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Derivations and representations of commutative algebras verifying a polynomial identity of degree five

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Abstract. In this paper we study a class of commutative non associative algebras satisfying a polynomial identity of degree five. We show that under the assumption of the existence of a non-zero idempotent, any commutative algebra verifying such an identity admits a Peirce decomposition. Using this decomposition we proceeded to the study of the derivations and representations of algebras of this class.

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Key Words and Phrases: Generalized Almost-Jordan algebra, Identity of degree five, Peirce decomposition, Idempotent, Derivation, Representation

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1. Introduction

Albert[1] was one of the authors who made a major contribution to the study of Jordan algebras in the algebra theory of population genetics. Following him, several authors [4] studied algebras verifying more general polynomial identities, such as almost Jordan algebras. The irreducible identities of degree five, not a consequence of commutativity, have all been determined by Osborn in [8], in characteristics other than 2, 3, and 5. In[7], he proceeds to study his first identity

$$2((x^2x)x)x - 3(x^2x^2)x + (x^2x)x^2 = 0. \tag{1}$$

More recently, in [2], A. Dembega study the second identity under certain conditions

$$\begin{aligned}&\beta_1[yx^4 - 4(yx^3)x + 6((yx^2)x)x - 3(((yx)x)x)x]+\\&\beta_2[-y(x^2.x^2) + 5(yx^3)x - 9((yx^2)x)x + 4(((yx)x)x)x + ((yx)x^2)x+\\&(yx^2)x^2 - (yx)x^3] + \beta_3[((yx^2)x)x - (((yx)x)x)x - (yx^2)x^2 + ((yx)x)x^2] = 0\end{aligned} \tag{2}$$

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where β_1, β_2 and $\beta_3 \in \mathbb{K}$, where $\mathbb{K}$ is a commutative field.
The aim of the this paper is to determine the representations and the derivations of commutative algebras A defined by

$$yx^4 - 4(yx^3)x + 6((yx^2)x)x - 3(((yx)x)x)x = 0, \tag{3}$$

corresponding to $\beta_1 = 1$, $\beta_2 = \beta_3 = 0$ in the identity (2). Such algebras can be illustrated by the following examples given in [2]. In the rest of the paper, $\mathbb{K}$ refers to the field $\mathbb{C}$ of complex numbers.

Example 1.1. Let A be the commutative five-dimensional algebra whose the multiplication table in the basis $\{e_1, e_2, e_3, e_4, e_5\}$ is $e_1^2 = e_2$, $e_1e_2 = e_3$, $e_2^2 = e_4$, $e_1e_3 = e_5$, the other products not mentioned being zero. This algebra verifies (3). Indeed, $A^2 =< e_2, e_3, e_4, e_5 >$, $A^3 =< e_3, e_4, e_5 >$, $A^4 =< e_5 >$ and $A^5 = 0$.

Example 1.2. Let A be the commutative 5-dimensional algebra whose the multiplication table in the basis $\{e, x_0, y_1, z_{\lambda_1}, w_{\lambda_2}\}$ is $e^2 = e$, $ey_1 = y_1$, $ez_{\lambda_1} = \lambda_1 z_{\lambda_1}$, $ew_{\lambda_2} = \lambda_2 w_{\lambda_2}$, $x_0 z_{\lambda_1} = z_{\lambda_1}$, $y_1 w_{\lambda_2} = w_{\lambda_2}$ with $\lambda_1 = \frac{3+i\sqrt{3}}{6}$ and $\lambda_2 = \frac{3-i\sqrt{3}}{6}$, all other products not mentioned being zero.

2. Peirce decomposition

In the rest of the paper we study commutative algebras verifying the identity (3) and admitting a non-zero idempotent e. By setting $x = e$ in (3), we obtain

$$[L_e(L_e - id_A)(3L_e^2 - 3L_e + I)](y) = 0, \tag{4}$$

where $y \in A$, $id_A : A \to A, x \mapsto x$ and $L_e : A \to A, y \mapsto ey$ is the multiplication by e. Consider $p(t) = -3t^4 + 6t^3 - 4t^2 + t$ which factorises to $p(t) = -t(t-1)(3t^2 - 3t + 1)$. Since $\mathbb{K}$ is algebraically closed, then the polynomial $3t^2 - 3t + 1$ has two roots λ_1 and λ_2, i.e $p(t) = -t(t-1)(t-\lambda_1)(t-\lambda_2)$, in which case $A = A_0 \oplus A_1 \oplus A_{\lambda_1} \oplus A_{\lambda_2}$, where $A_\mu = \{x \in A | ex = \mu x\}$, $\lambda_1 = \frac{3+i\sqrt{3}}{6}$ and $\lambda_2 = \frac{3-i\sqrt{3}}{6}$.

The products of the components of the Peirce decomposition are given by the following theorem.

Theorem 2.1. *([2],Théorème 5.1.1) Let A be an algebra satisfying the identity* (3)*. Suppose that the polynomial $3t^2 - 3t + 1 = 3(t-\lambda_1)(t-\lambda_2)$ in $\mathbb{K}[t]$. Then the Peirce decomposition of A is $A = A_0 \oplus A_1 \oplus A_{\lambda_1} \oplus A_{\lambda_2}$, where $\lambda_1 = \frac{3+i\sqrt{3}}{6}$ and $\lambda_2 = \frac{3-i\sqrt{3}}{6}$. We have:*

i) $A_i A_{\lambda_j} \subseteq A_{\lambda_j}$*, with $i \in \{0, 1\}$ and $j \in \{1; 2\}$;*

ii) $A_{\lambda_i} A_{\lambda_j} = \{0\}$*, with $i, j \in \{1; 2\}$;*

iii) $A_iA_i \subseteq A_i$ *with* $i \in \{0;1\}$*;*

iv) $A_0A_1 = \{0\}$.

The following theorem gives us the identities used in the rest of the document.

