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π-Modules

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Abstract. In this paper we characterize π-modules. Next we establish some equivalent conditions for an almost π-module to be a π-module.

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1. Introduction

Throughout this paper R denotes a commutative ring with identity and all modules are unital R-modules. $L(R)$ denotes the lattice of all ideals of R. Throughout this paper M denotes a unital R-module. In this paper we introduce and study the concepts of π-module and almost π-module. In Section 3, we prove that a faithful R-module M is a π-module if and only if R is a π-ring and M is a multiplication module if and only if every cyclic submodule of M is of the form IM, where I is a finite product of

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quasi-principal prime ideals of rank less than or equal to one (see Theorem 1). Using these results, we establish some equivalent conditions for an almost π-module to be a π-module (see Theorem 2).

2. Basic notions

For any $x \in R$, the principal ideal generated by x is denoted by (x). Recall that an ideal I of R is called a *multiplication ideal*, if for every ideal $J \subseteq I$, there exists an ideal K with $J = KI$. Multiplication ideals have been extensively studied - for example, see [2], [5] and [6]. An ideal I of R is called a *quasi-principal ideal* [16, Exercise 10, Page 147] (or a principal element of $L(R)$ [19]) if it satisfies the following identities (i) $(A \cap (B : I))I = AI \cap B$ and (ii) $(A + BI) : I = (A : I) + B$, for all $A, B \in L(R)$. It is well known that an ideal I of R is quasi-principal if and only if it is finitely generated and locally principal if and only if it is a finitely generated multiplication ideal [8, Theorem 3]. R is a *π-ring* if every principal ideal is a finite product of prime ideals of R. R is an *almost π-ring* if R_P is a π-ring, for every maximal ideal P of R. π-rings have been extensively studied - for example, see [13], [15] and [17]. By a *special principal ideal ring*, we mean a principal ideal ring R with exactly one prime ideal $P \neq R$, $P^n = (0)$ for some positive integer n, so the only ideals of R are $R, P, P^2, \ldots, P^n = (0)$.

A submodule N of M is *proper* if $N \neq M$. For any two submodules N and K of M, the ideal $\{a \in R \mid aK \subseteq N\}$ will be denoted by $(N : K)$. Thus $(O : M)$ is the annihilator of M. M is said to be a *faithful module* if $(O : M)$ is the zero ideal of R. M is said to be a *multiplication module* [9] if every submodule of M is of the form IM, for some ideal I of R. A submodule N of M is said to be a *multiplication submodule* if for every submodule $N_1 \subseteq N$, there exists an ideal J of R such that $N_1 = JN$. An R-module M is said to be *locally cyclic* if M_P is a cyclic R_P-module for all maximal ideals P of R.

A proper submodule N of M is said to be a *maximal submodule*, if it is not properly contained in any other proper submodule of M. A proper submodule N of M is a *prime submodule*, if for any $r \in R$ and $m \in M$, $rm \in N$ implies either $m \in N$ or $r \in (N : M)$. A proper submodule N of M is a *primary submodule*, if for any $r \in R$ and $m \in M$, $rm \in N$ implies either $m \in N$ or $r^n \in (N : M)$ for some positive integer n. By a *minimal prime submodule over* a submodule N of M (or a *prime submodule minimal over* N), we mean a prime submodule which is minimal in the collection of all prime submodules containing N. Minimal prime submodules over the zero submodule are simply called the minimal prime submodules. It is well known that maximal submodules and prime submodules exist in multiplication modules (for details, see [11]). It is well known that if M is a faithful multiplication R-module and P is a prime ideal of R such that $M \neq PM$, then PM is a prime submodule of M and every prime submodule of M is of the form PM for some prime ideal P of R (see [11, Corollary 2.11]). Also if M is a faithful and finitely generated multiplication R-module and P is a prime ideal of R, then by [11, Theorem 3.l], PM is a proper prime submodule of M. Further PM is minimal over a submodule N of M if and only if P is minimal over the ideal $(N : M)$ of R.

By a *multiplicative lattice* we mean a complete lattice L on which there is defined a commutative, associative multiplication which distributes over arbitrary joins (i.e., $a(\vee b_\alpha) = \vee_\alpha a b_\alpha$) and has compact greatest element 1 as a multiplicative identity [1]. An element e of a multiplicative lattice L is said to be *principal* if it satisfies the dual identities (i) $a \wedge be = ((a : e) \wedge b)e$ and (ii) $(a \vee be) : e = (a : e) \vee b$. A principally generated, compactly generated modular multiplicative lattice is called an *r-lattice*. An r-lattice L is said to be a *π-lattice* [1] if L is generated by a set S of elements (not necessarily principal) each of which is a finite product of prime elements. It should be mentioned that every principal ideal of R is quasi-principal and hence $L(R)$, the lattice of all ideals of R, is an r-lattice. Note that if R is a π-ring, then $L(R)$ is a π-lattice.

