in $R*M$, so ϕ_1 and ψ_1 are nonzero elements in $(R*M)[N]$. Moreover, from $\phi\varphi\psi = 0$ and ω compatible automorphism, λ a center element of N one can easily obtain that $\phi_1\varphi_1\psi_1 = 0$ for any $\varphi_1 \in (R * M)[N]$. Since R is strongly CN-reflexive. Then, $c_i\omega_{l_i}(\omega_\lambda(r\, a_j))f(l_i, h_j)(l_i h_j) = 0$. So $c_i\omega_{l_i}(\omega_\lambda(Ra_j)) = 0$. By a compatible automorphism, we have $a_j\omega_{h_j}(\omega_\lambda(Rc_i)) = 0$. Therefore, R is strongly CM-reflexive. □

Corollary 1. *[4, Proposition 3.1] Let M be a cancellative monoid and N an ideal of M. If R is strongly N-reflexive, then R is strongly M-reflexive.*

Suppose I is an ideal of R and $\omega : M \to \text{Aut}(R)$ is a monoid homomorphism. We define $\bar{\omega} : M \to \text{Aut}(R/I)$ as $\bar{\omega}_g(d + I) = \omega_g(d) + I$, where $d \in R$ and $g \in M$. It can be shown that $\bar{\omega}$ is a monoid homomorphism. Additionally, the twisting map $f : M \times M \to U(R)$ induces a twisting map $\bar{f} : M \times M \to U(R/I)$ given by $\bar{f}(x, y) = f(x, y) + I$. Furthermore, for every $\phi = \sum_{i=1}^{n} c_i l_i \in R * M$, we denote $\bar{\phi} = \sum_{i=1}^{n} \bar{c}_i l_i \in (R/I) * M$, where $\bar{c}_i = c_i + I$ for $1 \leq i \leq n$. It can be easily verified that the mapping $\theta : R \times M \to (R/I) \times M$ defined as $\theta(\phi) = \bar{\phi}$ is a ring homomorphism. In a proof presented [4], it was shown that when I is a reduced ideal of R and R/I is strongly M-reflexive, then R is strongly M-reflexive. Similarly, we can establish the following result.

Theorem 3. *Let M be a u.p.-monoid and I an ideal of R with twisting $f : M\times M \to U(R)$ and action $\omega : M \to Aut(R)$. If I is a reduced and R/I is strongly CM-reflexive, then R is strongly CM-reflexive.*

Proof. Let $\phi = \Sigma_{i=1}^{n} c_i l_i, \psi = \Sigma_{j=1}^{m} a_j h_j \in R * M$ satisfying $\phi(R * M)\psi = 0$. We will show that $c_i\omega_{l_i}(\omega_g(Ra_j)) = 0$ for any i and j.

Note that in $(R/I) * M$, $\bar{\phi} = \Sigma_{i=1}^{n} \bar{c}_i l_i, \bar{\psi} = \Sigma_{j=1}^{m} \bar{a}_j h_j \in (R/I) * M$, we have

$$\begin{aligned} \bar{0} = & \bar{\phi}((R/I) * M)\bar{\psi} \\ = & (\bar{c}_1 l_1 + \bar{c}_2 l_2 + \cdots + \bar{c}_n l_n)\bar{r} g \omega_{l_i}(\omega_g(\bar{a}_1 h_1 + \bar{a}_2 h_2 + \cdots + \bar{a}_m h_m)) f(l_i, h_j) l_i h_j \\ = & (c_1 + I)\bar{r} g \bar{\omega}_{l_1}(\omega_g(a_1 + I)) f(l_1, h_1) l_1 h_1 + (c_2 + I)\bar{r} g \bar{\omega}_{l_2}(\omega_g(a_2 + I)) f(l_2, h_2) l_2 h_2 \\ + & \cdots + (c_n + I)\bar{r} g \bar{\omega}_{l_n}(\omega_g(a_m + I)) f(l_n, h_m) l_n h_m. \end{aligned}$$

Thus we have $c_i\omega_{l_i}(\omega_g(Ra_j))f(l_i, h_j)(l_i h_j) \subseteq I$ for all i and j with $1 \leq i \leq n$ and $1 \leq j \leq m$ since R/I is strongly CM-reflexive.

