EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 16, No. 4, 2023, 2348-2367
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

The $\mathrm{SL}_2(\mathbb{R})$ Group Representations on Spaces of Holomorphic Functions on the Unit Disc

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Abstract. We can realise the representations of the group $\mathrm{SL}_2(\mathbb{R})$ on the unit disc. This is due to the isomorphism between the group $\mathrm{SL}_2(\mathbb{R})$ and the group $\mathrm{SU}(1,1)$. The discrete series representations for the group $\mathrm{SL}_2(\mathbb{R})$ given by

$$\pi_n(g)\varphi(z) = \varphi\left(\frac{dz-b}{a-cz}\right)(a-cz)^{-n}, \quad n \in \mathbb{Z}. \tag{1}$$

are on the Bergman space where $n \geq 2$ [5, 6, 10]. Lang [13, IX] studied the discrete series on the group $\mathrm{SL}(\mathbb{R})$ in the upper half-plane and on the unit disc. For $n = 1$, the $\mathrm{SL}_2(\mathbb{R})$ representation is called the mock discrete series. The representation space of the mock discrete series is the Hardy space [5, 6, 10]. In this paper we describe the $\mathrm{SL}_2(\mathbb{R})$ representation on the Dirichlet space.

2020 Mathematics Subject Classifications: 32A10, 31C25
Key Words and Phrases: $\mathrm{SL}_2(\mathbb{R})$ group, representations, Dirichlet space,$\mathrm{SU}(1,1)$ group

1. Introduction

The Lie group $\mathrm{SL}_2(\mathbb{R})$ consists of 2×2 matrices with real entries and a determinant equal to one

$$\mathrm{SL}_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1, \; a, b, c, d \in \mathbb{R} \right\}.$$

It acts on the upper half-plane by Möbius transformation

$$g \cdot z = \frac{az+b}{cz+d},$$

where $g \in \mathrm{SL}_2(\mathbb{R})$ and $z \in \{z \in \mathbb{C} : \mathrm{Im} z > 0\}$.

The group $\mathrm{SL}_2(\mathbb{R})$ contains the following three subgroups:

$$K = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : \theta \in \mathbb{R} \right\},$$

DOI: https://doi.org/10.29020/nybg.ejpam.v16i4.4923

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$$A = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} : \alpha > 0 \right\},$$
$$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\}.$$

The Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ is the set of all 2×2 real matrices of trace zero. It is a three-dimensional Lie algebra so we can choose a basis$\{Z, A, B\}$ of $\mathfrak{sl}_2(\mathbb{R})$ by setting

$$Z = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad A = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \quad B = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2}$$

Note that

$$[Z, A] = 2B, \quad [Z, B] = -2A, \quad [A, B] = -\frac{1}{2}Z. \tag{3}$$

2. The Group $\mathrm{SU}(1, 1)$

The Cayley transform of the upper-half plane to the unit disc $\mathbb{D}$ is defined by

$$w = \frac{z - i}{z + i}, \tag{4}$$

where $x \in \mathbb{D}$ and $z \in \{z \in \mathbb{C}, \mathrm{Im} z > 0\}$.
By the transformation (4) we can transfer the action of the group $\mathrm{SL}_2(\mathbb{R})$ from the upper half-plane to the action of the group $\mathrm{SU}(1, 1)$ on the unit disc, where

$$\mathrm{SU}(1, 1) = \left\{ \begin{pmatrix} \alpha & \overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1 \right\}.$$

Furthermore, the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ can be an element of the group $\mathrm{SU}(1, 1)$ by the following identity:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} = \begin{pmatrix} \alpha & \overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix}. \tag{5}$$

Next, any $g \in SU(1, 1)$ has a unique decomposition of the form

$$\begin{aligned} \begin{pmatrix} \alpha & \overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} &= |\alpha| \begin{pmatrix} 1 & \overline{\beta}\overline{\alpha}^{-1} \\ \beta\alpha^{-1} & 1 \end{pmatrix}\begin{pmatrix} \frac{\alpha}{|\alpha|} & 0 \\ 0 & \frac{\overline{\alpha}}{|\alpha|} \end{pmatrix} \\ &= \frac{1}{\sqrt{1 - |u|^2}}\begin{pmatrix} 1 & u \\ \overline{u} & 1 \end{pmatrix}\begin{pmatrix} \mathrm{e}^{i\theta} & 0 \\ 0 & \mathrm{e}^{-i\theta} \end{pmatrix}, \end{aligned} \tag{6}$$

where $\theta = \arg\alpha$, $u = \overline{\beta}\overline{\alpha}^{-1}$ and $|u| < 1$ (since $|\alpha|^2 - |\beta|^2 = 1$). Let $u = r\mathrm{e}^{i\phi}$, then the identity (6) describes an element $g \in \mathrm{SU}(1, 1)$ by a triplet of numbers (r, ϕ, θ) where $0 \le r < 1$ and $-\pi < \phi, \theta \le \pi$. The connection with the (α, β) coordinates is as follows:

$$\alpha = \frac{\mathrm{e}^{i\theta}}{\sqrt{1 - |r|^2}}, \quad \beta = \frac{r\mathrm{e}^{i(\theta - \phi)}}{\sqrt{1 - |r|^2}},$$

$$r = \left|\frac{\beta}{\alpha}\right|, \quad \phi = -\arg\frac{\beta}{\alpha}, \quad \theta = \arg\alpha.$$

Moreover, the decomposition (6) can be rewritten with the same variables as

$$\begin{pmatrix} \alpha & \overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} = \begin{pmatrix} e^{i\frac{\phi}{2}} & 0 \\ 0 & e^{-i\frac{\phi}{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{1-|r|^2}} & \frac{r}{\sqrt{1-|r|^2}} \\ \frac{r}{\sqrt{1-|r|^2}} & \frac{1}{\sqrt{1-|r|^2}} \end{pmatrix} \begin{pmatrix} e^{i(\theta-\frac{\phi}{2})} & 0 \\ 0 & e^{-i(\theta-\frac{\phi}{2})} \end{pmatrix} \tag{7}$$

The last presentation is a decomposition of the group $\mathrm{SU}(1,1)$ as the product KAK of its subgroups, which is called the Cartan decomposition. The base of the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ consists of the following three matrices:

$$\tilde{Z} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \tilde{A} = \begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} \text{ and } \tilde{B} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}. \tag{8}$$

The matrices $\tilde{Z}$, $\tilde{A}$ and $\tilde{B}$ satisfy the commutation relation (3). Also, the exponential map of each matrix generates a one-dimensional subgroup of the $\mathrm{SU}(1,1)$ group, that is

$$e^{\theta\tilde{Z}} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \tag{9}$$

$$e^{\theta\tilde{A}} = \begin{pmatrix} \cosh\frac{\theta}{2} & -i\sinh\frac{\theta}{2} \\ i\sinh\frac{\theta}{2} & \cosh\frac{\theta}{2} \end{pmatrix}, \tag{10}$$

$$e^{\theta\tilde{B}} = \begin{pmatrix} \cosh\frac{\theta}{2} & \sinh\frac{\theta}{2} \\ \sinh\frac{\theta}{2} & \cosh\frac{\theta}{2} \end{pmatrix}. \tag{11}$$

3. Induced Representation on the Unit Disc

In this section, we induce a representation of the group $\mathrm{SU}(1,1)$ from the subgroup K. Mainly, we use the references [8, 11].