For general background and terminology, the reader is referred to [16] and [20].

3. π-modules

In this section, we characterize π-modules. Next we establish several equivalent conditions for an almost π-module to be a π-module.

We shall begin with the following definition.

Definition 1. *An R-module M is said to be a π-module if every proper cyclic submodule N of M is of the form IM, where I is a finite product of prime ideals of R.*

Definition 2. *An R-module M is said to be an almost π-module if for any maximal ideal P of R, the R_P-module M_P is a π-module.*

Observe that π-rings and cyclic modules over π-rings are examples of π-modules. Almost π-rings are almost π-modules. Again note that π-modules are almost π-modules, but the converse need not be true.

Lemma 1. *Suppose M is a π-module. Then M is a multiplication module.*

Proof. The proof of the lemma follows from [11, Proposition 1.1].

Lemma 2. *Suppose M is a faithful π-module. Then (i) R contains only finitely many minimal prime ideals of R.*

(ii) M contains only finitely many minimal prime submodules.

(iii) M is finitely generated.

Proof. (i). As M is a faithful π-module, the zero ideal is a finite product of prime ideals and hence R contains only finitely many minimal prime ideals.

(ii). By Lemma 1, M is a multiplication module. As M is faithful, it follows that every minimal prime submodule is of the form PM for some minimal prime ideal P of R. So by (i), M contains only finitely many minimal prime submodules.

(iii). The result follows from (ii) and [11, Theorem 3.7].

Lemma 3. *Suppose M is a faithful π-module. Then every proper cyclic submodule of M has only finitely many minimal primes.*

Proof. Let $x \in M$. As M is a π-module, by definition, $Rx = P_1P_2\cdots P_nM$ for some prime ideals $P_1, P_2, \cdots, P_n$ of R. Let N be a prime submodule minimal over Rx. Note that by Lemma 1 and Lemma 2, M is a faithful and finitely generated multiplication R-module. So $N = PM$ for some prime ideal P of R. As $Rx \subseteq N$, by [11, Theorem 3.1], it follows that $P_i \subseteq P$ for some i, so $Rx \subseteq P_iM \subseteq PM = N$. As M is a faithful and finitely generated multiplication R-module, it follows that P_iM is a prime submodule and hence $P_iM = N$. Therefore Rx has only finitely many minimal primes. This completes the proof of the lemma.

Lemma 4. *Suppose M is a faithful cyclic R-module. Then R is a π-ring if and only if M is a π-module.*

Proof. The proof of the lemma follows from [14, Lemma 6] and [11, Theorem 3.1].

Lemma 5. *Suppose M is a faithful and finitely generated multiplication R-module. Then M is an almost π-module if and only if R is an almost π-ring.*

Proof. Let P be a maximal ideal of R. Consider the R_P-module M_P. As M is a finitely generated faithful multiplication R-module, it follows that M_P is a faithful cyclic R_P-module. So by Lemma 4, M_P is a π-module if and only if R_P is a π-ring. Therefore R is an almost π-ring if and only if M is an almost π-module. This completes the proof of the theorem.

Lemma 6. *Let M be a faithful π-module. If N is a minimal prime submodule, then N is a multiplication submodule.*

Proof. Note that by Lemma 1 and Lemma 2, M is a faithful and finitely generated multiplication module. Suppose N is a minimal prime submodule. Then $N = PM$ for some minimal prime ideal P of R. Suppose $Rx \subseteq PM$ for some $x \in M$. As M is a π-module, it follows that $Rx = IM$, where $I = P_1P_2...P_n$ and $P_i's$ are prime ideals of R. Since $Rx \subseteq PM$, by [11, Theorem 3.1], it follows that $P_i \subseteq P$ for some i. As P is a minimal prime ideal, it follows that $P = P_i$. Therefore $Rx = J(PM) = JN$ for some $J \in L(R)$. Consequently, N is a multiplication submodule.