By induction on both n and m, considering every g in M, and for $1 \leq i \leq n$ and $1 \leq j \leq m$. If we take $n = 1$. Then $(c_1 l_1)(R * M)(a_1 h_1 + a_2 h_2 + \cdots + a_m h_m) = 0$. Thus, $(c_1 l_1)(rg)(a_1 h_1) + (c_1 l_1) rg(a_2 h_2) + \cdots + (c_1 l_1) rg(a_m h_m) = c_1\omega_{l_1}(\omega_g(ra_1))f(l_1, h_1)(l_1 h_1) + c_1\omega_{l_1}(\omega_g(ra_2))f(l_1, h_2)(l_1 h_2) + \cdots + c_1\omega_{l_1}(\omega_g(ra_m))f(l_1, h_m)(l_1 h_m) = 0$ for any $r \in R, g \in M$. By Lemma 1, M is cancellative we have $l_1 h_i \neq l_1 h_j$ for any i and j with $1 \leq i \neq j \leq m$. Then $c_1\omega_{l_1}(\omega_g(ra_j))f(l_1, h_j)(l_1 h_j) = 0, j = 1, 2, \ldots, m$. Thus, $c_1\omega_{l_1}(\omega_g(Ra_j)) = 0$ for any j. If $m = 1$, then proof is similar.

Now suppose that $n \geq 2$ and $m \geq 2$. Since M is a *u.p.*-monoid, there exist i, j with $1 \leq i \leq n$ and $1 \leq j \leq m$ such that $l_i g h_j$ is uniquely presented by considering two subsets $K = \{l_1 g, l_2 g, \ldots, l_n g\}$ and $H = \{h_1, h_2, \ldots, h_m\}$ of the monoid M. Without loss of generality, we may assume that $i = 1$ and $j = 1$. We can deduce that $c_1\omega_{l_1}(\omega_g(Ra_1))f(l_1 g, h_1)l_1(g h_1) = 0$, which implies that $c_1\omega_{l_1}(\omega_g(Ra_1)) = 0$. Since ω_g and ω_{l_1} are automorphisms of R, we have $c_1\omega_{l_1}(Ra_1) = 0$. Let $b = c_k r a_q$, where $r \in R, 1 \leq k \leq n, 1 \leq q \leq m$. Then $b \in I$. Since $(a_1 b c_1)^2 = 0$ and I is reduced, we have $a_1 b c_1 = 0$. Thus,
$(a_1 b c_2 l_2 + a_1 b c_3 l_3 + \cdots + a_1 b c_n l_n)(R * M)(a_1 h_1 + a_2 h_2 + \cdots + a_m h_m)$
$= (a_1 b \lambda)(c_1 l_1 + c_2 l_2 + \cdots + c_n l_n)(R * M)(a_1 h_1 + a_2 h_2 + \cdots + a_m h_m) = 0$. By induction, we have $a_1 b c_i \omega_{l_1}(\omega_g(Ra_j)) = 0$ for $2 \leq i \leq n$ and $1 \leq j \leq m$. Thus, $(a_1 b c_i R)^2 = 0$. Since I is reduced and ω is automorphism, it follows that $a_1 b c_i \omega_{l_i}(R) = 0$. Note that $b c_i \omega_{l_i}(Ra_1) \subseteq I$. Thus $b c_i \omega_{l_i}(Ra_1) = 0$ for any i. Now we have

$(b c_1 l_1 + b c_2 l_2 + \cdots + b c_n l_n)(R * M)(a_1 h_1 + a_2 h_2 + \cdots + a_m h_m) = (b\lambda)(c_1 l_1 + c_2 l_2 + \cdots + c_n l_n)(R * M)(a_1 h_1 + a_2 h_2 + \cdots + a_m h_m) = 0$.