The one-dimensional compact subgroup K is defined as follows:

$$K = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \quad -\pi < \theta \leq \pi. \right\} \tag{12}$$

Using the decomposition (6) of any element $g \in \mathrm{SU}(1,1)$, we can identify the homogeneous space $X = \mathrm{SU}(1,1)/\mathrm{K}$ with the open unit disc $\mathbb{D}$. Let the section $\mathsf{s} : \mathbb{D} \to \mathrm{SU}(1,1)$ be defined as follows:

$$\mathsf{s} : u \mapsto \frac{1}{\sqrt{1-|u|^2}} \begin{pmatrix} 1 & u \\ \bar{u} & 1 \end{pmatrix}. \tag{13}$$

There is a natural projection map $\mathsf{p} : \mathrm{SU}(1,1) \to \mathbb{D}$, which assigns to an element of $\mathrm{SU}(1,1)$ its equivalence class in $\mathrm{SU}(1,1)/\mathrm{K}$:

$$\mathsf{p} : \begin{pmatrix} \alpha & \overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} \mapsto \frac{\overline{\beta}}{\overline{\alpha}}. \tag{14}$$

Mapping $\mathsf{r}: \mathrm{SU}(1,1) \to \mathrm{K}$ associates f to the natural projection p, and the section s is defined as follows:

$$\mathsf{r}: \begin{pmatrix} \alpha & \overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} \mapsto \begin{pmatrix} \frac{\alpha}{|\alpha|} & 0 \\ 0 & \frac{\overline{\alpha}}{|\alpha|} \end{pmatrix} \tag{15}$$

For the homogeneous space $\mathrm{SU}(1,1)/\mathrm{K}$ defines a left action denoted by "$\cdot$" as follows:

$$g : u \mapsto g \cdot u = \mathsf{p}(g * \mathsf{s}(u)), \tag{16}$$

where $*$ is the multiplication of the group $\mathrm{SU}(1,1)$.

The invariant measure $d\mu(u)$ on $\mathbb{D}$ comes from the decomposition $dg = d\mu(u)dk$, where dg and dk are the Haar measures on $G = \mathrm{SU}(1,1)$ and K respectively. The measure $d\mu(u)$ is given by

$$d\mu(u) = \frac{du \wedge d\bar{u}}{(1 - |u|^2)^2}. \tag{17}$$

Let $\chi_n : \mathbb{T} \to \mathbb{C}$ be a character of the subgroup $K \simeq \mathbb{T}$ defined as follows:

$$\chi_n(w) = w^n, \quad n \in \mathbb{Z}. \tag{18}$$

This character induces a representation of $\mathrm{SU}(1,1)$ constructed in the Hilbert space $L_2^{\chi_n}(SU(1,1))$, consisting of the functions $F_n : SU(1,1) \to \mathbb{C}$ with the property

$$F_n\left[\begin{pmatrix} \alpha & \overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix}\right] = \chi_n\left(\frac{\alpha}{|\alpha|}\right) F\left(\frac{\overline{\beta}}{\overline{\alpha}}\right), \tag{19}$$

where $F \in L_2(\mathbb{D})$. Then, the norm of the function F_n is defined as follows:

$$\|F_n\|^2 = \int_{\mathbb{D}} |F(u)|^2 \frac{du \wedge d\bar{u}}{(1 - |u|^2)^2}. \tag{20}$$

The space $L_2^{\chi_n}(\mathrm{SU}(1,1))$ is invariant under the left shift of the $\mathrm{SU}(1,1)$ group. The restriction of the left shift on $L_2^{\chi_n}(\mathrm{SU}(1,1))$ is the left regular representation of $\mathrm{SU}(1,1)$, which can be written as follows:

$$[\Lambda(g)F_n](g') = F_n(g^{-1} * g'), \tag{21}$$

where $*$ is a matrix multiplication.

The lifting map $\mathcal{L}_{\chi_n} : L_2(\mathbb{D}) \to L_2^{\chi_n}(SU(1,1))$ for the subgroup K and its character χ_n is defined as follows:

$$\begin{aligned} [\mathcal{L}_{\chi_n} f] \begin{pmatrix} \alpha & \overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} &= \chi_n\left(\mathsf{r}\begin{pmatrix} \alpha & \overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix}\right) f\left(\mathsf{p}\begin{pmatrix} \alpha & \overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix}\right) \\ &= \left(\frac{\bar{\alpha}}{|\alpha|}\right)^n f\left(\frac{\overline{\beta}}{\overline{\alpha}}\right). \end{aligned} \tag{22}$$

The pulling map is given by the following:

$$\mathcal{P} : L_2^{\chi_n}(SU(1,1)) \to L_2(\mathbb{D}),$$

$$\mathcal{P}(F(w,\bar{w})) = F(\mathsf{s}(w)),$$

such that $\mathcal{P}\circ\mathcal{L}_{\chi_n} = \mathbb{I}$ and $\mathcal{L}_{\chi_n}\circ\mathcal{P} = \mathbb{I}$.

Therefore, the representation $\pi_n : L_2(\mathbb{D}) \to L_2(\mathbb{D})$, which is induced by the character χ_n is given by the following:

$$\left[\pi_n \begin{pmatrix} \alpha & \overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix}\right] = \mathcal{P}\circ\Lambda \begin{pmatrix} \alpha & \overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} \circ \mathcal{L}_{\chi_n}.$$

By simple calculation, we get:

$$\begin{aligned} [\pi_n(g)f](w,\bar{w}) &= \frac{(\overline{\alpha}-\overline{\beta}w)^n}{|\overline{\alpha}-\overline{\beta}w|^n} f\left(\frac{\overline{\alpha}w-\overline{\beta}}{\alpha-\beta w}\,,\ \frac{\alpha\overline{w}-\beta}{\overline{\alpha}-\overline{\beta}\overline{w}}\right) \\ &= \left(\frac{\overline{\alpha}-\overline{\beta}w}{\alpha-\beta\bar{w}}\right)^{\frac{n}{2}} f\left(\frac{\overline{\alpha}w-\overline{\beta}}{\alpha-\beta w}\,,\ \frac{\alpha\overline{w}-\beta}{\overline{\alpha}-\overline{\beta}\overline{w}}\right). \end{aligned} \tag{23}$$

[9] For $n\in\mathbb{Z}$, an n-peeling is an isometry $\mathcal{P}_n : L_2(\mathbb{D}, dw) \to L_2(\mathbb{D}, (1-|w|^2)^{n-2} dw\wedge d\bar{w})$ defined as follows:

$$\mathcal{P}_n : f(w) \mapsto [\mathcal{P}_n f](w) = \frac{f(w)}{(1-|w|^2)^{\frac{n}{2}}}, \quad w = u+iv. \tag{24}$$

The representation (23) is intertwined $\breve{\pi}_n\circ\mathcal{P}_n = \mathcal{P}_n\circ\pi_n$ by the n-peeling with the following representation:

$$[\breve{\pi}_n(g)f](w) = (\overline{\alpha}-\overline{\beta}w)^{-n} f\left(\frac{\alpha w-\beta}{\overline{\alpha}-\overline{\beta}w}\right), \tag{25}$$

which is unitary in $L_2(\mathbb{D}, (1-|w|^2)^{n-2} dw\wedge d\bar{w})$. The demonstration of the intertwining properties is based on the following analogue of identity for the unit disc :

$$1-\left|\frac{\alpha w-\beta}{\overline{\alpha}-\overline{\beta}w}\right| = \frac{1-|w|^2}{|\overline{\alpha}-\overline{\beta}w|^2}.$$

The matrix $\begin{pmatrix} \alpha & \overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} \in \mathrm{SU}(1,1)$ is transformed to $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ by the identity (5). Therefore, the representation $\breve{\rho}_n^K$ can be transformed to a holomorphic representation of the group $SL_2(\mathbb{R})$:

$$[\breve{\pi}_n(g)f](z) = (d-bz)^{-n} f\left(\frac{az-c}{d-bz}\right), \tag{26}$$

which is unitary on the upper half-plane where $z = x+iy \in \mathbb{R}^2_+$ with the measure $d\mu(g) = \frac{dxdy}{y^2}$.