Theorem 2.2. *([2], Théorème 5.1.2) Let A be an algebra satisfying the identity* (3)*, whose Peirce decomposition of A is written $A = A_0 \oplus A_1 \oplus A_{\lambda_1} \oplus A_{\lambda_2}$, then*

i) $x_0(y_0x_{\lambda_k}) + y_0(x_0x_{\lambda_k}) = 2\overline{\lambda_k}x_{\lambda_k}(x_0y_0)$*;*

ii) $x_1(y_1x_{\lambda_k}) + y_1(x_1x_{\lambda_k}) = (1 - 3(\overline{\lambda_k})^3)x_{\lambda_k}(x_1y_1)$*;*

iii) $x_1(x_{\lambda_k}x_0) = x_0(x_1x_{\lambda_k})$,

where $\overline{\lambda_k}$ is the conjugate of λ_k, and $x_i, y_i \in A_i$, $x_{\lambda_k} \in A_{\lambda_k}$ for $i \in \{0;1\}$ and $k \in \{1;2\}$.

3. Derivations

Let A be a $\mathbb{K}$-algebra satisfying (3). We say that an endomorphism d of A is a derivation of A if for all $x, y \in A$, $d(xy) = d(x)y + xd(y)$. The set $Der_{\mathbb{K}}(A)$ of all derivations of A with the Lie bracket $[,]$, is a Lie subalgebra of $End_{\mathbb{K}}(A)^-$, where $End_{\mathbb{K}}(A)$ is the associative algebra of endomorphisms of A. The Lie bracket is defined by: $[d, d'] = dd' - d'd$ ([5], [6]). The following theorems characterise the derivations of an algebra satisfying (3).

Theorem 3.1. *Let $A = A_0 \oplus A_1 \oplus A_{\lambda_1} \oplus A_{\lambda_2}$ be the Peirce decomposition of an algebra satisfying* (3)*. If d is a derivation of A, then d satisfies the following properties, for $i \in \{0, 1, \lambda_1, \lambda_2\}$; $\alpha \in \{\lambda_1, \lambda_2\}$; $\beta \in \{0, 1\}$:*

i) $d(e) = 0$*;*

ii) $d(x_i) \in A_i$*;*

iii) $f_i = d_{/A_i}$*;*

iv) $f_\alpha(x_\beta y_\alpha) = x_\beta f_\alpha(y_\alpha) + y_\alpha f_\beta(x_\beta)$.

Proof. *i)* Since e is an idempotent of A then $2ed(e) = d(e)$. By setting $d(e) = x_0 + x_1 + x_{\lambda_1} + x_{\lambda_2}$, we have $-x_0 + x_1 + (2\lambda_1 - 1)x_{\lambda_1} + (2\lambda_2 - 1)x_{\lambda_2} = 0$ i.e $x_0 = x_1 = x_{\lambda_1} = x_{\lambda_2} = 0$; hence $d(e) = 0$.
ii) and *iii)* Let $x_i \in A_i$ for $i \in \{0, 1, \lambda_1, \lambda_2\}$; then $ex_i = ix_i$ i.e $ed(x_i) = id(x_i)$ because $d(e) = 0$. So we have $d(x_i) \in A_i$, and $f_i = d_{/A_i}$.

iv) We have $x_\beta y_\alpha \in A_\alpha$ for $\alpha \in \{\lambda_1, \lambda_2\}$ and $\beta \in \{0, 1\}$ according to the products of the Peirce subspaces of A, since $d(x_\beta y_\alpha) = x_\beta d(y_\alpha) + y_\alpha d(x_\beta)$ then, applying *iii)*, we have $f_\alpha(x_\beta y_\alpha) = x_\beta f_\beta(y_\alpha) + y_\alpha f_\alpha(x_\beta)$.

Theorem 3.2. *Let $A = A_0 \oplus A_1 \oplus A_{\lambda_1} \oplus A_{\lambda_2}$ be the Peirce decomposition of an algebra satisfying* (3)*. Let d and d' be derivations of A, then the derivation $[d, d']$ satisfies for $i \in \{0; 1; \lambda_1; \lambda_2\}$; $\alpha \in \{\lambda_1; \lambda_2\}$ and $\beta \in \{0; 1\}$ the following conditions:*

i) $[d, d'](e) = 0$;

ii) $[d, d'](x_i) \in A_i$;

iii) $f_{[d,d']}(x_i) = [d, d']_{/A_i} = f_d(f_{d'}(x_i)) - f_{d'}(f_d(x_i))$.

Proof. We have $d(e) = 0$ by the theorem(3.1), so $[d, d'](e) = d(d'(e)) - d'(d(e)) = 0$. Hence i).

Also $[d, d'](x_i) = f_d(f_{d'}(x_i)) - f_{d'}(f_d(x_i))$, hence ii) and iii).

4. Representations

We are interested here in the representations of algebras defined by the identity (3) ([4]). Following Eilenberg [3], if A is a commutative $\mathbb{K}$-algebra belonging to a class $\mathcal{C}$ and M a vector space on $\mathbb{K}$, a linear application ρ: $A \longrightarrow End(M)$ is a representation of A in the class $\mathcal{C}$ if the extension $S = A \oplus M$ of M with multiplication given by $(a + m)(b + n) = ab + \rho(a)(n) + \rho(b)(m)$ for all $a, b \in A, m, n \in M$; belong to the class $\mathcal{C}$. In the following section, we give some results on the representations of algebras verifying the identity (3).