By applying the induction hypothesis, it follows that $b c_i \omega_{l_i}(\omega_g(Ra_j)) = 0$ for all $1 \leq i \leq n$ and $2 \leq j \leq m$. Thus, we have $c_i\omega_{l_i}(\omega_g(ra_j)) = 0$ for all i, j and all $r \in R$. Particularly, we have $b c_k \omega_{l_k}(\omega_g(ra_q)) = 0$ and so $b^2 = 0$. Thus $b = 0$. This shows that $c_k\omega_{l_k}(\omega_g(Ra_q)) = 0$ for any $1 \leq k \leq n$ and $1 \leq q \leq m$. Consequently, we can see that $a_j\omega_{h_j}(\omega_g(Rc_i)) = 0$ for all $g \in M$, $1 \leq j \leq m$, and $1 \leq i \leq n$. Therefore, R is strongly CM-reflexive. □

The notion of complete M-compatibility is important in the following result [18].

Corollary 2. *Assuming R is a ring that is completely M-compatible, where M is a monoid with twisting $f : M \times M \to U(R)$ and action $\omega : M \to Aut(R)$, and I is an ideal of R such that I is reduced and R/I is CM-quasi-Armendariz, then R is strongly CM-reflexive.*

Proof. As CM-quasi-Armendariz rings are strongly CM-reflexive, the result can be obtained from Theorem 3. □

Corollary 3. *Suppose that R is a completely M-compatible ring, where M is a monoid with twisting $f : M \times M \to U(R)$ and action $\omega : M \to Aut(R)$. Let I be an ideal of R such that I is reduced and R/I is CM-Armendariz. Then, R is strongly CM-reflexive.*

Proof. Since CM-Armendariz is a CM-quasi-Armendariz, the result can be derived from Corollary 2. □

Proposition 2. *Assuming R is a ring that is both M-compatible and CM-quasi-Armendariz, where M is a monoid with twisting $f : M \times M \to U(R)$ and action $\omega : M \to Aut(R)$, then R is strongly CM-reflexive if and only if $R * M$ is strongly CM-reflexive.*

Proof. To prove a necessary condition is sufficient. Let $\phi = \Sigma_{i=1}^{n} c_i l_i, \psi = \Sigma_{j=1}^{m} a_j h_j \in R * M$ satisfying $\phi(R * M)\psi = 0$. Since R is CM-quasi-Armendariz, we have $c_i \omega_{l_i}(\omega_g(Ra_j))f(l_i, h_j)(l_i h_j) = 0$ for all i, j. This implies that $c_i \omega_{l_i}(\omega_g(Ra_j)) = 0$ for all i, j since R is M-compatible. Because R is a reflexive ring, $a_j R c_i = 0$. Then, $a_j \omega_{h_j}(\omega_g(Rc_i)) = 0$ for all i, j, and hence for any $r \in R, g \in M$, we have

$$\psi(R * M)\phi = \Sigma_{j=1}^{m} \Sigma_{i=1}^{n} a_j \omega_{h_j}(\omega_g(r\, c_i)) f(h_j, l_i)(h_j l_i) = 0.$$

Thus, $a_j \omega_{h_j}(\omega_g(r\, c_i)) = 0$ since R is M-compatible and CM-quasi-Armendariz. Therefore, R is strongly CM-reflexive. □

Every left APP-ring is quasi-Armendariz, but not conversely [19, 20].

Proposition 3. *Let M be a strictly totally ordered monoid with twisting $f : M \times M \to U(R)$ and action $\omega : M \to Aut(R)$. Let R be an M-compatible left APP-ring. Then R is strongly CM-reflexive if and only if $R * M$ is strongly CM-reflexive.*

Proof. If R is a left APP-ring, then it is M-quasi-Armendariz [21]. Therefore, the result follows from Proposition 2. □

Corollary 4. *Let R be a ring, M be a monoid with twisting $f : M \times M \to U(R)$ and action $\omega : M \to Aut(R)$. If R is a reduced, then R is strongly CM-reflexive.*

Proof. Since R is reduced, it is quasi-Armendariz. Therefore, the result can be derived from Proposition 2. □