Lemma 7. *Let M be a faithful π-module. If N is a prime submodule minimal over a cyclic submodule of M, then N is either minimal or a multiplication submodule with $rankN = 1$.*

Proof. Observe that M is a faithful and finitely generated multiplication module. Suppose N is a prime submodule minimal over a cyclic submodule of M. Then $N = PM$ for some prime ideal P of R. Suppose PM is non-minimal. Then P is non-minimal. Let $P_0 \supseteq P$ be a maximal ideal of R. Suppose PM is minimal over a cyclic submodule Ry of M. Then by [18, Lemma 1.4], P is minimal over a quasi-principal ideal $(Ry : M)$ of R. Note that by Lemma 5, R is an almost π-ring. Therefore by [12, Theorem 46.8 and Corollary 46.10, Page 576-577], for every maximal ideal Q of R, R_Q is either a π-domain or a special principal ideal ring. As R_{P_0} is a π-domain and P_{P_0} is a prime minimal over a non-zero principal element of R_{P_0}, by [15, Theorem 4.2 and Corollary 4.3], P_{P_0} is principal and rank $P = 1$. Again note that P is locally principal. Observe that by [12, Corollary 46.9, Page 577], every prime ideal contains a unique minimal prime ideal. Let P' be a minimal prime ideal contained in P. Then $P'P = P'$ locally and hence globally. Now we show that PM is a multiplication submodule. Suppose $Rx \subseteq PM$ for some $x \in M$. As M is a π-module, it follows that $Rx = IM$, where $I = P_1P_2...P_n$ and $P_i's$ are prime ideals of R. Since $Rx \subseteq PM$, it follows that $P_i \subseteq P$, for some i. So either $P' = P_i$ or $P = P_i$. Therefore $Rx = J(PM) = JN$ for some $J \in L(R)$.

As M is a π-module, it follows that $N = PM$ is a multiplication submodule. Since rank $P = 1$, by [11, Theorem 3.1], rank $N = 1$. This completes the proof of the lemma.

Lemma 8. *Let M be a faithful π-module. Then every cyclic submodule is a finite intersection of primary submodules.*

Proof. Let Rx be a cyclic submodule of M and let $I = (Rx : M)$. As M is a faithful π-module, by Lemma 3, it follows that Rx has only finitely many minimal primes. As M is a faithful and finitely generated multiplication module, it follows that every prime submodule is of the form PM for some prime ideal P of R. Let $P_i{}'s$ for $i = 1, 2, ..., m$ be the distinct prime ideals of R such that $P_1M, P_2M, ..., P_mM$ are the distinct prime submodules which are minimal over Rx. It can be easily seen that a prime ideal P of R is minimal over I if and only if $P = P_i$ for some i. As R is an almost π-ring, by [12, Corollary 46.10, Page 577], it follows that the non-maximal minimal primes are unbranched and idempotent. By Lemma 6 and Lemma 7, each P_iM is a multiplication submodule and rank $P_i \leq 1$ for $i = 1, 2, ..., m$. Again by [11, Theorem 3.1] and [18, Lemma 1.4], each P_i is a multiplication ideal. Without loss of generality, assume that $P_1, P_2, ..., P_s$ are the rank one multiplication prime ideals, $P_{s+1}, P_{s+2}, ..., P_{s+t}$ are the non maximal minimal primes and $P_{s+t+1}, P_{s+t+2}, ..., P_m$ are the minimal primes which are also maximal. Since $P_1, P_2, ..., P_s$ are the rank one multiplication prime ideals, by [5, Theorem 3], these are quasi-principal ideals. Therefore by [3, Theorem 2.2], there exist positive integers $n_i{}'s$ for $i = 1, 2, ..., s$, such that $I \subseteq P_i^{n_i}$ and $I \not\subseteq P_i^{n_i+1}$. Since each R_{P_i} $(s + t + 1 \leq i \leq m)$ is a special principal ideal ring, there exist positive integers $n_j{}'s$ for $(s + t + 1 \leq j \leq m)$ such that $I_{P_j} = (P_j^{n_j})_{P_j}$. Observe that by [5, Corollary] and [6, Lemma 1], the powers of P_i $(1 \leq i \leq m)$ are multiplication P_i-primary ideals. Let $J = P_1^{n_1} \cap P_2^{n_2} \cap ... \cap P_s^{n_s} \cap P_{s+1} \cap ... \cap P_{s+t} \cap P_{s+t+1}^{n_{s+t+1}} \cap ... \cap P_m^{n_m}$. Now we claim that $I = J$. Let Q be a maximal prime ideal of R. If $P_j \subseteq Q$ for some $j \in \{s + 1, s + 2, ..., s + t\}$, then $I_Q = J_Q = 0_Q$ as R_Q is a π-domain. Without loss of generality, assume that $P_1, P_2, ..., P_t \subseteq Q$ for $(1 \leq t < s)$ and $P_j \nsubseteq Q$ for $(t+1 \leq j \leq s)$. Note that R_Q is a π-domain and I_Q is a non zero principal ideal of R_Q. Therefore by [15, Theorem 4.2 and Corollary 4.3], I_Q is a finite product of the rank one principal prime ideals minimal over it. Again using Theorem 3 of [7], it can be easily shown that $I_Q = ({P_1}^{n_1})_Q \cap ({P_2}^{n_2})_Q \cap ... \cap ({P_t}^{n_t})_Q$. Therefore $I_Q = J_Q$ since $({P_j}^{n_j})_Q = R_Q$ for $(t+1 \leq j \leq s)$ and $(P_k)_Q = R_Q$ for $(s+1 \leq k \leq m)$. If $P_j \subseteq Q$ for $(s+t+1 \leq j \leq m)$, then $I_Q = J_Q$. This shows that $I_Q = J_Q$ for all maximal prime ideals Q containing I. Further, if $I \nsubseteq Q$, then $I_Q = J_Q = R_Q$. Consequently, $I = J$ and hence $Rx = IM = JM$. Since J is a finite intersection of primary ideals, by [11, Theorem 1.6] and [21, Corollary 1], Rx is a finite intersection of primary submodules. This completes the proof of the lemma.