4. Actions of Ladder Operators

In this section, we study the left and right actions of the ladder operators for the representation π_n given by (23). First, the derived representations are given as follows:

$$\begin{aligned}[Ef](w,\overline{w}) &= \frac{d}{dt}\pi_n(\mathrm{e}^{t\tilde{Z}})f(w,\overline{w})|_{t=0} \\ &= [-inI - 2iw\partial_w + 2i\overline{w}\partial_{\bar{w}}]f(w,\overline{w}),\end{aligned} \tag{27}$$

$$\begin{aligned}[A_1f](w,\overline{w}) &= \frac{d}{dt}\pi_n(\mathrm{e}^{t\tilde{A}})f(w,\overline{w})|_{t=0} \\ &= \left[\frac{ni}{4}(w+\overline{w})I + \frac{i}{2}(1+w^2)\partial_w - \frac{i}{2}(1+\overline{w}^2)\partial_{\bar{w}}\right]f(w,\overline{w}),\end{aligned} \tag{28}$$

$$\begin{aligned}[B_1f](w,\overline{w}) &= \frac{d}{dt}\pi_n(\mathrm{e}^{t\tilde{B}})f(w,\overline{w})|_{t=0} \\ &= \left[\frac{n}{4}(w-\bar{w})I + \frac{1}{2}(w^2-1)\partial_w + \frac{1}{2}(\overline{w}^2-1)\partial_{\bar{w}}\right]f(w,\overline{w}).\end{aligned} \tag{29}$$

The ladder operators are defined as

$$\begin{aligned}L_+ &= B_1 - iA_1 = \frac{n}{2}wI + w^2\partial_w - \partial_{\overline{w}}, \\ L_- &= B_1 + iA_1 = \frac{-n}{2}\overline{w}I + \bar{w}^2\partial_{\overline{w}} - \partial_w,\end{aligned}$$

and satisfy the following relations:

$$[E, L_\pm] = \pm 2iL_\pm, \quad [L_+, L_-] = -iE. \tag{30}$$

The Casimir operator is given by

$$\begin{aligned}d\pi_n(C) &= E^2 - 2[L_+L_- + L_-L_+] \\ &= (w\overline{w}-1)[n^2I + 2nw\partial_w - 2n\bar{w}\partial_{\overline{w}} + 4(w\overline{w}-1)\partial_w\partial_{\overline{w}}].\end{aligned} \tag{31}$$

The Casimir operator in the polar coordinate $w = re^{i\theta}$ is as follows:

$$d\pi_n(C) = (r^2-1)(n^2I - 2in\partial_\theta) - (r^2-1)^2(\partial_r^2 + r^{-1}\partial_r + r^{-2}\partial_\theta^2). \tag{32}$$

Lemma 1. *The operator* (27) *has two eigenfunctions:*

(i) For $m \neq 2, 4, 6, 8......,$

$$\begin{aligned}f_{-\frac{m}{2},n}(w,\overline{w}) = w^{-\frac{m}{2}}(1-w\bar{w})^{\frac{1\pm\sqrt{1-\mu}}{2}}F\bigg(&\frac{1}{2}[1+n-m\pm\sqrt{1-\mu}], \\ &\frac{1}{2}[1-n\pm\sqrt{1-\mu}], 1-\frac{m}{2}, w\overline{w}\bigg),\end{aligned} \tag{33}$$

(ii) For $m \neq -2, -4, -6, -8......$,

$$\tilde{f}_{-\frac{m}{2},n}(w,\overline{w}) = \overline{w}^{\frac{m}{2}}(1-w\overline{w})^{\frac{1\pm\sqrt{1-\mu}}{2}} F\bigg(\frac{1}{2}[1-n+m\pm\sqrt{1-\mu}], \frac{1}{2}[1+n\pm\sqrt{1-\mu}], 1+\frac{m}{2}, w\overline{w}\bigg), \quad (34)$$

where F *is a hypergeometric function.*

Proof.

To find the eigenfunction of the subgroup K, we will solve the following partial differential equation by using the method of characteristics:

$$[Ef](w,\overline{w}) = [-inI - 2iw\partial_w + 2i\overline{w}\partial_{\bar{w}}]f(w,\bar{w}) = 0.$$

We can write the characteristics for this equation as follows:

$$\frac{df}{inf} = \frac{dw}{-2iw} = \frac{d\overline{w}}{2i\overline{w}}.$$

$$\frac{dw}{-2iw} = \frac{d\bar{w}}{2i\bar{w}} \Rightarrow \quad C_1 = w\overline{w}.$$

We need to obtain another integral curve that involves f. This is possible from the following equation:

$$\frac{df}{inf} = \frac{dw}{-2iw}, \quad \text{we get} \quad C_2 = w^{\frac{n}{2}} f.$$

Then, the general solution of (4) is of the form $C_2 = \phi(C_1)$, that is

$$f(w,\overline{w}) = w^{-\frac{n}{2}}\phi(w\bar{w}).$$

Now, for $m \in \mathbb{Z}$ the eigenfunction is given by

$$f_{-\frac{m}{2}}(w,\overline{w}) = w^{-\frac{m}{2}}\phi(w\overline{w}), \quad (35)$$

which satisfies

$$[Ef_m](w,\overline{w}) = i(m-n)f_m(w,\overline{w}). \quad (36)$$

Therefore, the eigenvalue of the operator E is $m-n$.

Next,let $w = re^{i\theta}$. Then the eigenfunction (35) will be given by

$$f_{-\frac{m}{2}}(r,\theta) = (re^{i\theta})^{-\frac{m}{2}}\phi(r^2).$$

The Casimir operator (32) is applied to $f_{-\frac{m}{2}}(r,\theta)$

$$[d\pi_n(C)f_{-\frac{m}{2},n}](r,\theta) = (re^{i\theta})^{-\frac{m}{2}}\big[(r^2-1)(n^2-nm)\phi(r^2) - 2(r^2-1)^2((-m+2)\phi'(r^2) + 2r^2\phi''(r^2))\big]. \quad (37)$$

To find the value of ϕ in (35), we need to solve the differential equation

$$[d\pi_n(C)f](r,\theta) = \mu f(r,\theta).$$

That is,

$$[(r^2-1)(n^2-nm)-\mu]\phi(r^2) - 2(r^2-1)^2(-m+2)\phi'(r^2) - 4(r^2-1)^2 r^2\phi''(r^2) = 0. \quad (38)$$

Let $x = r^2$. Then we get

$$[(x-1)(n^2-nm)-\mu]\phi(x) - 2(x-1)^2(-m+2)\phi'(x) - 4(x-1)^2x^2\phi''(x) = 0. \quad (39)$$

Now, let

$$\phi(x) = x^{\alpha}(1-x)^{\beta}\psi(x).$$

Then by substitute $\phi(x)$ in (39), we get

$$\alpha = \frac{m}{2} \text{ or } 0, \quad \beta = \frac{1\pm\sqrt{1-\mu}}{2}.$$

Hence, we have two solutions:

(i) $\phi(x) = (1-x)^{\frac{1\pm\sqrt{1-\mu}}{2}}\psi(x)$,

(ii) $\phi(x) = x^{\frac{m}{2}}(1-x)^{\frac{1\pm\sqrt{1-\mu}}{2}}\psi(x)$.