3. Some results on ring extensions of Crossed product type

Let Δ be a multiplicative monoid consisting of central regular elements of R. Then, the set $\Delta^{-1}R := \{u^{-1}c | u \in \Delta, c \in R\}$ forms a ring. Suppose $\omega : M \to Aut(R)$ is a monoid homomorphism such that $\omega_h(\Delta) \subseteq \Delta$ for every $h \in M$. Then, ω can be extended to $\bar{\omega} : M \to Aut(\Delta^{-1}R)$ defined by $\bar{\omega}_h(u^{-1}c) = \omega_h(u)^{-1}\omega_h(c)$. If $f : M \times M \to U(R)$ is a twisted function, then it can be viewed as a twisted function from $M \times M$ to $U(\Delta^{-1}R)$ by noting that $U(R) \subseteq U(\Delta^{-1}R)$.

Theorem 4. *Assuming R is an M-compatible ring, where M is a cancellative monoid with twisting $f : M \times M \to U(R)$ and action $\omega : M \to Aut(R)$, then R is strongly CM-reflexive if and only if $\Delta^{-1}R$ is strongly CM-reflexive, where Δ is the multiplicative subset of R consisting of all elements that are not zero divisors modulo M.*

Proof. It is enough showing necessary. Assume that R is strongly CM-reflexive. Let $\phi = \Sigma_{i=1}^{n} u^{-1}c_i l_i, \psi = \Sigma_{j=1}^{m} v^{-1}a_j h_j$ be elements in $\Delta^{-1}R*M$ satisfying $\phi\varphi\psi = 0$, where $\varphi = \Sigma_{k=1}^{q} \lambda^{-1} b_k \ell_k$ is any nonzero element in $\Delta^{-1}R*M$. Then, we have $\alpha = (u_n u_{n-1} \dots u_1)\phi, \theta = (\lambda_q \lambda_{q-1} \dots \lambda_1)\varphi, \beta = (v_m v_{m-1} \dots v_1)\psi$ are in $R*M$. Since R is strongly CM-reflexive and $\alpha\theta\beta = 0$ we have

$$(u_n u_{n-1} \dots u_1 u_i^{-1} c_i)\omega_{l_i}(\omega_g(b(v_m v_{m-1} \dots v_1 v_j^{-1})a_j))f(l_i, h_j)(l_i h_j)(v_j u_i)^{-1} = 0$$

for all i, j and $b \in R$. It follows that $c_i\omega_{l_i}(\omega_g(Ra_j))f(l_i, h_j)(l_i h_j) = 0$ for any $g \in M$, because Δ is a multiplicative monoid consisting of central regular elements of R and all $u_i, v_j, \lambda_k \in \Delta$. Hence, $(u_i^{-1}c_i)\omega_{l_i}(\omega_g(Rv_j^{-1})a_j)) = c_i\omega_{l_i}(\omega_g(Ra_j))(\omega_{l_i}(v_j)u_i)^{-1} = 0$ for all i, j and ω is automorphism. Therefore, $\Delta^{-1}R$ is strongly CM-reflexive. □

The following statement describes how the strongly CM-reflexive property of a ring R is related to the property of its subrings, which are created by a central idempotent.

Proposition 4. *The following conditions are equivalent for a ring R, a monoid M with twisting $f : M \times M \to U(R)$, an action $\omega : M \to Aut(R)$, and a central idempotent e of R such that $\omega_g(e) = e$:*
(1) *R is strongly CM-reflexive.*
(2) *eR and $(1-e)R$ are strongly CM-reflexive.*

Proof. (1) ⇒ (2). It is easy.
(2) ⇒ (1). Assume that both eR and $(1-e)R$ are strongly CM-reflexive. Let $\phi = \Sigma_{i=1}^{n} c_i l_i, \psi = \Sigma_{j=1}^{m} a_j h_j \in R*M$ satisfying $\phi(R*M)\psi = 0$. Let

$$\phi_1 = \Sigma_{i=1}^{n} e\, c_i l_i, \psi_1 = \Sigma_{j=1}^{m} e\, a_j h_j, \phi_2 = \Sigma_{i=1}^{n}(1-e)c_i l_i, \psi_2 = \Sigma_{j=1}^{m}(1-e)a_j h_j.$$