4.1. General results

Lemma 4.1. *Let A be an algebra verifying the identity* (3) *and ρ: $A \longrightarrow End(M)$ a linear application. The application ρ is a representation of A if and only if for all $a, b \in A$ the following identities are verified:*

$$\rho_{a^4} - 4\rho_a\rho_{a^3} + 6\rho_a^2\rho_{a^2} - 3\rho_a^4 = 0; \tag{5}$$

$$\rho_b\rho_a\rho_{a^2} + 2\rho_b\rho_a^3 + \rho_b\rho_{a^3} - 4\rho_a\rho_b\rho_{a^2} - 8\rho_a\rho_b\rho_a^2 - 4\rho_{a^3b} + 12\rho_a^2\rho_b\rho_a + 6\rho_a\rho_{a^2b} + 6\rho_{a(a^2b)} - 3\rho_a^3\rho_b - 3\rho_a^2\rho_{ab} - 3\rho_a\rho_{a(ab)} - 3\rho_{a(a(ab))} = 0 \tag{6}$$

where $\rho_a := \rho(a) \in End(M)$ and for any $a \in A$.

Proof. The application ρ is a representation of A if and only if for all $x = a + m$, and $y = b + n \in A \oplus M$, the equality (3) is satisfied.

Since:
$yx^4 = a^4b + \rho_b\rho_a\rho_{a^2}(m) + 2\rho_b\rho_a^3(m) + \rho_{a^4}(n) + \rho_b\rho_{a^3}(m)$;
$x(yx^3) = a(a^3b) + \rho_a\rho_b\rho_{a^2}(m) + 2\rho_a\rho_b\rho_a^2(m) + \rho_a\rho_{a^3}(n) + \rho_{a^3b}(m)$;
$x(x(yx^2)) = a(a(a^2b)) + 2\rho_a^2\rho_b\rho_a(m) + \rho_a^2\rho_{a^2}(n) + \rho_a\rho_{a^2b}(m) + \rho_{a(a^2b)}(m)$;

$x(x(x(yx))) = a(a(a(ab))) + \rho_a^4(n) + \rho_a^3\rho_b(m) + \rho_a^2\rho_{ab}(m) + \rho_a\rho_{a(ab)}(m) + \rho_{a(a(ab))}(m)$;
Equality (3) becomes:

$$[a^4b-4a(a^3b)+6a(a(a^2b))-3a(a(a(ab)))]+[\rho_b\rho_a\rho_{a^2}+2\rho_b\rho_a^3+\rho_b\rho_{a^3}-4\rho_a\rho_b\rho_{a^2}-8\rho_a\rho_b\rho_a^2-4\rho_{a^3b} + 12\rho_a^2\rho_b\rho_a + 6\rho_a\rho_{a^2b} + 6\rho_{a(a^2b)} - 3\rho_a^3\rho_b - 3\rho_a^2\rho_{ab} - 3\rho_a\rho_{a(ab)} - 3\rho_{a(a(ab))}](m) + [\rho_{a^4} - 4\rho_a\rho_{a^3} + 6\rho_a^2\rho_{a^2} - 3\rho_a^4](n) = 0. \quad (7)$$

Since a and b are elements of the algebra A, then:

$$a^4b - 4a(a^3b) + 6a(a(a^2b)) - 3a(a(a(ab))) = 0.$$

We then obtain,

$$[\rho_b\rho_a\rho_{a^2}+2\rho_b\rho_a^3+\rho_b\rho_{a^3}-4\rho_a\rho_b\rho_{a^2}-8\rho_a\rho_b\rho_a^2-4\rho_{a^3b}+12\rho_a^2\rho_b\rho_a+6\rho_a\rho_{a^2b}+6\rho_{a(a^2b)}-3\rho_a^3\rho_b - 3\rho_a^2\rho_{ab} - 3\rho_a\rho_{a(ab)} - 3\rho_{a(a(ab))}](m) + [\rho_{a^4} - 4\rho_a\rho_{a^3} + 6\rho_a^2\rho_{a^2} - 3\rho_a^4](n) = 0. \quad (8)$$

Then finally:

$$\rho_{a^4} - 4\rho_a\rho_{a^3} + 6\rho_a^2\rho_{a^2} - 3\rho_a^4 = 0;$$

$$\rho_b\rho_a\rho_{a^2}+2\rho_b\rho_a^3+\rho_b\rho_{a^3}-4\rho_a\rho_b\rho_{a^2}-8\rho_a\rho_b\rho_a^2-4\rho_{a^3b}+12\rho_a^2\rho_b\rho_a+6\rho_a\rho_{a^2b}+6\rho_{a(a^2b)}-3\rho_a^3\rho_b - 3\rho_a^2\rho_{ab} - 3\rho_a\rho_{a(ab)} - 3\rho_{a(a(ab))} = 0.$$

Proposition 4.2. *Let A be an algebra verifying the identity* (3). *Suppose that A has an idempotent $e \neq 0$. Let $\rho : A \longrightarrow End(M)$ be a representation of A. Then:*

$$M = M_0 \oplus M_1 \oplus M_{\lambda_1} \oplus M_{\lambda_2},$$

with $M_i = \{m \in M, \rho_e(m) = im\}$ for $i \in \{0, 1, \lambda_1, \lambda_2\}$ where $\lambda_1 = \frac{3+i\sqrt{3}}{6}$ and $\lambda_2 = \frac{3-i\sqrt{3}}{6}$ are roots of the polynomial $3t^2 - 3t + 1$.

Proof. Putting $a = e$ into the identity (5), we obtain $-\rho_e(\rho_e - I)(3\rho_e^2 - 3\rho_e + I) = 0$.
The kernel lemma gives us: $M = M_0 \oplus M_1 \oplus M_{\lambda_1} \oplus M_{\lambda_2}$, with $M_i = \{m \in M|\rho_e(m) = im\}$ for $i \in \{0, 1, \lambda_1, \lambda_2\}$.

Let's study the action of A on M. In this section, we'll focus on the products of $A_i.M_j$, for $i; j \in \{0, 1, \lambda_1, \lambda_2\}$.