By substituting the first solution in the differential equation(39), we get

$$x(1-x)\psi''(x) + \left(1-\frac{m}{2}-(1\pm\sqrt{1-\mu}+1-\frac{m}{2})x\right)\psi'(x) + \left[\frac{\mu}{2} + \left(\frac{-1\mp\sqrt{1-\mu}}{2}\right)\left(1-\frac{m}{2}\right) + \frac{1}{4}\left(n^2-nm\right)\right]\psi(x) = 0. \quad (40)$$

This is a hypergeometric differential equation that takes the following form:

$$x(1-x)\psi''(x) + [c-(a+b+1)x]\psi'(x) - ab\psi(x) = 0.$$

By simple calculation, we get

$$a = \frac{1}{2}[1+n-m\pm\sqrt{1-\mu}],$$
$$b = \frac{1}{2}[1-n\pm\sqrt{1-\mu}],$$
$$c = 1-\frac{m}{2}.$$

Then, $\psi(x) = F(a, b, c, x)$, and the solution of (38) is

$$\phi(r^2) = (1 - r^2)^{\frac{1 \pm \sqrt{1-\mu}}{2}} F(a, b, c, r^2).$$

The hypergeometric function is given by

$$F(a, b, c, r^2) = 1 + \sum_{k=1}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{(r^2)^k}{k!}.$$

Finally, the eigenfunction is given by

$$f_{-\frac{m}{2},n}(w, \overline{w}) = w^{-\frac{m}{2}} (1 - w\overline{w})^{\frac{1 \pm \sqrt{1-\mu}}{2}} F\Big(\frac{1}{2}[1 + n - m \pm \sqrt{1-\mu}], \frac{1}{2}[1 - n \pm \sqrt{1-\mu}], 1 - \frac{m}{2}, w\overline{w}\Big),$$

where $m \neq 2, 4, 6, 8.......$

Following the same calculation for the second solution, we get the eigenfunction

$$\tilde{f}_{-\frac{m}{2},n}(w, \overline{w}) = \overline{w}^{\frac{m}{2}} (1 - w\bar{w})^{\frac{1 \pm \sqrt{1-\mu}}{2}} F\Big(\frac{1}{2}[1 - n + m \pm \sqrt{1-\mu}], \frac{1}{2}[1 + n \pm \sqrt{1-\mu}], 1 + \frac{m}{2}, w\overline{w}\Big),$$

where $m \neq -2, -4, -6, -8.......$

The commutator relation $[E, L_{\pm}] = \pm 2iL_{\pm}$, implies that

$$[Ef_{-\frac{m}{2},n}(w, \bar{w})] = \mathrm{e}^{(m-n)i\theta} f_{-\frac{m}{2},n}(w, \bar{w}).$$

Hence $E = (m - n)i$. Additionally, by using the relation $[L_+, L_-] = -iE$, and (31), we get the following identities:

$$4L_+L_- = E^2 - 2iE - d\pi_n(C),$$

$$4L_-L_+ = E^2 + 2iE - d\pi_n(C).$$

Then, for $d\pi_n(C) = \mu I$,

$$L_+L_- = -\frac{1}{4}[(m - n - 1)^2 + \mu - 1], \tag{41}$$

$$L_-L_+ = -\frac{1}{4}[(m - n + 1)^2 + \mu - 1]. \tag{42}$$

Now, since $L_+^* = -L_-$, we have

$$\begin{aligned} \|L_-\| = \|L_-^* L_-\|^{\frac{1}{2}} &= \| - L_+L_-\|^{\frac{1}{2}} \\ &= \frac{1}{2}[(m - n - 1)^2 + \mu - 1]^{\frac{1}{2}}. \end{aligned} \tag{43}$$

Similarly,

$$\|L_+\| = \frac{1}{2}[(m-n+1)^2 + \mu - 1]^{\frac{1}{2}}. \tag{44}$$

Let $1-\mu = (n-1)^2$, where n is an integer. The functions (33) are given by

$$f_{-\frac{m}{2},n}(w,\overline{w}) = w^{\frac{-m}{2}}(1-w\overline{w})^{\frac{n}{2}}. \tag{45}$$

The functions $f_{-\frac{m}{2},n}(w,\overline{w}) = w^{\frac{-m}{2}}(1-w\overline{w})^{\frac{n}{2}}$, are L_2 summable for $n>1$ and $m \le 0$.

Proof. Let $w = re^{i\theta}$, then $f_{-\frac{m}{2},n}(re^{i\theta}, re^{-i\theta}) = (re^{i\theta})^{\frac{-m}{2}}(1-r^2)^{\frac{n}{2}}$. The measure is $d\mu = \frac{rdr\wedge d\theta}{(1-r^2)^2}$. Then,

$$\begin{aligned}
\|f_{-\frac{m}{2},n}\|^2 &= \int_0^{2\pi}\int_0^1 |f_{-\frac{m}{2},n}(re^{i\theta}, re^{-i\theta})|^2 \frac{rdr\wedge d\theta}{(1-r^2)^2} \\
&= \int_0^{2\pi}\int_0^1 \left|(re^{i\theta})^{\frac{-m}{2}}(1-r^2)^{\frac{n}{2}}\right|^2 \frac{rdr\wedge d\theta}{(1-r^2)^2} \\
&= \int_0^{2\pi}\int_0^1 r^{-m}(1-r^2)^{n-2} rdrd\theta \\
&\le \pi\int_0^1 (1-r^2)^{n-2}2rdr, \quad \text{for} \quad m \le 0 \\
&= -\pi\frac{(1-r^2)^{n-1}}{n-1}\bigg|_0^1 \\
&= \frac{\pi}{n-1} < \infty.
\end{aligned}$$

Hence, $f_{-\frac{m}{2},n}$ are L_2 summable if $n>1$ and $m \le 0$.