clear that $\phi_1, \psi_1 \in (eR)*M$ and $\phi_2, \psi_2 \in ((1-e)R)*M$. Since e is a central idempotent of R such that $\omega_g(e) = e$ for each $g \in M$ and for any $r \in R$ we have

$$\begin{aligned}
& \phi_1((eR)*M)\psi_1 \\
= \;& ec_1(er)\omega_{l_1}(\omega_g(ea_1))f(l_1, h_1)l_1h_1 + \dots + ec_n(er)\omega_{l_n}(\omega_g(ea_m))f(l_n, h_m)l_nh_m \\
= \;& ec_1(er)\omega_{l_1}(\omega_g(e)\omega_{l_1}(\omega_g(a_1))f(l_1, h_1)l_1h_1 + \cdots \\
+ \;& ec_n(er)\omega_{l_n}(\omega_g(e))\omega_{l_n}(\omega_g(a_m))f(l_n, h_m)l_nh_m \\
= \;& ec_1e(er)\omega_{l_1}(a_1)f(l_1, h_1)l_1h_1 + \dots + ec_ne(er)\omega_{l_n}(a_m)f(l_n, h_m)l_nh_m \\
= \;& ec_1e^2(r)\omega_{l_1}(a_1)f(l_1, h_1)l_1h_1 + \dots + e^2c_n(r)\omega_{l_n}(a_m)f(l_n, h_m)l_nh_m \\
= \;& ec_1e(r)\omega_{l_1}(a_1)f(l_1, h_1)l_1h_1 + \dots + ec_n(r)\omega_{l_n}(a_m)f(l_n, h_m)l_nh_m \\
= \;& e^2c_1r\omega_{l_1}(a_1)f(l_1, h_1)l_1h_1 + \dots + e^2c_nr\omega_{l_n}(a_m)f(l_n, h_m)l_nh_m \\
= \;& ec_1r\omega_{l_1}(a_1)f(l_1, h_1)l_1h1 + \dots + ec_nr\omega_{l_n}(a_m)f(l_n, h_m)l_nh_m \\
= \;& e[c_1r\omega_{l_1}(a_1)f(l_1, h_1)l_1h_1 + \dots + c_nr\omega_{l_n}(a_m)f(l_n, h_m)l_nh_m] \\
= \;& e\phi(R*M)\psi = 0,
\end{aligned}$$

$$\begin{aligned}
& \phi_2((1-e)R*M)\psi_2 \\
= \; & (1-e)c_1((1-e)r)\omega_{l_1}(\omega_g((1-e)a_1))f(l_1,h_1)l_1h_1+\cdots \\
+ \; & (1-e)c_n((1-e)r)\omega_{l_n}(\omega_g((1-e)(1-e)a_m))f(l_n,h_m)l_nh_m \\
= \; & (1-e)c_1((1-e)r)\omega_{l_1}((1-e)a_1)f(l_1,h_1)l_1h_1 \\
+ \; & \cdots+(1-e)c_n(1-e)r\omega_{l_n}((1-e)a_m)f(l_n,h_m)l_nh_m \\
= \; & (1-e)[c_1r\omega_{l_1}(a_1)f(l_1,h_1)l_1h_1+\cdots+c_nr\omega_{l_n}(a_m)f(l_n,h_m)l_nh_m] \\
= \; & (1-e)\phi(R*M)\psi=0.
\end{aligned}$$

Because eR and $(1-e)R$ are strongly CM-reflexive subrings of R, we conclude that $\psi_1((eR)*M)\phi_1=0, \psi_2(((1-e)R)*M)\phi_2=0$. Therefore, we have

$$\begin{aligned}
\psi(R*M)\phi = \; & \psi_1((eR)*M)\phi_1+\psi_2(((1-e)R)*M)\phi_2 \\
= \; & e\psi(R*M)\phi+(1-e)\psi(R*M)\phi=0.
\end{aligned}$$