Theorem 4.3. *Let A be an algebra verifying the identity* (3) *admitting an idempotent $e \neq 0$. Let $\rho : A \longrightarrow End(M)$ be a representation of A. Then the action of A on M satisfies the following relations:*

i) $A_i.M_{\lambda_j} \subseteq M_{\lambda_j}$, *where $i \in \{0; 1\}$ and $j \in \{1; 2\}$;*

ii) $A_{\lambda_j}.M_i \subseteq M_{\lambda_j}$, *where $i \in \{0, 1\}$ and $j \in \{1, 2\}$;*

iii) $A_{\lambda_i}.M_{\lambda_j} = \{0\}$, *with* $i, j \in \{1, 2\}$;

iv) $A_i.M_i \subseteq M_i$ *with* $i \in \{0, 1\}$;

v) $A_i.M_j = \{0\}$ *with* $i, j \in \{0, 1\}$, $i \neq j$.

Let $A = A_0 \oplus A_1 \oplus A_{\lambda_1} \oplus A_{\lambda_2}$ be the Peirce decomposition of an algebra satisfying (3) relative to an idempotent e and that of the modulus $M = M_0 \oplus M_1 \oplus M_{\lambda_1} \oplus M_{\lambda_2}$ then the Peirce decomposition of the extension S is $S = S_0 \oplus S_1 \oplus S_{\lambda_1} \oplus S_{\lambda_2}$ where $S_k = \{a + m \in S : e(a + m) = k(a + m)\}$ and $k = \{0, 1, \lambda_1, \lambda_2\}$ such that the following result is verified.

Proposition 4.4. *The subspaces* S_i *of the extension* S *satisfy the relations:*

i) $S_i.S_{\lambda_j} \subseteq S_{\lambda_j}$, *and* $S_{\lambda_j}.S_i \subseteq S_{\lambda_j}$ *with* $i \in \{0, 1\}$ *and* $j \in \{1, 2\}$;

ii) $S_{\lambda_i}.S_{\lambda_j} = \{0\}$, *with* $i, j \in \{1, 2\}$;

iii) $S_i.S_i \subseteq S_i$ *with* $i \in \{0, 1\}$;

iv) $S_i.S_j = \{0\}$ *with* $i, j \in \{0, 1\}$, $i \neq j$.

Proof. This follows from the fact that the algebra S must belong to the same class as A, i.e it must verify the identity (3). Consequently, its subspaces verify the same properties as those of A.

4.2. Irreducible representations

As in the case of groups, rings or vector spaces, a submodule is a non-empty part of a module, stable for the laws of modules, hence the following definition:

Definition 4.5. Let A be a $\mathbb{K}$-algebra and $\rho : A \longrightarrow End(M)$ a representation of A.

i) Let N be a subspace of M, N is a submodule of M if and only if $A.N \subseteq N$.

ii) M is an irreducible module or ρ is an irreducible representation of A, if $M \neq 0$ and the only submodules of M are its trivial submodules.

iii) An A-module M is said to be simple or irreducible if M is not the null module and there are no submodules outside $\{0\}$ and M.

Lemma 4.6. *Let* A *be an algebra satisfying the identity* (3)*, then* M_{λ_1} *and* M_{λ_2} *are submodules of* M*.*

Proof. Using Theorem (4.3), we observe that for all $a \in A$ and $m \in M_\lambda$, we have $a.m \in M_\lambda$ for any $\lambda \in \{\lambda_1, \lambda_2\}$.

Proposition 4.7. *Let A be an algebra satisfying the identity (3) admitting an idempotent $e \neq 0$ and $\rho : A \longrightarrow End(M)$ an irreducible representation of A. Then one of the conditions below holds:*

i) $M = M_0$ *or* $M = M_1$*;*

ii) $M = M_{\lambda_1}$*;*

iii) $M = M_{\lambda_2}$.

Proof. Using the Lemma 4.6, we have M_{λ_1} and M_{λ_2} are submodules of M. Since M is irreducible, then: $M_{\lambda_1} = M$ or $M_{\lambda_1} = \{0\}$. In other words, $M_{\lambda_2} = M$ or $M_{\lambda_2} = \{0\}$.

Theorem 4.8. *Let A be an algebra verifying the identity (3). Suppose that A has an idempotent $e \neq 0$ and M an irreducible module. If $M = M_i$, with $i \in \{0; 1\}$, then- $\forall a, b \in A; m \in M$*

1) $(a, b, m) = (a_i, b_i, m_i)$*;*

2) $(a, m, b) = (a_i, m_i, b_i)$.

Proof. Suppose $M = M_i$, $i \in \{0; 1\}$ then $M_{\lambda_1} = M_{\lambda_2} = \{0\}$. We'll do the proof for $M = M_0$; for the case where $M = M_1$, the proof is done simply. Let $a, b \in A$ be such that $a = a_0 + a_1 + a_{\lambda_1} + a_{\lambda_2}$ and $b = b_0 + b_1 + b_{\lambda_1} + b_{\lambda_2}$, where $a_l, b_l \in A_l$, $l \in \{0; 1; \lambda_1; \lambda_2\}$. Take $m \in M = M_0$, so $m = m_0$, where $m_0 \in M_0$, *i.e* $\rho_e(m_0) = 0$. Let's calculate (a, b, m_0), (a, m_0, b) $\forall a, b \in A$ and $m \in M$. We have:

$$
\begin{aligned}
&(a, b, m) = (a_0 + a_1 + a_{\lambda_1} + a_{\lambda_2}, b_0 + b_1 + b_{\lambda_1} + b_{\lambda_2}, m_0)\\
&= (a_0, b_0, m_0) + (a_0, b_1, m_0) + (a_0, b_{\lambda_1}, m_0) + (a_0, b_{\lambda_2}, m_0) + (a_1, b_0, m_0) + (a_1, b_1, m_0)\\
&+ (a_1, b_{\lambda_1}, m_0) + (a_1, b_{\lambda_2}, m_0) + (a_{\lambda_1}, b_0, m_0) + (a_{\lambda_1}, b_1, m_0) + (a_{\lambda_1}, b_{\lambda_1}, m_0)\\
&+ (a_{\lambda_1}, b_{\lambda_2}, m_0) + (a_{\lambda_2}, b_0, m_0) + (a_{\lambda_2}, b_1, m_0) + (a_{\lambda_2}, b_{\lambda_1}, m_0) + (a_{\lambda_2}, b_{\lambda_2}, m_0)\\
&= (a_0b_0)m_0 - a_0(b_0m_0) + (a_0b_1)m_0 - a_0(b_1m_0) + (a_0b_{\lambda_1})m_0 - a_0(b_{\lambda_1}m_0) + (a_0b_{\lambda_2})m_0\\
&- a_0(b_{\lambda_2}m_0) + (a_1b_0)m_0 - a_1(b_0m_0) + (a_1b_1)m_0 - a_1(b_1m_0) + (a_1b_{\lambda_1})m_0 - a_1(b_{\lambda_1}m_0)\\
&+ (a_1b_{\lambda_2})m_0 - a_1(b_{\lambda_2}m_0) + (a_{\lambda_1}b_0)m_0 - a_{\lambda_1}(b_0m_0) + (a_{\lambda_1}b_1)m_0 - a_{\lambda_1}(b_1m_0)\\
&+ (a_{\lambda_1}b_{\lambda_1})m_0 - a_{\lambda_1}(b_{\lambda_1}m_0) + (a_{\lambda_1}b_{\lambda_2})m_0 - a_{\lambda_1}(b_{\lambda_2}m_0) + (a_{\lambda_1}b_0)m_0 - a_{\lambda_2}(b_0m_0)\\
&+ (a_{\lambda_2}b_1)m_0 - a_{\lambda_2}(b_1m_0) + (a_{\lambda_2}b_{\lambda_1})m_0 - a_{\lambda_2}(b_{\lambda_1}m_0) + (a_{\lambda_2}b_{\lambda_2})m_0 - a_{\lambda_2}(b_{\lambda_2}m_0)\\
&= (a_0, b_0, m_0)
\end{aligned}
$$

This result is found by using the multiplication table of A (see theorem (2.1)) and the action of A on M (see theorem (4.3)), for all $m \in M$, we have: $(a, b, m) = (a_0, b_0, m_0)$. Similarly, if $m \in M = M_0$, then we have:

$$
\begin{aligned}
&(a, m, b) = (a_0 + a_1 + a_{\lambda_1} + a_{\lambda_2}, m_0, b_0 + b_1 + b_{\lambda_1} + b_{\lambda_2})\\
&= (a_0, m_0, b_0) + (a_0, m_0, b_1) + (a_0, m_0, b_{\lambda_1}) + (a_0, m_0, b_{\lambda_2}) + (a_1, m_0, b_0) + (a_1, m_0, b_1)
\end{aligned}
$$

$$
\begin{aligned}
&+ (a_1, m_0, b_{\lambda_1}) + (a_1, m_0, b_{\lambda_2}) + (a_{\lambda_1}, m_0, b_0) + (a_{\lambda_1}, m_0, b_1) + (a_{\lambda_1}, m_0, b_{\lambda_1})\\
&+ (a_{\lambda_1}, m_0, b_{\lambda_2}) + (a_{\lambda_2}, m_0, b_0) + (a_{\lambda_2}, m_0, b_1) + (a_{\lambda_2}, m_0, b_{\lambda_1}) + (a_{\lambda_2}, m_0, b_{\lambda_2})\\
&= (a_0m_0)b_0 - a_0(m_0b_0) + (a_0m_0)b_1 - a_0(m_0b_1) + (a_0m_0)b_{\lambda_1} - a_0(m_0b_{\lambda_1}) + (a_0m_0)b_{\lambda_2}\\
&- a_0(m_0b_{\lambda_2}) + (a_1m_0)b_0 - a_1(m_0b_0) + (a_1m_0)b_1 - a_1(m_0b_1) + (a_1m_0)b_{\lambda_1} - a_1(m_0b_{\lambda_1})\\
&+ (a_1m_0)b_{\lambda_2} - a_1(m_0b_{\lambda_2}) + (a_{\lambda_1}m_0)b_0 - a_{\lambda_1}(m_0b_0) + (a_{\lambda_1}m_0)b_1 - a_{\lambda_1}(m_0b_1)\\
&+ (a_{\lambda_1}m_0)b_{\lambda_1} - a_{\lambda_1}(m_0b_{\lambda_1}) + (a_{\lambda_1}m_0)b_{\lambda_2} - a_{\lambda_1}(m_0b_{\lambda_2}) + (a_{\lambda_2}m_0)b_0 - a_{\lambda_2}(m_0b_0)\\
&+ (a_{\lambda_2}m_0)b_1 - a_{\lambda_2}(m_0b_1) + (a_{\lambda_2}m_0)b_{\lambda_1} - a_{\lambda_2}(m_0b_{\lambda_1}) + (a_{\lambda_2}m_0)b_{\lambda_2} - a_{\lambda_2}(m_0b_{\lambda_2})\\
&= (a_0, m_0, b_0)
\end{aligned}
$$

Theorem 4.9. *Let A be an algebra verifying the identity (3). Suppose that A has an idempotent $e \neq 0$ and M an irreducible module. If $M = M_\lambda$, with $\lambda \in \{\lambda_1, \lambda_2\}$ then*

$$(i)(a,b,m) = 0 \quad ,\forall a,b \in A; m \in M \qquad (ii)\rho_b\rho_c = \frac{3\lambda^2 - 1}{9\lambda^3}\rho_{cb} \quad ,\forall a,b \in A; m \in M \tag{9}$$