By simple calculation, we get

$$[L_+ f_{-\frac{m}{2},n}](w,\overline{w}) = \left(n - \frac{m}{2}\right) f_{-\frac{m}{2}+1,n}(w,\overline{w}), \tag{46}$$

$$[L_- f_{-\frac{m}{2},n}](w,\overline{w}) = \frac{m}{2} f_{-\frac{m}{2}-1,n}(w,\overline{w}). \tag{47}$$

At $m=0$, we have the function $f_{0,n}(w,\overline{w}) = (1-w\overline{w})^{\frac{n}{2}}$. Then, $L_- f_{0,n}(w,\overline{w}) = 0$, which means that $f_{0,n}(w,\overline{w})$ is the vacuum of the operator L_-. This is represented by the following diagram: Next, let $1-\mu = (n+1)^2$. Then, the functions (34) are given by

$$0 \xleftarrow[L_-]{} f_{0,n} \underset{L_-}{\overset{L_+}{\rightleftarrows}} f_{1,n} \underset{L_-}{\overset{L_+}{\rightleftarrows}} f_{2,n} \underset{L_-}{\overset{L_+}{\rightleftarrows}} \cdots$$

$$\tilde{f}_{-\frac{m}{2},n}(w,\overline{w}) = \overline{w}^{\frac{m}{2}}(1-w\overline{w})^{\frac{-n}{2}}, \tag{48}$$

which are L_2 summable for $n < -1$ and $m \geq 0$; that is,

$$\|\tilde{f}_{-\frac{m}{2},n}\|^2 = \int_{\mathbb{D}} \left|\tilde{f}_{-\frac{m}{2},n}(w,\overline{w})\right|^2 \frac{dw \wedge d\overline{w}}{(1-|w|^2)} < \infty, \quad n < -1.$$

$$[L_+\tilde{f}_{-\frac{m}{2},n}](w,\overline{w}) = \frac{-m}{2}\tilde{f}_{-(\frac{m}{2}+1),n}(w,\overline{w}), \tag{49}$$

$$[L_-\tilde{f}_{-\frac{m}{2},n}](w,\overline{w}) = \left(\frac{m}{2} - n\right)\tilde{f}_{-(\frac{m}{2}-1),n}(w,\overline{w}). \tag{50}$$

At $m = 0$, we get the function $\tilde{f}_{0,n}(w,\overline{w}) = (1-w\overline{w})^{\frac{-n}{2}}$. We can then see that $[L_+\tilde{f}_{0,n}](w,\overline{w}) = 0$, which means that $\tilde{f}_{0,n}$ is the vacuum of the operator L_+. This is represented by the following diagram: The Lie derivatives of the representation π_n are

$$\cdots \underset{L_-}{\overset{L_+}{\rightleftarrows}} \tilde{f}_{-2,n} \underset{L_-}{\overset{L_+}{\rightleftarrows}} \tilde{f}_{-1,n} \underset{L_-}{\overset{L_+}{\rightleftarrows}} \tilde{f}_{0,n} \xrightarrow{L_+} 0$$

$$\mathfrak{L}^{\tilde{Z}} = -\partial_\theta, \tag{51}$$

$$\mathfrak{L}^{\tilde{A}} = \frac{-r}{2}\sin(\phi - 2\theta)\partial_\theta - \frac{1}{2}(1-u\bar{u})\left[e^{2i\theta}\partial_u + e^{-2i\theta}\partial_{\bar{u}}\right], \tag{52}$$

$$\mathfrak{L}^{\tilde{B}} = \frac{r}{2}\cos(\phi - 2\theta)\partial_\theta - \frac{i}{2}(1-u\bar{u})\left[e^{2i\theta}\partial_u - e^{-2i\theta}\partial_{\bar{u}}\right]. \tag{53}$$

The right ladder operators are then represented by

$$\mathfrak{L}_+ = \mathfrak{L}^{\tilde{A}+i\tilde{B}} = e^{-2i\theta}\left[\frac{i}{2}u\partial_\theta - (1-u\bar{u})\partial_{\bar{u}}\right], \tag{54}$$

$$\mathfrak{L}_- = \mathfrak{L}^{\tilde{A}-i\tilde{B}} = -e^{2i\theta}\left[\frac{i}{2}\bar{u}\partial_\theta + (1-u\bar{u})\partial_u\right]. \tag{55}$$

Proof. The Lie derivative $\mathfrak{L}^X$ for an element X of the Lie algebra $\mathfrak{su}(1,1)$ is given by

$$[\mathfrak{L}^X F](g) = \frac{d}{dt}F(g\exp tX)|_{t=0}, \tag{56}$$

for any differentiable function F on SU(1,1) and $g = \begin{pmatrix} \alpha & \bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}$.

We know that the space $L_2^{\chi_n}(\mathrm{SU}(1,1))$ consists of the functions $F_n : \mathrm{SU}(1,1) \to \mathbb{C}$ with the property

$$F_n\left[\begin{pmatrix} \alpha & \overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix}\right] = \chi_n\left(\frac{\alpha}{|\alpha|}\right) F\left(\frac{\overline{\beta}}{\overline{\alpha}}, \frac{\beta}{\alpha}\right),$$

where $F \in L_2(\mathbb{D})$.

Hence, for $v = \frac{\alpha}{|\alpha|} = \mathrm{e}^{i\theta}$ and $u = \frac{\overline{\beta}}{\overline{\alpha}} = r\mathrm{e}^{i\phi}$, we have

$$\begin{aligned}[\mathfrak{L}^X F_n](g) &= \frac{d}{dt} F_n(g \exp tX)\Big|_{t=0} \\ &= \frac{d}{dt}\chi_n(v(t))F\left(u(t), \bar{u}(t)\right)\Big|_{t=0} \\ &= \frac{\partial \chi_n}{\partial v}\frac{dv(t)}{dt}\Big|_{t=0} + \frac{\partial F}{\partial u}\frac{du(t)}{dt}\Big|_{t=0} + \frac{\partial F}{\partial \bar{u}}\frac{d\bar{u}(t)}{dt}\Big|_{t=0}.\end{aligned} \tag{57}$$

From section 2 we have $\tilde{Z}$, $\tilde{A}$ and $\tilde{B} \in \mathfrak{su}(1,1)$ given by (8). Then, the Lie derivatives corresponding to the subgroups $\exp t\tilde{Z}$,(9), $\exp t\tilde{A}$,(10) and $\exp t\tilde{B}$(11) are obtained through the differentiation of the right action of these subgroups as follows:

$$\begin{aligned}[\mathfrak{L}^{\tilde{Z}} F_n](g) &= \frac{d}{dt} F_n(g \exp t\tilde{Z})\Big|_{t=0} \\ &= \frac{d}{dt} F_n \begin{pmatrix} \alpha \mathrm{e}^{it} & \bar{\beta}\mathrm{e}^{-it} \\ \beta \mathrm{e}^{it} & \bar{\alpha}\mathrm{e}^{-it}\end{pmatrix}\Big|_{t=0} \\ &= \frac{d}{dt}\chi_n\left(\frac{\alpha \mathrm{e}^{it}}{|\alpha \mathrm{e}^{it}|}\right) F\left(\frac{\bar{\beta}\mathrm{e}^{-it}}{\bar{\alpha}\mathrm{e}^{-it}}, \frac{\beta \mathrm{e}^{-it}}{\alpha \mathrm{e}^{-it}}\right)\Big|_{t=0} \\ &= \frac{d}{dt}\chi_n(\mathrm{e}^{i(\theta+t)})F(u,\bar{u})\Big|_{t=0} \\ &= -\frac{\partial F}{\partial_\theta},\end{aligned}$$

where $\alpha = \frac{\mathrm{e}^{i\theta}}{\sqrt{1-|r|^2}}$ and $\beta = \frac{r\mathrm{e}^{i(\theta-\phi)}}{\sqrt{1-|r|^2}}$.