Therefore, R is strongly CM-reflexive, which concludes the proof. □

Proposition 5. *Let R be a ring and M is a strictly ordered monoid with a twisting $f: M\times M\to U(R)$ and an action $\omega: M\to Aut(R)$. Assume that R is CM-quasi-Armendariz. Let e be a nonzero idempotent in R such that $\omega_g(e)=e$ for all $g\in M$. Then, the subring eRe is strongly CM-reflexive.*

Proof. The proof is a variant of the proof given in Proposition 2.9 [17]. Let $\phi = c_1l_1+c_2l_2+\cdots+c_nl_n$ and $\psi=a_1h_1+a_2h_2+\cdots+a_mh_m\in(eRe)*M$ satisfy $\phi((eRe)*M)\psi=0$. Since M is a strictly totally ordered monoid, we can assume that $l_i\preceq l_j$ and $h_i\preceq h_j$ whenever $i<j$. Since R is CM-quasi-Armendariz, then so is eRe. Thus, we have $c_i\omega_{l_i}(\omega_g((eRe)a_j))f(l_i,h_j)(l_ih_j)=0$ for all i,j. This implies that $c_i\omega_{l_i}(\omega_g((eRe)a_j))=0$ for all i,j since R is M-compatible and ω is an automorphism. Therefore, by Proposition 2, eRe is strongly CM-reflexive. □

Corollary 5. *[20, Proposition 3.7] Let $e\in R$ be an idempotent. If R is a left APP, then eRe is a left APP-ring.*

Corollary 6. *[22, Corollary 3.19] Let M be a strictly totally ordered monoid and $\omega: M\to End(R)$ a monoid homomorphism. Assume that e be an idempotent. If R is left APP, then eRe is (M,ω)-quasi-Armendariz.*

Proposition 6. *Let M be a strictly totally ordered monoid with twisting $f: M\times M\to U(R)$ and action $\omega: M\to Aut(R)$. Assume that e be an idempotent. If R is a left APP, then eRe is strongly CM-reflexive.*

Proof. By Corollary 5, eRe is a left APP. So, eRe is (M,ω)-quasi-Armendariz by Corollary 6. Thus, the result follows from Proposition 5. □

Let I be an index set and R_i be a ring for each $i\in I$. Let M be a strictly ordered monoid and $\omega^i: M\to End(R_i)$ a monoid homomorphism. Then the mapping $\omega: M\to End(\prod_{i\in I}R_i)$ is a monoid homomorphism given by $\omega_g(\{r_i\}_{i\in I})=\{(\omega^i)_g(r_i)\}_{i\in I}\}$ for all $g\in M$.

Proposition 7. *Let R_i be a ring for each i in a finite index set I, and let M be a monoid with a twisting $f : M \times M \to \bigcup_{i \in I} U(R_i)$ and an action $\omega^i : M \to Aut(R_i)$ on each R_i. Suppose that each R_i is strongly CM-reflexive. Then, the direct product $R = \prod_{i \in I} R_i$, equipped with the product action $\omega = \prod_{i \in I} \omega^i$, is strongly CM-reflexive.*

Proof. Let $R = \prod_{i \in I} R_i$ be the direct product of rings $(R_i)_{i \in I}$ and R_i is is strongly CM-reflexive for each $i \in I$. Denote the projection $R \to R_i$ as Π_i. Suppose that $\phi, \psi \in R * M$ are such that $\phi(R * M)\psi = 0$. Set $\phi_i = \prod_i \phi$, $\psi_i = \prod_i \psi$ and $\varphi_i = \prod_i \varphi$. Then $\phi_i, \psi_i \in R_i * M$. For any $u, v \in M$, assume $\phi(u) = (c_i^u)_{i \in I}$, $\psi(v) = (a_i^v)_{i \in I}$. Now, for any $r \in R$ and any $g \in M$,