Proof. Suppose $M = M_\lambda$, $\lambda \in \{\lambda_1, \lambda_2\}$ then $M_0 = M_1 = \{0\}$. Let $a, b \in A$ be such that $a = a_0 + a_1 + a_{\lambda_1} + a_{\lambda_2}$ and $b = b_0 + b_1 + b_{\lambda_1} + b_{\lambda_2}$, where $a_i, b_i \in A_i$, $i \in \{0; 1; \lambda_1; \lambda_2\}$. Take $m \in M = M_\lambda$, so $m = m_\lambda$, where $m_\lambda \in M_\lambda$, *i.e* $\rho_e(m_\lambda) = \lambda m_\lambda$. Let's calculate (a, b, m_λ), (a, m_λ, b) $\forall a, b \in A$ and $m \in M$. We have:

$$
\begin{aligned}
&(a,b,m) = (a_0 + a_1 + a_{\lambda_1} + a_{\lambda_2}, b_0 + b_1 + b_{\lambda_1} + b_{\lambda_2}, m_\lambda)\\
&= (a_0, b_0, m_\lambda) + (a_0, b_1, m_\lambda) + (a_0, b_{\lambda_1}, m_\lambda) + (a_0, b_{\lambda_2}, m_\lambda) + (a_1, b_0, m_\lambda) + (a_1, b_1, m_\lambda)\\
&+ (a_1, b_{\lambda_1}, m_\lambda) + (a_1, b_{\lambda_2}, m_\lambda) + (a_{\lambda_1}, b_0, m_\lambda) + (a_{\lambda_1}, b_1, m_\lambda) + (a_{\lambda_1}, b_{\lambda_1}, m_\lambda)\\
&+ (a_{\lambda_1}, b_{\lambda_2}, m_\lambda) + (a_{\lambda_2}, b_0, m_\lambda) + (a_{\lambda_2}, b_1, m_\lambda) + (a_{\lambda_2}, b_{\lambda_1}, m_\lambda) + (a_{\lambda_2}, b_{\lambda_2}, m_\lambda)
\end{aligned}
$$

$$
\begin{aligned}
&= (a_0b_0)m_\lambda - a_0(b_0m_\lambda) + (a_0b_1)m_\lambda - a_0(b_1m_\lambda) + (a_0b_{\lambda_1})m_\lambda - a_0(b_{\lambda_1}m_\lambda) + (a_0b_{\lambda_2})m_\lambda\\
&- a_0(b_{\lambda_2}m_\lambda) + (a_1b_0)m_\lambda - a_1(b_0m_\lambda) + (a_1b_1)m_\lambda - a_1(b_1m_\lambda) + (a_1b_{\lambda_1})m_\lambda - a_1(b_{\lambda_1}m_\lambda)\\
&+ (a_1b_{\lambda_2})m_\lambda - a_1(b_{\lambda_2}m_\lambda) + (a_{\lambda_1}b_0)m_\lambda - a_{\lambda_1}(b_0m_\lambda) + (a_{\lambda_1}b_1)m_\lambda - a_{\lambda_1}(b_1m_\lambda)\\
&+ (a_{\lambda_1}b_{\lambda_1})m_\lambda - a_{\lambda_1}(b_{\lambda_1}m_\lambda) + (a_{\lambda_1}b_{\lambda_2})m_\lambda - a_{\lambda_1}(b_{\lambda_2}m_\lambda) + (a_{\lambda_2}b_0)m_\lambda - a_{\lambda_2}(b_0m_\lambda)\\
&+ (a_{\lambda_2}b_1)m_\lambda - a_{\lambda_2}(b_1m_\lambda) + (a_{\lambda_2}b_{\lambda_1})m_\lambda - a_{\lambda_2}(b_{\lambda_1}m_\lambda) + (a_{\lambda_2}b_{\lambda_2})m_\lambda - a_{\lambda_2}(b_{\lambda_2}m_\lambda)\\
&= (a_0b_0)m_\lambda - a_0(b_0m_\lambda) - a_0(b_1m_\lambda) - a_1(b_0m_\lambda) + (a_1b_1)m_\lambda - a_1(b_1m_\lambda)\\
&= (a_0, b_0, m_\lambda) + (a_1, b_1, m_\lambda) - a_0(b_1m_\lambda) - a_1(b_0m_\lambda)
\end{aligned}
$$

Similarly, if $\forall m \in M = M_\lambda$, we have:

$$
\begin{aligned}
&(a,m,b) = (a_0 + a_1 + a_{\lambda_1} + a_{\lambda_2}, m_\lambda, b_0 + b_1 + b_{\lambda_1} + b_{\lambda_2})\\
&= (a_0, m_\lambda, b_0) + (a_0, m_\lambda, b_1) + (a_0, m_\lambda, b_{\lambda_1}) + (a_0, m_\lambda, b_{\lambda_2}) + (a_1, m_\lambda, b_0) + (a_1, m_\lambda, b_1)
\end{aligned}
$$