Similarly, it is easy to determine that

$$\begin{aligned}[\mathfrak{L}^{\tilde{A}} F_n](g) &= \frac{d}{dt} F_n(g \exp t\tilde{A})\Big|_{t=0} \\ &= \frac{-r}{2}\sin(\phi - 2\theta)\frac{\partial F}{\partial_\theta} - \frac{1}{2}(1-u\bar{u})\left[\mathrm{e}^{2i\theta}\frac{\partial F}{\partial_u} + \mathrm{e}^{-2i\theta}\frac{\partial F}{\partial_{\bar{u}}}\right].\end{aligned}$$

$$\begin{aligned}[\mathfrak{L}^{\tilde{B}} F_n](g) &= \frac{d}{dt} F_n(g \exp t\tilde{B})\Big|_{t=0} \\ &= \frac{r}{2}\cos(\phi - 2\theta)\frac{\partial F}{\partial_\theta} - \frac{i}{2}(1-u\bar{u})\left[\mathrm{e}^{2i\theta}\frac{\partial F}{\partial_u} - \mathrm{e}^{-2i\theta}\frac{\partial F}{\partial_{\bar{u}}}\right].\end{aligned}$$

The function $f_{-\frac{m}{2},n}$ given by (45) is an eigenfunction with an eigenvalue in for the operator $\mathfrak{L}^{\tilde{Z}}$. That is, for

$$F_n(g) = \mathrm{e}^{int} f_{-\frac{m}{2},n}(w, \overline{w}),$$

we have

$$\mathfrak{L}^{\tilde{Z}} \mathrm{e}^{int} f_{-\frac{m}{2},n}(w,\overline{w}) = in\mathrm{e}^{int} f_{-\frac{m}{2},n}(w,\overline{w}). \tag{58}$$

Moreover, $\tilde{f}_{-\frac{m}{2},n}$ (48) is an eigenfunction with an eigenvalue in for the operator $\mathfrak{L}^{\tilde{Z}}$.

Lemma 2. *We have*

$$\mathfrak{L}^{\tilde{A}\pm i\tilde{B}} : f_{-\frac{m}{2},n} \to f_{-\frac{m}{2},n\pm 2},$$

and

$$\mathfrak{L}^{\tilde{A}\pm i\tilde{B}} : \tilde{f}_{-\frac{m}{2},n} \to \tilde{f}_{-\frac{m}{2},n\pm 2}.$$

Proof. From the commutator relations $[\mathfrak{L}^{\tilde{Z}}, \mathfrak{L}^{\tilde{A}\pm i\tilde{B}}] = \pm 2i\mathfrak{L}^{\tilde{A}\pm i\tilde{B}}$, for the eigenfunction $f_{-\frac{m}{2},n}$ given by (45), we can see that

$$\begin{aligned}[\mathfrak{L}^{\tilde{Z}} \mathfrak{L}^{\tilde{A}\pm i\tilde{B}}]\mathrm{e}^{int} f_{-\frac{m}{2},n} &= \mathfrak{L}^{\tilde{A}\pm i\tilde{B}}(\mathfrak{L}^{\tilde{Z}}\mathrm{e}^{int} f_{-\frac{m}{2},n}) \pm 2i\mathfrak{L}^{\tilde{A}\pm i\tilde{B}}\mathrm{e}^{int} f_{-\frac{m}{2},n} \\ &= \mathfrak{L}^{\tilde{A}\pm i\tilde{B}}(ni\mathrm{e}^{int} f_{-\frac{m}{2},n}) \pm 2i\mathfrak{L}^{\tilde{A}\pm i\tilde{B}}\mathrm{e}^{int} f_{-\frac{m}{2},n} \\ &= (n\pm 2)i\mathfrak{L}^{\tilde{A}\pm i\tilde{B}}\mathrm{e}^{int} f_{-\frac{m}{2},n}.\end{aligned} \tag{59}$$

Similarly,for the eigenfunction $\tilde{f}_{-\frac{m}{2},n}$ (48), we have

$$[\mathfrak{L}^{\tilde{Z}} \mathfrak{L}^{\tilde{A}\pm i\tilde{B}}]\mathrm{e}^{int} \tilde{f}_{-\frac{m}{2},n} = (n\pm 2)i\mathfrak{L}^{\tilde{A}\pm i\tilde{B}}\mathrm{e}^{int} \tilde{f}_{-\frac{m}{2},n}.$$

The vacuum $f_{0,n}(w,\overline{w}) = (1 - w\overline{w})^{\frac{n}{2}}$ is annihilated by the operator $\mathfrak{L}^{\tilde{A}+i\tilde{B}}$. That is, $[\mathfrak{L}^{\tilde{A}+i\tilde{B}}\mathrm{e}^{int} f_{0,n}](w,\overline{w}) = 0$. Then, all the vectors $f_{j,n} = (L_+)^j f_{0,n}$ are vacuums of the operator $\mathfrak{L}^{\tilde{A}+i\tilde{B}}$ due to the commutation of the left and right actions:

$$\begin{aligned}\mathfrak{L}^{\tilde{A}+i\tilde{B}} f_{j,n} &= \mathfrak{L}^{\tilde{A}+i\tilde{B}}(L_+)^j f_{0,n} \\ &= (L_+)^j \mathfrak{L}^{\tilde{A}+i\tilde{B}} f_{0,n} = 0.\end{aligned} \tag{60}$$

For each vacuum $f_{0,n}$, the collection of vectors $f_{j,n} = (L_+)^j f_{0,n}$ forms an orthogonal basis of an irreducible component with the respective ladder operators (46) and (47). The left and the right actions for the eigenfunctions $f_{m,n}$ (45) jointly create the two-dimensional lattice structure that can be seen in the following diagram:

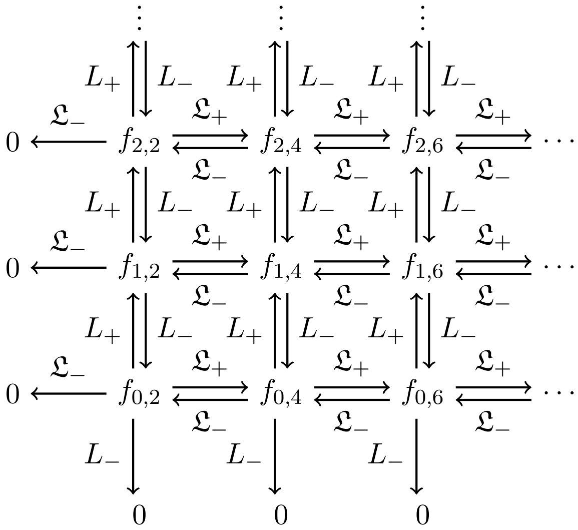

Figure 1: The left and the right actions of the ladder operators for $f_{\frac{-m}{2},n}$.