$$\begin{aligned}
\phi(R * M)\psi &= \sum_{(u,v) \in X_s(\phi, c_r\psi)} \phi(u)\omega_u(\omega_g(r\psi(v)))f(u_m, v_n)u_m v_n \\
&= \sum_{(u,v) \in X_s(\phi, c_r\psi)} (c_i^u)_{i \in I}((\prod_{i \in I} \omega^i)_u(\omega_g(r_i a_i^v))f(u_m^i, v_n^i)u_m^i v_n^i)_{i \in I} \\
&= \sum_{(u,v) \in X_s(\phi, c_r\psi)} (c_i^u)_{i \in I}(\prod_{i \in I} \omega_u^i)(\omega_g(r_i a_i^v)f(u_m^i, v_n^i)u_m^i v_n^i)_{i \in I} \\
&= \sum_{(u,v) \in X_s(\phi, c_r\psi)} (c_i^u \omega_u^i(\omega_g(r_i a_i^v))f(\phi_i, \psi_i)u_m^i v_n^i)_{i \in I} \\
&= \sum_{(u,v) \in X_s(\phi, c_r\psi)} (\phi_i(u)\omega_u^i(r_i\psi_i(v)))f(\phi_i, \psi_i)u_m^i v_n^i)_{i \in I} \\
&= \Big(\sum_{(u,v) \in X_s(\phi, c_r\psi)} \phi_i(u)\omega_u^i(\omega_g(r_i\psi_i(v))\Big)f(\phi_i, \psi_i)u_m^i v_n^i)_{i \in I} \\
&= \Big(\sum_{(u,v) \in X_s(\phi_i, c_{r_i}\psi_i)} \phi_i(u)\omega_u^i(\omega_g(r_i\psi_i(v)))\Big)f(\phi_i, \psi_i)u_m^i v_n^i)_{i \in I} \\
&= (\phi_i(R_i * M)\psi_i)_{i \in I}.
\end{aligned}$$

Since $\phi(R * M)\psi = 0$, we have $\phi_i(R_i * M)\psi_i = 0$.

Now it follows $\phi_i(u)\omega_u^i(\omega_g(r_i\psi_i(v))) = 0$ for any $r \in R$, any $u, v, g \in M$ and any $i \in I$, since R_i is strongly CM-reflexive. Hence, for any $u, v \in M$,

$$\psi(v)\omega_v(\omega_g(r\phi(u))) = (\psi_i(v)\omega_v^i(\omega_g(r_i\phi_i(u))))_{i \in I} = 0$$

since I is finite. Thus, $\psi(v)\omega_v(\omega_g(r\phi(u))) = 0$ by the compatibility of ω. Therefore, $\psi(R * M)\phi = 0$. This means that R is strongly CM-reflexive. □

Theorem 5. *Assuming that R is an M-compatible ring and M is a cancellative monoid with a twisting map $f : M \times M \to U(R)$ and an action map $\omega : M \to Aut(R)$, and considering R as a right Ore ring with the classical right quotient ring Q, the R is strongly CM-reflexive if and only if Q is strongly CM-reflexive.*

Proof. It is enough showing necessary. Assume that R is strongly CM-reflexive. Let $\phi = \Sigma_{i=1}^m \alpha_i l_i, \psi = \Sigma_{k=1}^p \gamma_k h_k$ be elements in $Q * M$ satisfying $\phi\varphi\psi = 0$, where $\varphi = \Sigma_{j=1}^n \beta_j g_j$ is any nonzero element in $Q * M$. By Proposition 2.1.16 [23], we may assume that $\alpha_i = a_i u^{-1}, \beta_j = b_j v^{-1}$ and $\gamma_k = c_k w^{-1}$ with regular $u, v, w \in R$. Also, Proposition 2.1.16 [23], for each j and k, there exist $d_j, e_k \in R$ and regular $s, t \in R$ such that $u^{-1}b_j = d_j s^{-1}$