Theorem 1. *Suppose M is a faithful R-module. Then the following statements on M are equivalent:*

(i) M is a π-module.

(ii) R is a π-ring and M is a multiplication module.

(iii) Every cyclic submodule of M is of the form IM, where I is a finite product of quasi-principal prime ideals of rank less than or equal to one.

Proof. (i)⇒(ii). Suppose (i) holds. Then M is a faithful and finitely generated multiplication module. By Lemma 5, R is an almost π-ring. By [13, Theorem 6], it is enough if we show that every prime ideal of R of rank less than or equal to one is finitely generated. Note that if P is a minimal prime ideal of R, then PM is a minimal prime submodule, so by [14, Lemma 7], Lemma 6 and Lemma 8, PM is a finitely generated multiplication submodule. Therefore by [18, Lemma 1.4], $P = (PM : M)$ is a finitely generated multiplication ideal. Let P be a rank one prime ideal. As R is an almost π-ring, it follows that P is locally principal. So by [14, Lemma 6], PM is locally cyclic and hence by [14, Lemma 7] and Lemma 8, PM is a finitely generated multiplication submodule. Consequently, P is a finitely generated multiplication ideal. Therefore R is a π-ring.

(ii)⇒(iii). Suppose (ii) holds. Let $x \in M$. As M is a multiplication module, it follows that $Rx = (Rx : M)M$. Since R is a π-ring, it follows that R contains only finitely many minimal prime ideals and so M contains only finitely many minimal prime submodules. So by [11, Theorem 3.7], M is finitely generated. Again by [18, Lemma 1.4], $(Rx : M)$ is a quasi-principal ideal of R (i.e., a principal element of R). As R is a π-ring, it follows that $L(R)$ is a π-lattice, so by [4, Theorem 2], $(Rx : M)$ is a finite product of quasi-principal prime ideals of rank less than or equal to one. Therefore (iii) holds and (iii)⇒(i) follows from the definition. This completes the proof of the theorem.

Theorem 2. *Suppose M is a faithful R-module. Then the following statements on M are equivalent:*

(i) M is a π-module.

(ii) M is an almost π-module in which every finitely generated multiplication submodule is a finite intersection of primary submodules.

(iii) M is an almost π-module in which every cyclic submodule is a finite intersection of primary submodules.

(iv) M is finitely generated and an almost π-module in which every prime submodule of rank less than or equal to one is finitely generated.

(v) M is a multiplication module in which every minimal prime submodule is a finitely generated multiplication submodule and every non minimal prime submodule contains a non minimal finitely generated multiplication prime submodule.

Proof. (i)⇒(ii). Suppose (i) holds. Clearly, M is an almost π-module. Let N be a finitely generated multiplication submodule. As M is a faithful and finitely generated multiplication module, by [18, Lemma 1.4], $(N : M)$ is a principal element of $L(R)$.

So by Theorem 1 and [13, Theorem 6], $(N : M)$ is a finite intersection of primary ideals and hence by [11, Theorem 1.6] and [21, Corollary 1], $N = (N : M)M$ is a finite intersection of primary submodules.

(ii)⇒(iii) is obvious.