Furthermore, the function $\tilde{f}_{0,n}(w,\overline{w}) = (1-w\overline{w})^{\frac{-n}{2}}$ is a vacuum of the operator $\mathfrak{L}^{\tilde{A}-i\tilde{B}}$. That is, $[\mathfrak{L}^{\tilde{A}-i\tilde{B}}\mathrm{e}^{int}\tilde{f}_{0,n}](w,\overline{w}) = 0$. Then, all the vectors $\tilde{f}_{k,n} = (L_-)^k\tilde{f}_{0,n}$ are vacuums of the operator $\mathfrak{L}^{\tilde{A}-i\tilde{B}}$ due to the commutation of the left and right actions:

$$\begin{aligned}\mathfrak{L}^{\tilde{A}-i\tilde{B}}\tilde{f}_{k,n} &= \mathfrak{L}^{\tilde{A}-i\tilde{B}}(L_-)^k f_{0,n}\\ &= (L_-)^k\mathfrak{L}^{\tilde{A}-i\tilde{B}}\tilde{f}_{0,n} = 0.\end{aligned} \tag{61}$$

For each $\tilde{f}_{0,n}$, the collection of vectors $\tilde{f}_{k,n} = (L_-)^k f_{0,n}$ forms an orthogonal basis of an irreducible component with the respective ladder operators (49) and (50). The left and the right actions for the functions $\tilde{f}_{m,n}$ (48) jointly create the two-dimensional lattice structure that can be seen in the following diagram:

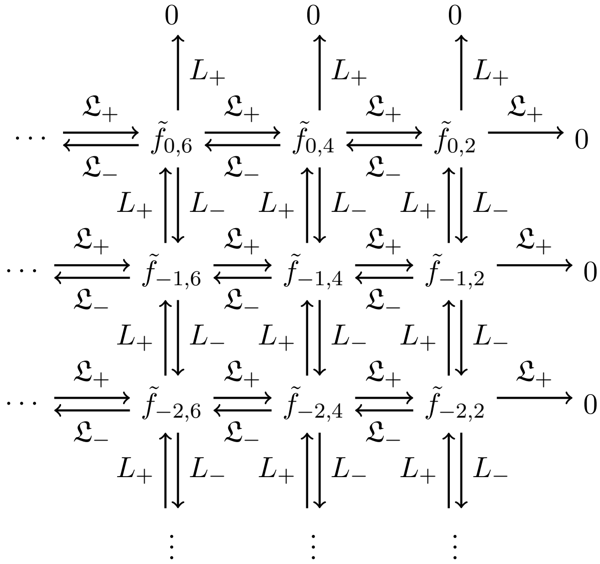

Figure 2: The left and the right actions of the ladder operators for $\tilde{f}_{\frac{-m}{2},n}$.

5. Representation on the Dirichlet Space

The Dirichlet space, the Hardy space and the Bergman space are the three classical spaces of holomorphic functions in the unit disc. In the present section, we find the $\mathfrak{su}(1,1)$ module (which is the space of the derived representation) on the Dirichlet space.

[4] The Dirichlet space $\mathcal{D}$ on the unit disc $\mathbb{D} = \{w : |w| < 1\}$ consists of the holomorphic functions $f(w)$ on $\mathbb{D}$, for which the following semi-norm is finite:

$$\mathcal{D}(f) := \left(\frac{1}{\pi} \int_{\mathbb{D}} |f'(w)|^2 dxdy \right)^{\frac{1}{2}}, \quad w = x + iy. \tag{62}$$

For $g = \begin{pmatrix} \alpha & \overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} \in SU(1,1)$, the SU(1,1) representation on the Dirichlet space is defined as follows:

$$[\breve{\pi}_0(g)f](w) = f\left(\frac{\overline{\alpha}w - \overline{\beta}}{\alpha - \beta z} \right). \tag{63}$$

The semi-norm $\mathcal{D}(f)$ is not a norm because $\mathcal{D}(f) = 0$ whenever f is a constant. Then, $\breve{\pi}_0$ is a non-unitary representation.

Lemma 3. *The Dirichlet space has two $\mathfrak{su}(1,1)$ vector module:*

- *the lowest weight vector module $V_{0+2m} = \{w_{0,m} : m = 0, 1, 2, 3.....\}$, with the following ladder operators*

$$L_+ w_{0,m} = imw_{0,m+1}, \quad m \in \mathbb{Z}_+ - \{0\},$$
$$L_- w_{0,m} = imw_{0,m-1}, \quad m \in \mathbb{Z}_+ - \{0\},$$

$$L_+ w_{0,0} = 0,$$

- *the highest weight vector module* $\bar{V}_{0+2m} = \{w_{0,m} : m = 0, 1, 2, 3.....\}$,*with the following ladder operators*

$$\begin{aligned} L_+ w_{0,m} &= imw_{0,m+1}, \quad m \in \mathbb{Z}_+ - \{0\}, \\ L_- w_{0,m} &= imw_{0,m-1}, \quad m \in \mathbb{Z}_+ - \{0\}, \\ L_- w_{0,0} &= 0, \end{aligned}$$

Proof.

The representation $\breve{\pi}_0$ (63) is the SU(1, 1) representation $\breve{\pi}_n$, for $n = 0$. The representation $\breve{\pi}_n$ is defined as follows:

$$[\breve{\pi}_n(g)f](w) = f\left(\frac{\overline{\alpha}w - \overline{\beta}}{\alpha - \beta w}\right)(\alpha - \beta w)^{-n}, \tag{64}$$

where $n \in \mathbb{Z}$. The derived representations for the basis $\{\tilde{Z}, \tilde{A}, \tilde{B}\}$ (8) are as follows:

$$\begin{aligned} E = d\breve{\pi}_n^{\tilde{Z}} &= \frac{d}{dt}\breve{\pi}_n(\mathrm{e}^{t\tilde{Z}})f(w)|_{t=0} = [-inI - 2iw\partial_w]f(w), \\ A_1 = d\breve{\pi}_n^{\tilde{A}} &= \frac{d}{dt}\breve{\pi}_n(\mathrm{e}^{t\tilde{A}})f(w)|_{t=0} = \frac{i}{2}[nwI + (1 + w^2)\partial_w]f(w), \\ B_1 = d\breve{\pi}_n^{\tilde{B}} &= \frac{d}{dt}\breve{\pi}_n(\mathrm{e}^{t\tilde{B}})f(w)|_{t=0} = \frac{1}{2}[nwI + (w^2 - 1)\partial_w]f(w). \end{aligned}$$

The commutator relations are

$$[E, A_1] = 2B_1, \ [E, B_1] = -2A_1, \ [A_1, B_1] = -\frac{1}{2}E.$$

The ladder operators are defined as

$$L_+ = A_1 + iB_1 = inwI + iw^2\partial_w, \quad L_- = A_1 - iB_1 = i\partial_w,$$

and

$$[E, L_+] = -2iL_+, \ [E, L_-] = 2iL_- \quad \text{and} \quad [L_+, L_-] = iE. \tag{65}$$

The Casimir operator is

$$d\breve{\pi}_n(C) = d\breve{\pi}_n(\tilde{Z}^2 - 4\tilde{A}^2 - 4\tilde{B}^2) = -n^2 + 2n. \tag{66}$$

The representation $\breve{\pi}_n$ on $L^2(\mathbb{D})$ is irreducible, and V_{n+2m} is the one-dimensional subspace generated by $w_{n,m}$ [13]. Indeed,

$$\breve{\pi}_n\left(\begin{pmatrix} \mathrm{e}^{i\theta} & 0 \\ 0 & \mathrm{e}^{-i\theta} \end{pmatrix}\right)(w_{n,m}) = \mathrm{e}^{-i\theta(n+2m)}w_{n,m}.$$