and $(vs)^{-1}c_k = e_k t^{-1}$. Suppose $\phi_1 = \Sigma_{i=1}^{m} a_i l_i, \varphi_1 = \Sigma_{j=1}^{n} b_j g_j, \varphi_2 = \Sigma_{j=1}^{n} d_j g_j, \psi_1 = \Sigma_{k=1}^{p} c_k h_k, \psi_2 = \Sigma_{k=1}^{p} e_k h_k \in R * M$. Since M is a cancellative monoid by Lemma 1. Thus, $g_i s h_1 \neq g_j s h_1$ for $g_i \neq g_j$. Then, we have $0 = \phi\varphi\psi = \Sigma_{i=1}^{m}\Sigma_{k=1}^{p}(a_i u^{-1})\omega_{l_i}(\omega_g(R\, c_k w^{-1}))$ $f(l_i, h_k)(l_i h_k) = \Sigma_{i=1}^{m}\Sigma_{k=1}^{p} a_i \omega_{l_i}(\omega_g(Re_k)) f(l_i, h_k)(l_i h_k)(\omega_{l_i}(t)w)^{-1} = 0 = \phi_1\varphi_2\psi_2(wt)^{-1}$. Therefore, $\phi_1\varphi_2\psi_2 = 0$. Since R is strongly CM-reflexive, then $\psi_2\varphi_2\phi_1 = 0$. This implies that $\phi_1 u\varphi_2\psi_2 = \phi_1\varphi_1 s\psi_2 = 0$ since $u^{-1}b_j = d_j s^{-1}$, then $s\psi_2\varphi_1\phi_1 = 0$ and $(vs)\psi_2\varphi_1\phi_1 = 0$, so $\psi_1\varphi_1\phi_1 = 0$ since $(vs)^{-1}c_k = e_k t^{-1}$. Using Proposition 2.1.16 [23] again, for each i, j there exist $\varphi_i, \phi_j \in R * M$ and regular element $q, p \in R$ such that $w^{-1}b_j = \phi_j q^{-1}$ and $(vq)^{-1}a_i = \varphi_i p^{-1}$. Let $\phi_2 = \Sigma_{i=1}^{m}\varphi_i l_i, \varphi_3 = \Sigma_{j=1}^{n}\phi_j g_j$. Then,
$q\psi_1\varphi_1\phi_1 = \Sigma_{i=1}^{m}\Sigma_{k=1}^{p} q(c_k w^{-1})\omega_{h_k}(\omega_g(Ra_i u^{-1})) f(h_k, l_i)(h_k l_i) = \Sigma_{i=1}^{m}\Sigma_{k=1}^{p} q(c_k)\omega_{h_k}(\omega_g(Ra_i)) \times$ $(\omega_{h_k}(u)w)^{-1} = 0$ since $\psi_1\varphi_1\phi_1 = 0$. Thus, for all k, i we have $c_k\omega_{h_k}(\omega_g(Ra_i)) = 0$, and it follows that $\phi_1 w\varphi_3 q\psi_2 = \Sigma_{i=1}^{m}\Sigma_{k=1}^{p} w a_i \omega_{l_i}(\omega_g(Re_k)) = 0$ since $w^{-1}b_j = \phi_j q^{-1}$. Then, $\psi_1\varphi_3\phi_1 w = 0$ since R is strongly CM-reflexive, and so $\psi_1\varphi_3\phi_1 = 0$. Therefore, $\psi_1\varphi_3\phi_1 p = \Sigma_{k=1}^{p}\Sigma_{i=1}^{m} c_k\omega_{h_k}(\omega_g(Ra_i)) f(h_k, l_i)(h_k l_i)p = \psi_1\varphi_3\phi_2(vq) = \Sigma_{k=1}^{p}\Sigma_{j=1}^{n} c_k\omega_{h_k}(\omega_g(Rd_j))(pv) = 0$, and thus $\psi_1\varphi_3\phi_2 = 0$. Therefore,

$$\psi\varphi\phi = \Sigma_{k=1}^{p}\Sigma_{i=1}^{m}(c_k w^{-1})\omega_{h_k}(\omega_g(R\, a_i u^{-1})) = \Sigma_{k=1}^{p}\Sigma_{i=1}^{m} c_k\omega_{h_k}(\omega_g(Ra_i))(\omega_{h_k}(u)w)^{-1} = 0.$$

Thus, $c_k\omega_{h_k}(\omega_g(Ra_i))(up)^{-1} = 0. Therefore, \mathrm{Q} is strongly \mathrm{CM}-reflexive.$ □

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