$$
\begin{aligned}
&+ (a_1, m_\lambda, b_{\lambda_1}) + (a_1, m_\lambda, b_{\lambda_2}) + (a_{\lambda_1}, m_\lambda, b_0) + (a_{\lambda_1}, m_\lambda, b_1) + (a_{\lambda_1}, m_\lambda, b_{\lambda_1})\\
&+ (a_{\lambda_1}, m_\lambda, b_{\lambda_2}) + (a_{\lambda_2}, m_\lambda, b_0) + (a_{\lambda_2}, m_\lambda, b_1) + (a_{\lambda_2}, m_\lambda, b_{\lambda_1}) + (a_{\lambda_2}, m_\lambda, b_{\lambda_2})\\
&= (a_0 m_\lambda)b_0 - a_0(m_\lambda b_0) + (a_0 m_\lambda)b_1 - a_0(m_\lambda b_1) + (a_0 m_\lambda)b_{\lambda_1} - a_0(m_\lambda b_{\lambda_1}) + (a_0 m_\lambda)b_{\lambda_2}\\
&- a_0(m_\lambda b_{\lambda_2}) + (a_1 m_\lambda)b_0 - a_1(m_\lambda b_0) + (a_1 m_\lambda)b_1 - a_1(m_\lambda b_1) + (a_1 m_\lambda)b_{\lambda_1} - a_1(m_\lambda b_{\lambda_1})\\
&+ (a_1 m_\lambda)b_{\lambda_2} - a_1(m_\lambda b_{\lambda_2}) + (a_{\lambda_1} m_\lambda)b_0 - a_{\lambda_1}(m_\lambda b_0) + (a_{\lambda_1} m_\lambda)b_1 - a_{\lambda_1}(m_\lambda b_1)\\
&+ (a_{\lambda_1} m_\lambda)b_{\lambda_1} - a_{\lambda_1}(m_\lambda b_{\lambda_1}) + (a_{\lambda_1} m_\lambda)b_{\lambda_2} - a_{\lambda_1}(m_\lambda b_{\lambda_2}) + (a_{\lambda_2} m_\lambda)b_0 - a_{\lambda_2}(m_\lambda b_0)\\
&+ (a_{\lambda_2} m_\lambda)b_1 - a_{\lambda_2}(m_\lambda b_1) + (a_{\lambda_2} m_\lambda)b_{\lambda_1} - a_{\lambda_2}(m_\lambda b_{\lambda_1}) + (a_{\lambda_2} m_\lambda)b_{\lambda_2} - a_{\lambda_2}(m_\lambda b_{\lambda_2})\\
&= (a_0 m_\lambda)b_0 - a_0(m_\lambda b_0) + (a_0 m_\lambda)b_1 - a_0(m_\lambda b_1) + (a_1 m_\lambda)b_0 - a_1(m_\lambda b_0) + (a_1 m_\lambda)b_1\\
&- a_1(m_\lambda b_1)\\
&= (a_0, m_\lambda, b_0) + (a_0, m_\lambda, b_1) + (a_1, m_\lambda, b_0) + (a_1, m_\lambda, b_1)
\end{aligned}
$$

Partial linearisation of the terms of the identity (6) gives us:

$$
\begin{aligned}
&\rho_b\rho_a\rho_{a^2} = \rho_b\rho_c\rho_{a^2} + 2\rho_b\rho_a\rho_{ac};\\
&2\rho_b\rho_a^3 = 2(\rho_b\rho_c\rho_a^2 + \rho_b\rho_a\rho_c\rho_a + \rho_b\rho_a^2\rho_c);\\
&\rho_b\rho_{a^3} = \rho_b\rho_{ca^2} + 2\rho_b\rho_{a(ac)};\\
&4\rho_a\rho_b\rho_{a^2} = 4(\rho_c\rho_b\rho_{a^2} + 2\rho_a\rho_b\rho_{ac});\\
&8\rho_a\rho_b\rho_a^2 = 8(\rho_c\rho_b\rho_a^2 + \rho_a\rho_b\rho_c\rho_a + \rho_a\rho_b\rho_a\rho_c);\\
&4\rho_{a^3b} = 4(\rho_{(ca^2)b} + 2\rho_{(a(ac))b});\\
&12\rho_a^2\rho_b\rho_a = 12(\rho_a\rho_c\rho_b\rho_a + \rho_c\rho_a\rho_b\rho_a + \rho_a^2\rho_b\rho_c);\\
&6\rho_a\rho_{a^2b} = 6(\rho_c\rho_{a^2b} + 2\rho_a\rho_{(ca)b});\\
&6\rho_{a(a^2b)} = 6(\rho_{c(a^2b)} + 2\rho_{a((ac)b)});\\
&3\rho_a^3\rho_b = 3(\rho_c\rho_a^2\rho_b + \rho_a\rho_c\rho_a\rho_b + \rho_a^2\rho_c\rho_b);\\
&3\rho_a^2\rho_{ab} = 3(\rho_c\rho_a\rho_{ab} + \rho_a\rho_c\rho_{ab} + \rho_a^2\rho_{cb});\\
&3\rho_a\rho_{a(ab)} = 3(\rho_c\rho_{a(ab)} + \rho_a\rho_{c(ab)} + \rho_a\rho_{a(cb)});\\
&3\rho_{a(a(ab))} = 3(\rho_{c(a(ab))} + \rho_{a(c(ab))} + \rho_{a(a(cb))}).
\end{aligned}
$$

Putting it all together gives:

$$
\begin{aligned}
&[\rho_b\rho_c\rho_{a^2} + 2\rho_b\rho_a\rho_{ac} + 2\rho_b\rho_c\rho_a^2 + 2\rho_b\rho_a\rho_c\rho_a + 2\rho_b\rho_a^2\rho_c + \rho_b\rho_{ca^2} + 2\rho_b\rho_{a(ac)} - 4\rho_c\rho_b\rho_{a^2}\\
&-8\rho_a\rho_b\rho_{ac} - 8\rho_c\rho_b\rho_a^2 - 8\rho_a\rho_b\rho_c\rho_a - 8\rho_a\rho_b\rho_a\rho_c - 4\rho_{(ca^2)b} - 8\rho_{(a(ac))b} + 12\rho_a\rho_c\rho_b\rho_a + 12\rho_c\rho_a\rho_b\rho_a + 12\rho_a^2\rho_b\rho_c\\
&+6\rho_c\rho_{a^2b} + 12\rho_a\rho_{(ca)b} + 6\rho_{c(a^2b)} + 12\rho_{a((ac)b)} - 3\rho_c\rho_a^2\rho_b - 3\rho_a\rho_c\rho_a\rho_b - 3\rho_a^2\rho_c\rho_b - 3\rho_c\rho_a\rho_{ab} - 3\rho_a\rho_c\rho_{ab} - 3\rho_a^2\rho_{cb}\\
&- 3\rho_c\rho_{a(ab)} - 3\rho_a\rho_{c(ab)} - 3\rho_a\rho_{a(cb)} - 3\rho_{c(a(ab))} - 3\rho_{a(c(ab))} - 3\rho_{a(a(cb))}](m) = 0. \quad (10)
\end{aligned}
$$

