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On The Farthest Point Problem In Hilbert Spaces

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Abstract. In an attempt to solve the farthest point problem, we introduce a new class of uniquely remotal sets. Namely, the class of uniquely distant sets. Then, we prove that in a separable Hilbert space, every uniquely distant set is a singleton.

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1. Introduction

Let X be a normed space, and E be a closed bounded subset of X. Define the real valued function $r(.,E) : X \to \mathbb{R}$ by

$$r(x, E) = \sup\{\|x - e\| : e \in E\},$$

which is known as the farthest distance function. The set E is said to be remotal if for every $x \in X$, there exists $e \in E$ such that $r(x, E) = \|x - e\|$. In this case, we denote the set $\{e \in E : r(x, E) = \|x - e\|\}$ by $F(x, E)$. It is clear that $F(., E) : X \to E$ is a multi-valued function. However, if $F(., E) : X \to E$ is a single-valued function, then E is called uniquely remotal, also known as max-Chebyshev set. In this case, we denote $F(x, E)$ by $F(x)$, if no confusion arises.

The study of remotal and uniquely remotal sets has attracted many mathematicians in the last decades, due to its connection to the geometry of Banach spaces. We refer the reader to [1], [10], [11], [12], [13], [15], [8], [2] and [9] for samples of these studies. However, uniquely remotal sets are of special interest.

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One of the most interesting and challenging problems in this field, is known as the farthest point problem, which is stated as:

Q) If E is a uniquely remotal set in a normed space X, does it follow that E is a singleton?

This problem became more important when Klee [14] proved that singletoness of closed uniquely remotal sets is equivalent to convexity of Chebyshev sets in Hilbert spaces (which is an open problem too, in the theory of nearest points). Since then, a considerable work has been done to answer this question, and many partial results have been obtained in the positive direction.

An element c in a normed space X is called a Chebyshev center of $E \subset X$ if

$$r(c, E) = \inf_{x \in X} r(x, E).$$

Whether a set has a Chebyshev center or not is still an open question. However, in inner product spaces, any closed bounded set does have a Chebyshev center [7]. Chebyshev Centers of sets have played a major role in the study of uniquely remotal sets. We refer the reader to [4], [5], [3] and[6].

For $x, y \in X$, $[x, y]$ is the line segment joining x and y. In [16] it was proved that if E is a uniquely remotal subset of a normed space, admitting a Chebyshev center c, and if F, restricted to the line segment $[c, F(c)]$ is continuous at c, then E is a singleton.

Partially continuity was introduced in [17] as follows:

Let $G : A \subset X \to X$ be a function, and let $a \in A$. Then G is partially continuous at a if there exists a nonconstant sequence $(a_n) \subset A$, such that $a_n \to a$ and $G(a_n) \to G(a)$.

In [17], the following result was proved:

Theorem 1. *Let E be a closed subset of a normed space X, admitting a Chebyshev center c. If E is uniquely remotal, and if $F : [c, F(c)] \to E$ is partially continuous at c, then E is a singleton.*

One of the important notions that appeared in the literature while studying remotal and uniquely remotal sets is the so-called storngly remotal set. It was first introduced by Kalil et al in [8] for general Banach spaces as follows:

Let X be a Banach space and E be a nonempty closed convex bounded subset of X. Let $M = \{\phi : \phi : [0, \infty) \to [0, \infty) \text{ be strictly increasing function } , \phi(0) = 0\}$, and $N = \{\psi : \psi : X \to (0, 1) \text{ such that } \psi(x) \leq \psi(y) \text{ whenever } ||x|| < ||y||\}$. E is called stongly remotal in X if there exist $\phi \in M$ and $\psi \in N$, with $\inf_{y \in X} \psi(y) > 0$, such that for each $x \in X$ there exists $y \in E$ such that for all $z \in E$, the following inequality holds

$$\phi(||x-y||) \geq \phi(||x-z||) + \psi(y)\phi(||y-z||) \tag{1}$$

It was also proved in [8] that strong remotality of E, with the associated functions ϕ and ψ, is equivalent to saying that:

For every $x \in X$, there exists $y \in E$ such that

$$\inf_{z\in E\setminus\{y\}} \left\{ \frac{\phi(||x-y||) - \phi(||x-z||)}{\phi(||y-z||)} \right\} > 0. \tag{2}$$

clearly, strong remotal sets in Hilbert space can be defined similarly. We provide the definition below with $\phi(t) = t^2$.