(iii)⇒(iv). Suppose (iii) holds. By (iii) and Lemma 1, M is locally cyclic. So by [14, Lemma 7], M is a finitely generated multiplication module. Let N be a prime submodule of rank less than or equal to one. As M is a faithful and finitely generated multiplication module, $N = PM$ for some prime ideal P of rank less than or equal to one. By the proof of (i)⇒(ii) of Theorem 1, P is finitely generated and hence N is finitely generated. Therefore (iv) holds.

(iv)⇒(v). Suppose (iv) holds. By (iv), M is finitely generated and locally cyclic. So by [9, Proposition 5], M is a finitely generated multiplication module. So by Lemma 5, R is an almost π-ring. As M is a faithful and finitely generated multiplication module, by hypothesis and [11, Theorem 3.1], every prime ideal of rank less than or equal to one is finitely generated. So by [13, Theorem 6], the minimal prime ideals of R are quasi-principal ideals and every non minimal prime ideal contains a non minimal quasi-principal prime ideal. Now the result follows from [18, Lemma 1.4] and [11, Theorem 3.1]. Therefore (v) holds.

(v)⇒(i). Suppose (v) holds. By hypothesis, the minimal prime submodules are finitely generated, so M contains only finitely many minimal prime submodules and hence M is finitely generated (for details, see [10, Theorem 2]). As M is a faithful and finitely generated multiplication module, by (v), [11, Theorem 3.1] and [18, Lemma 1.4], the minimal prime ideals of R are quasi-principal and every non minimal prime ideal contains a non minimal quasi-principal prime ideal. Now the result follows from Theorem 1 and [13, Theorem 6]. This completes the proof of the theorem.

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References

[1] D.D. Anderson, Abstract commutative ideal theory without chain condition, Algebra Universalis, 6: 131-145 (1976).

[2] D.D. Anderson, Multiplication ideals, multiplication rings, and the ring $R(X)$, Canad. Jour. of Mathematics. 28: 760-768 (1976).

[3] D.D. Anderson, J. Matijevic and W. Nichols, The Krull Intersection Theorem II, Pacific Journal of Mathematics, 66: 15-22 (1976).

[4] D.D. Anderson, Multiplicative lattices in which every principal element is a product of prime elements, Algebra Universalis, 8: 330-335 (1978).

[5] D.D. Anderson, Some remarks on multiplication ideals, Math. Japon. 25: 463-469 (1980).

[6] D.D. Anderson, Noetherian rings in which every ideal is a product of primary ideals, Canad. Math. Bull. 23(4): 457-459 (1980).

[7] D.D. Anderson and L.A. Mahaney, On primary factorizations, Journal of Pure and Applied Algebra, 54: 141-154 (1988).

[8] D.D. Anderson and E.W. Johnson, Dilworth's Principal elements, Algebra Universalis, 36: 99-109 (1996).

[9] A. Barnard, Multiplication modules, Journal of Algebra, 71: 174-178 (1981).

[10] M. Behboodi and H. Koohy, On minimal prime submodules, Far East J. Math. Sci. (FJMS). 6: 83-88 (2002).

[11] Z. El-Bast and P.F. Smith, Multiplication modules, Communications in Algebra, 16(4): 755-779 (1988).

[12] R.W. Gilmer, Multiplicative ideal theory, Marcel Decker, 1972.

[13] C. Jayaram, Almost π-lattices, Czechoslovak Mathematical Journal, 54(129): 119-130 (2004).

[14] C. Jayaram and Ünsal Tekir, *Q*-modules, Turkish Journal of Mathematics, 33: 215-225 (2009).

[15] B.G. Kang, On the converse of a well-known fact about Krull domains, Journal of Algebra, 124: 284-299 (1989).

[16] M.D. Larsen and P.J. McCarthy, Multiplicative theory of ideals, Academic Press, New York, 1971.

[17] K.B. Levitz, A characterization of general ZPI-rings, Proc. Amer. Math. Soc. 32: 376-380 (1972).

[18] G.M. Low and P.F. Smith, Multiplication modules and ideals, Communications in Algebra, 18(12): 4353-4375 (1990).

[19] P.J. McCarthy, Principal elements of lattices of ideals, Proc. Amer. Math. Soc. 30: 43-45 (1971).

[20] D.G. Northcott, Lessons on Rings, Modules and Multiplicities, Cambridge University Press, London, 1968.

[21] Shahabaddin Ebrahimi Atani, Fethi Çallıalp and Ünsal Tekir, A Short Note On Primary Submodules Of Multiplication modules, International Journal of Algebra, 1: 381-384 (2007).