Hence, V_{n+2m} is an eigenspace of K with an eigenvalue $\mathrm{e}^{-i\theta(n+2m)}$, which is the character of the subgroup K. Then

$$\breve{\pi}_n(\exp t\tilde{Z}) = \mathrm{e}^{-i(n+2m)t} I \quad \text{on} \quad V_{n+2m},$$

and the derived representation is given by

$$E = -i(n+2m)I \quad \text{on} \quad V_{n+2m}.$$

From the commutator relation (65), we have

$$\begin{aligned} E(L_+ w_{n,m}) &= L_+(E w_{n,m}) - 2i L_+ w_{n,m} \\ &= L_+(-i(n+2m)) - 2i L_+ = -i(n+2m+2)L_+, \end{aligned}$$

$$\begin{aligned} E(L_- w_{n,m}) &= L_-(E w_{n,m}) + 2i L_- w_{n,m} \\ &= L_-(-i(n+2m)) + 2i L_+ = -i(n+2m-2)L_-. \end{aligned}$$

Therefore, the ladder operator $L_\pm$ acts as follows:

$$L_+ : V_{n+2m} \to V_{n+2m+2}, \quad L_- : V_{n+2m} \to V_{n+2m-2}.$$

$$\cdots \underset{L_-}{\overset{L_+}{\rightleftarrows}} V_{n+2m-2} \underset{L_-}{\overset{L_+}{\rightleftarrows}} V_{n+2m} \underset{L_-}{\overset{L_+}{\rightleftarrows}} V_{n+2m+2} \underset{L_-}{\overset{L_+}{\rightleftarrows}} \cdots$$

$V_{n+2m} = \{w_{n,m} : m = 0, 1, 2, 3.....\}$ is the lowest weight module and is given as follows:

$$\begin{aligned} E w_{n,m} &= -(n+2m) i w_{n,m}, \\ L_+ w_{n,m} &= A_1 w_{n,m} + i B_1 w_{n,m} = (n+m) i w_{n,m+1}, \quad m \in \mathbb{Z}_+ - \{0\}, \\ L_- w_{n,m} &= A_1 w_{n,m} - i B_1 w_{n,m} = m i w_{n,m-1}, \quad m \in \mathbb{Z}_+ - \{0\} \\ L_- w_{n,0} &= 0, \\ d\breve{\pi}_n(C) w &= (-n^2 + 2n) w, \quad w \in V_{n+2m}. \end{aligned}$$

The vector $w_{n,0}$ is called the lowest weight vector.

$$0 \underset{L_-}{\longleftarrow} w_{n,0} \underset{L_-}{\overset{L_+}{\rightleftarrows}} w_{n,1} \underset{L_-}{\overset{L_+}{\rightleftarrows}} w_{n,2} \underset{L_-}{\overset{L_+}{\rightleftarrows}} \cdots$$

$\bar{V}_{n+2m} = \{w_{n,m} : m = 0, 1, 2, 3.....\}$ is the highest weight module and is given as follows:

$$\begin{aligned} E w_{n,m} &= -(n-2m) i w_{n,m}, \\ L_- w_{n,m} &= A_1 w_{n,m} + i B_1 w_{n,m} = i(n+m) w_{n,m-1}, \quad m \in \mathbb{Z}_+ - \{0\}, \\ L_+ w_{n,m} &= A_1 w_{n,m} - i B_1 w_{n,m} = i m w_{n,m+1}, \quad m \in \mathbb{Z}_+ - \{0\} \end{aligned}$$

$$\cdots \underset{L_-}{\overset{L_+}{\rightleftarrows}} w_{n,2} \underset{L_-}{\overset{L_+}{\rightleftarrows}} w_{n,1} \underset{L_-}{\overset{L_+}{\rightleftarrows}} w_{n,0} \xrightarrow[L_+]{} 0$$

$$\begin{aligned} &L_+ w_{n,0} = 0, \\ &d\breve{\pi}_n(C)w = (-n^2 + 2n)w, \quad w \in V_{n-2m}. \end{aligned}$$

The vector $w_{n,0}$ is called the highest weight vector.

The vector module V_{n+2m} is unitarisable if and only if $n > 0$, and $\bar{V}_{n+2m}$ is unitarisable if and only if $n < 0$ [7, p.96].

Next, for the Dirichlet space the $\mathfrak{su}(1,1)$ vector module is V_{0+2m}, which is given as follows:

$$\begin{aligned} Ew_{0,m} &= -2imw_{0,m}, \\ L_+ w_{0,m} &= imw_{0,m+1}, \quad m \in \mathbb{Z}_+ - \{0\}, \\ L_- w_{0,m} &= imw_{0,m-1}, \quad m \in \mathbb{Z}_+ - \{0\}, \\ L_+ w_{0,0} &= 0, \\ d\breve{\pi}_0(C) &= 0. \end{aligned}$$

This is shown in the following figure:

$$0 \xleftarrow[L_-]{} [w_{0,0}] \xleftarrow[L_-]{} w_{0,1} \underset{L_-}{\overset{L_+}{\rightleftarrows}} w_{0,2} \underset{L_-}{\overset{L_+}{\rightleftarrows}} \cdots$$

In addition, $w_{0,0}$ is the highest weight vector for the vector module $\bar{V}_{0+2m}$ which is given by

$$\begin{aligned} Ew_{0,m} &= -2imw_{0,m}, \\ L_+ w_{0,m} &= imw_{0,m+1}, \quad m \in \mathbb{Z}_+ - \{0\}, \\ L_- w_{0,m} &= imw_{0,m-1}, \quad m \in \mathbb{Z}_+ - \{0\}, \\ L_- w_{0,0} &= 0, \\ d\breve{\pi}_0(C) &= 0. \end{aligned}$$

and presented in the following figure:

$$\cdots \underset{L_-}{\overset{L_+}{\rightleftarrows}} w_{0,2} \underset{L_-}{\overset{L_+}{\rightleftarrows}} w_{0,1} \xrightarrow{L_+} [w_{0,0}] \xrightarrow[L_+]{} 0$$

6. Conclusion

This paper considers the discrete series representation of the $SL_2(\mathbb{R})$ group π_n defined by (1) on spaces of holomorphic functions on the unit disc. The group $\mathrm{SL}_2(\mathbb{R})$ is more convenient for complex analysis in the upper half-plane. However, the group $\mathrm{SU}(1,1)$ of 2×2 matrices, with complex entries and a determinant equal to one, is well suited in unit disc $\mathbb{D}$. The Cayley transform (4) defines an isomorphism of the group $\mathrm{SL}_2(\mathbb{R})$ with the group $\mathrm{SU}(1,1)$. We present the action of the ladder operator of representation on the $\mathfrak{su}(1,1)$ Lie algebra. This action can be realised on the Bergman space for $n \geq 2$, on the Hardy space for $n = 1$ and on the Dirichlet space for $n = 0$. The vector module of the representation on the Dirichlet space has been described. It is worth to studying the $\mathfrak{su}(1,1)$ vector module on other spaces of holomorphic function, for instance the Poly-Bergman space [16].

Acknowledgements

I would like to thank Dr. Vladimir Kisil for all his guidance, valuable advises and comments. Also, my deep gratitude to my family for their continuous love, help and support. I would like to thank my husband for his patience and kindness. Thanks for my kids who have made me stronger and filled me with happiness.

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