Definition 1. *Let E be a non-empty closed convex bounded set in a Hilbert space H. Then E is called strongly remotal in H if for every $x \in H$ there exists $y \in E$ such that*

$$\inf_{z\in E\setminus\{y\}} \left\{ \frac{\|x-y\|^2 - \|x-z\|^2}{\|y-z\|^2} \right\} > 0 \tag{3}$$

It is easy to see that inequality 1, with $\phi(t) = t^2$, can be written as

$$||x-y||^2 \geq ||x-z||^2 + \psi(y)||y-z||^2. \tag{4}$$

The main purpose of this article is to give an answer for the farthest point problem in separable Hilbert spaces, considering a special class of uniquely remotal sets that satisfy a certain condition.

2. Main Result

In this section, we introduce the so-called uniquely distant sets, and prove that every uniquely distant set in a separable Hilbert space is a singleton. First we prove the following result that plays an important role in the proof of our main result.

Theorem 2. *Let E be a non-empty strongly remotal subset of a separable Hilbert space H. Then the mapping $F : H \to E$, defined by $F(x) = F(x, E)$ is continuous on H.*

Proof. Let $x \in H$ be such that $r(x, E) = r_0$. Let (x_n) be a sequence in H such that $x_n \to x$. We claim that $F(x_n) \to F(x)$.

E is strongly remotal, so

$$\|x - F(x)\|^2 \geq \|x-z\|^2 + \psi(z)\|F(x) - z\|^2, \tag{5}$$

for all $z \in E$.

Let $z_n = F(x_n)$. Then,

$$\|x - F(x)\|^2 \geq \|x - F(x_n)\|^2 + \psi(y)\|F(x) - F(x_n)\|^2 \text{ for all } n. \tag{6}$$

This implies

$$\|F(x) - F(x_n)\|^2 \leq \frac{1}{\psi(y)}[\ \|x - F(x)\|^2 - \|x - F(x_n)\|^2] \tag{7}$$

Since $F(x_n)$ is the farthest point in E from x_n we have $\|x_n - F(x_n)\| \geq \|x_n - F(x)\|$, it follows that

$$\|x_n - F(x)\| \leq \|x_n - F(x_n)\| \leq \|x_n - x\| + \|x - F(x_n)\|, for\ all\ n \in \mathbb{N} \tag{8}$$

Taking the limits on both sides of 8 as $n \to \infty$, we obtain

$$\|x - F(x)\| \leq \lim_{n\to\infty} \|x - F(x_n)\| \tag{9}$$

Inequality 7 is true for all n, which implies

$$\lim_{n\to\infty} \|F(x) - F(x_n)\|^2 \leq \frac{1}{\psi(y)}[\|x - F(x)\|^2 - \lim_{n\to\infty} \|x - F(x_n)\|^2] \tag{10}$$

Substitute 9 in inequality 10 to get

$$\lim_{n\to\infty} \|F(x) - F(x_n)\|^2 = 0$$

So $F(x_n) \to F(x)$, which completes the proof.

Theorems 2 and 1 imply the following important result:

Theorem 3. *Every strongly remotal set in a Hilbert space is a singleton.*

Remark 1. *Theorems 1 and 2 imply Garkavi's result in [7], which states that a Chebyshev center in a Hilbert space exists and is unique.*

Next, we introduce the so-called uniquely distant sets.

Definition 2. *Let H be a Hilbert space and $E \subset H$ be a closed bounded subset. Then E is said to be* ***uniquely distant set*** *in H if the following two conditions are satisfied:*

(i) E is uniquely remotal

(ii) If $x \in H$, and y is the farthest point from x in E, then for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$r(x, E \backslash B(y, \delta)) \leq r(x, E) - \varepsilon.$$

Now, we are ready to prove the main result of this paper.

Theorem 4. *Every uniquely distant subset of a separable Hilbert space is a singleton.*

Proof.
Let E be a nonsingleton uniquely distant subset in a Hilbert space H. Using Theorem 2.6 in [10], we can assume, without loss of generality, that E is convex. Let $x \in H$ be any element, and y is a farthest element of x in E. Now, assume with no loss of generality that $\|x - y\| = r(x, E) = 1$.

We claim that

$$\inf\{\frac{\|x-y\|^2 - \|x-z\|^2}{\|y-z\|^2} : z \in E \backslash \{y\}\} > 0.$$

Assume on the contrary that

$$\inf\{\frac{\|x-y\|^2 - \|x-z\|^2}{\|y-z\|^2} : z \in E \backslash \{y\}\} = 0$$

Then, there exists a sequence $z_n \in E$ such that

$$\lim_{n\to\infty} \frac{\|x-y\|^2 - \|x-z_n\|^2}{\|y-z_n\|^2} = 0 \tag{11}$$

Since E is bounded, this implies that

$$\lim_{n\to\infty} \|x - z_n\| = \|x - y\| \tag{12}$$

We claim that $z_n \to y$. If not, then there exists an open ball $B(y, \delta)$ such that $z_n \notin B(y, \delta) \cap E$ for all n. The set $E \backslash B(y, \delta) \subsetneq E$ is a closed set.