If we put $a = e$ in (10) then

$$
[\rho_b\rho_c\rho_{e^2} + 2\rho_b\rho_e\rho_{ec} + 2\rho_b\rho_c\rho_e^2 + 2\rho_b\rho_e\rho_c\rho_e + 2\rho_b\rho_e^2\rho_c + \rho_b\rho_{ce^2} + 2\rho_b\rho_{e(ec)} - 4\rho_c\rho_b\rho_{e^2}
$$

$$-8\rho_e\rho_b\rho_{ec}-8\rho_c\rho_b\rho_e^2-8\rho_e\rho_b\rho_c\rho_e-8\rho_e\rho_b\rho_e\rho_c-4\rho_{(ce^2)b}-8\rho_{(e(ec))b}+12\rho_e\rho_c\rho_b\rho_e+12\rho_c\rho_e\rho_b\rho_e+12\rho_e^2\rho_b\rho_c$$
$$+6\rho_c\rho_{e^2b}+12\rho_e\rho_{(ce)b}+6\rho_{c(e^2b)}+12\rho_{e((ec)b)}-3\rho_c\rho_e^2\rho_b-3\rho_e\rho_c\rho_e\rho_b-3\rho_e^2\rho_c\rho_b-3\rho_c\rho_e\rho_{eb}-3\rho_e\rho_c\rho_{eb}-3\rho_e^2\rho_{cb}$$
$$-3\rho_c\rho_{e(eb)}-3\rho_e\rho_{c(eb)}-3\rho_e\rho_{e(cb)}-3\rho_{c(e(eb))}-3\rho_{e(c(eb))}-3\rho_{e(e(cb))}](m)=0. \quad (11)$$

We will demonstrate (i) and (ii) by taking $a,b\in A_1$ and $m\in M_\lambda$, with $\lambda\in\{\lambda_1,\lambda_2\}$. If $M=M_{\lambda_i}, i=\overline{1,2}$, we have $\rho_e=\lambda_i id_M$. By replacing $b,c\in A_1$ and $m\in M$ in (11) we obtain:

$$[\lambda\rho_b\rho_c+2\lambda\rho_b\rho_c+2\lambda^2\rho_b\rho_c+2\lambda^2\rho_b\rho_c+2\lambda^2\rho_b\rho_c+\rho_b\rho_c+2\rho_b\rho_c-4\lambda\rho_c\rho_b-8\lambda\rho_b\rho_c-8\lambda^2\rho_c\rho_b-8\lambda^2\rho_b\rho_c$$
$$-8\lambda^2\rho_b\rho_c-4\rho_{cb}-8\rho_{cb}+12\lambda^2\rho_c\rho_b+12\lambda^2\rho_c\rho_b+12\lambda^2\rho_b\rho_c+6\rho_c\rho_b+12\lambda\rho_{cb}+6\rho_{cb}+12\rho_{cb}$$
$$-3\lambda^2\rho_c\rho_b-3\lambda^2\rho_c\rho_b-3\lambda^2\rho_c\rho_b-3\lambda\rho_c\rho_b-3\lambda\rho_c\rho_b-3\lambda^2\rho_{cb}-3\rho_c\rho_b-3\lambda\rho_{cb}-3\lambda\rho_{cb}-3\rho_{cb}-3\rho_{cb}-3\rho_{cb}](m)=0.$$

After reduction we have:

$$(2\lambda^2-5\lambda+3)\rho_b\rho_c+(7\lambda^2-10\lambda+3)\rho_c\rho_b+(-3\lambda^2+6\lambda-3)\rho_{cb}=0. \quad (12)$$

Since $3\lambda^2=3\lambda-1$, then

$$\frac{1}{3}(-9\lambda+7)\rho_b\rho_c+\frac{1}{3}(-9\lambda+2)\rho_c\rho_b+\frac{1}{3}(9\lambda-6)\rho_{cb}=0.$$
$$(-9\lambda+7)\rho_b\rho_c+(-9\lambda+2)\rho_c\rho_b+(9\lambda-6)\rho_{cb}=0 \quad (13)$$

By interchanging b and c in (13) we obtain:

$$(-9\lambda+7)\rho_c\rho_b+(-9\lambda+2)\rho_b\rho_c+(9\lambda-6)\rho_{bc}=0 \quad (14)$$

The difference between (13) and (14) gives:

$$5\rho_b\rho_c-5\rho_c\rho_b=0\Leftrightarrow\rho_b\rho_c=\rho_c\rho_b \quad (15)$$

So (14) becomes

$$-3(2\lambda-1)\rho_c\rho_b+(3\lambda-2)\rho_{bc}=0\Leftrightarrow-9\lambda^3\rho_b\rho_c+(3\lambda^2-1)\rho_{cb}=0.$$

hence

$$\rho_b\rho_c=\frac{3\lambda^2-1}{9\lambda^3}\rho_{cb}$$

Remark 4.10. Taking $a,b\in A_0$ and $m\in M_\lambda$, then replacing them in (11), we have:

$$(9\lambda-2)\rho_b\rho_c+(9\lambda-7)\rho_c\rho_b-9\lambda^2\rho_{cb}=0 \quad (16)$$

Here again, by interchanging b and c and then subtracting, we obtain $\rho_b\rho_c=\rho_c\rho_b\Rightarrow(a,b,m)=0$. On the other hand, here we have: $\rho_b\rho_c=\frac{1}{3\lambda}\rho_{cb}$.

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