Since E is uniquely distant then for any $\varepsilon > 0$, we have $B(y, \delta)$ such that $r(x, E \backslash B(y, \delta)) \leq 1 - \varepsilon$. In this case equation 12 is not true. Thus, we must have $\lim_{n\to\infty} z_n = y$.

Now, let $[p, q]$ be the line segment joining p and q. Since H is a Hilbert space, it follows that for each z_n there exists w_n in $[x, y]$ such that $[z_n, w_n]$ is orthogonal to $[x, y]$. Assume that $\|z_n - w_n\| = d_n$, and $\|y - w_n\| = a_n$ and $\|x - w_n\| = b_n$. From basic geometry in Hilbert spaces we have:

$\|x-y\|^2 = 1 = (a_n+b_n)^2$, $\|x-z_n\|^2 = d_n^2+b_n^2$, and $\|y-z_n\|^2 = d_n^2+a_n^2$. Thus

$$\frac{\|x-y\|^2 - \|x-z_n\|^2}{\|y-z_n\|^2} = \frac{(a_n+b_n)^2-(d_n^2+b_n^2)}{d_n^2+a_n^2}. \tag{13}$$

But

$$\begin{aligned}(a_n+b_n)^2-(d_n^2+b_n^2) &= a_n^2+b_n^2+2a_nb_n-(d_n^2+b_n^2)\\ &= 2a_n^2+2a_nb_n-(d_n^2+a_n^2).\end{aligned}$$

Since E is uniquely remotal and $r(x,E) = \|x-y\|$, so $\|x-y\| > \|x-z_n\|$ for all n, It follows that

$$2a_n^2+2a_nb_n-(d_n^2+a_n^2) > 0$$

But $2a_n^2+2a_nb_n = 2a_n(a_n+b_n) = 2a_n$. Thus

$$2a_n-(d_n^2+a_n^2) > 0. \tag{14}$$

Equations 13 and 14 gives

$$\frac{\|x-y\|^2 - \|x-z_n\|^2}{\|y-z_n\|^2} = \frac{2a_n-(d_n^2+a_n^2)}{d_n^2+a_n^2} = \frac{2a_n}{d_n^2+a_n^2} - 1. \tag{15}$$

From 14 we have $\frac{2a_n}{d_n^2+a_n^2} - 1 > 0$, which is equivalent to

$$\frac{2a_n}{d_n^2+a_n^2} > 1. \tag{16}$$

Now, one should remark that as z_n approaches y, then $(a_n,d_n) \to (0,0)$. Equation 16 implies that a_n does not converge to zero before d_n, otherwise we obtain $\frac{2a_n}{d_n^2+a_n^2} \to 0$, which contradicts 16.

Therefore, we can assume that d_n converges to 0 as a function of a_n, since $\|y-z_n\|^2 = d_n^2+a_n^2$. Also, by sequential criterion, assume with no loss of generality that d_n is a differentiable function of a_n.

Now, we can think of the following limit:

$$\lim_{z\to y} \frac{\|x-y\|^2 - \|x-z_n\|^2}{\|y-z_n\|^2} = \lim_{(a_n,d_n)\to(0,0)} \left(\frac{2a_n}{d_n^2+a_n^2} - 1\right), \tag{17}$$

as $\lim_{w\to 0}\left(\frac{2w}{P^2(w)+w^2} - 1\right)$. Using L'Hopital's Rule we get:

$$\lim_{(a_n,d_n)\to(0,0)}\left(\frac{2a_n}{d_n^2+a_n^2}-1\right)=\lim_{(a_n,d_n)\to(0,0)}\left(\frac{2}{2d_n.d_n'+2a_n}-1\right). \tag{18}$$

Clearly, the limit in 18 is certainly bounded away from zero noting that by unique remotality, d_n' can not equal to infinity. Otherwise, $\|x-z_n\|$ will reach $\|x-y\|$ before z_n reaches y. Hence E is strongly remotal, and by Theorem 3, E is a singleton. This completes the proof.

Clearly, one can see that every uniquely distant set is uniquely remotal. However, we could not find a uniquely remotal set that is not uniquely distant. So, we believe that they are equivalent, and we conjecture the following:

Conjecture: Every uniquely remotal set in Hilbert space is uniquely distant.

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