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Numerical Solutions of Some Classes of Partial Differential Equations of Fractional Order

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Abstract. This paper explores the solutions of certain fractional partial differential equations using two methods; the first method involves separation of variables, which is a common technique for solving partial differential equations. However, since many equations cannot be separated in this way, the tensor product of Banach spaces method is applied to find the atomic solutions. To solve the resulting ordinary differential equations, the reproducing Kernel Hilbert space method is used to find numerical solutions, which are then used to find the numerical solution of the partial differential equation. The residual errors indicate that this method is effective and powerful. In summary, this paper presents a study on the solutions of certain fractional partial differential equations using two methods and demonstrates the effectiveness of these methods in finding numerical solutions.

2020 Mathematics Subject Classifications: 26A33

Key Words and Phrases: Separation of variables, Tensor product of Banach spaces, Atomic solution, Conformable derivatives

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1. Introduction

The concept of fractional calculus, which extends on classical calculus by including derivatives and integrals of non-integer order, are developed by several researchers such as; Riemann-Liouville derivative [15], Caputo derivative [13], Caputo Fabrizo derivative [14], Atangana-Baleanu derivative [6] and recently conformable fractional derivative by [22]. The conformable fractional derivative has been shown to have some advantages over these other types of fractional derivatives, including better preservation of certain mathematical properties such as the chain rule and the product rule [19].
Partial differential equations with fractional order have a wide range of applications in various fields such as; fluid dynamics in porous media [10], electromagnetic theory [8], quantum mechanics [24], finance [17], heat transfer [26], biology and medicine [25] and

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others. Some studies obtained solutions to several fractional partial differential equations, for instance, the fractional wave-type equations are solved by Fourier Series by [9], [7] used the Sumudu decomposition method (SDM) to find approximate solutions of two-dimensional fractional partial differential equations, [18] used a relatively new method for resolving partial differential equations (PDEs) with fractional derivatives is the Tensor Product of Banach Spaces (TPBS), which is used in this research.
Tensor products of Banach spaces, which are mathematical structures that generalize vector spaces and include norms and metrics, form the foundation of this approach. One of the main challenges of the TPBS method is the construction of a suitable tensor product basis, which requires a careful choice of Banach spaces and fractional derivative operators, another challenge is the accurate approximation of the fractional derivative operator, which can be numerically unstable and require specialized numerical techniques. Moreover, the reproducing Kernel Hilbert space method is used [11] to find the numerical solution for a certain equation to get a complete atomic solution of a fractional equation.
This paper is organized as follows: Section 2 provides some basic definitions of conformable derivative, while the Fractional Fourier series solution for the fractional heat type equation is discussed in Section 3. In Section 4 the Atomic Banach Space Method is used to solve two examples of linear fractional PDEs. Finally, a conclusion is given.

2. Preliminaries

In [22], a definition of the so-called α-conformable fractional derivative was introduced as follows:
Let $\alpha \in (0,1)$, and $f : E \subset (0,\infty) \to \mathbb{R}$. For $x \in E$, then:

$$D^{\alpha}(f)(x) = \lim_{\epsilon \to 0} \frac{x + \epsilon x^{1-\alpha} - f(x)}{\epsilon} \tag{1}$$

If the limit exists, then it is called α-conformable fractional derivatives of f at x. If $f$is α-differentiable on $(0, r)$ for some $r > 0$, and $\lim_{x\to 0^+} D^{\alpha}(f)(x)$ exists, then $D^{\alpha}(f)(0) = \lim_{x\to 0^+} D^{\alpha}(f)(x)$.
For $\alpha \in (0, 1]$, and f, g are differentiable are α-differentiable at some point t, one can easily see that the conformable derivative satisfies:

1. $D^{\alpha}(af + bg) = aD^{\alpha}(f) + bD^{\alpha}(g)$, for all $a, b \in \mathbb{R}$.
2. $D^{\alpha}(\lambda) = 0$, for all constant functions $f(t) = \lambda$.
3. $D^{\alpha}(fg) = fD^{\alpha}(g) + gD^{\alpha}(f)$.
4. $D^{\alpha}(\frac{f}{g}) = \dfrac{gD^{\alpha}(f) - fD^{\alpha}(g)}{g^2}$, $g(t) \neq 0$.

We list here the fractional derivatives of certain functions,

1. $D^\alpha(t^p) = pt^{p-\alpha}$.
2. $D^\alpha\Big(\sin(\frac{1}{\alpha}t^\alpha)\Big) = \cos(\frac{1}{\alpha}t^{\alpha)}$.
3. $D^\alpha\Big(\cos(\frac{1}{\alpha}t^\alpha)\Big) = -\sin(\frac{1}{\alpha}t^\alpha)$.
4. $D^\alpha\Big(e^{\frac{1}{\alpha}t^\alpha}\Big) = e^{\frac{1}{\alpha}t^\alpha}$.

For the geometric meaning of α-conformable fractional derivative we refer the reader to [21]. On putting $\alpha = 1$ in these derivatives, we get the corresponding classical rules for ordinary derivatives. For more on fractional calculus and its applications, we refer to ([1], [2], [3], [5], [4], and [23]).

3. Separation of Variables

Consider the following Heat fractional differential equation:

$$D_t^\alpha u - kD_x^{2\beta}u = D_x^\beta u, \ (0 < \alpha, \beta < 1) \tag{2}$$

which gives the temperature $u(x,t)$ in a body of homogeneous material, k is the thermal diffusivity. As an important application, let us first consider the temperature in a long thin bar or wire of constant cross-section and homogeneous material, which is oriented along the x-axis (Figure 1) and is perfectly insulated laterally, so that the heat flows in the x-direction only. Then u depends only on x and time t.
We shall solve equation (2) for some important types of boundaries and initial conditions. We begin with the case in which the ends $x = 0$ and $x = L$ of the bar are kept at temperature zero, so that we have the boundary conditions: $u(0,t) = 0$ and $u(L,t) = 0$ for all t, and the initial temperature in the bar at time $t = 0$ is $f(x)$, so that we have the initial condition $u(x,0) = f(x)$, where $f(x)$ is given.

Figure 1: Bar under consideration.

3.1. Solution of the Heat Equation (2)

Let

$$u(x,t) = X(x)T(t)$$

Substitute in equation (2) to get

$$X(x)T^\alpha(t) - kX^{2\beta}(x)T(t) = X^\beta(x)T(t)$$

Simplify, then we have

$$X(x)T^{\alpha}(t) = \Big(kX^{2\beta}(x) + X^{\beta}(x)\Big)T(t)$$

From which we obtain

$$\frac{kX^{2\beta}(x) + X^{\beta}(x)}{X(x)} = \frac{T^{\alpha}(t)}{T(t)} = \lambda$$

Since x and t are independent variables, then

$$\frac{kX^{2\beta}(x) + X^{\beta}(x)}{X(x)} = \lambda$$

and

$$\frac{T^{\alpha}(t)}{T(t)} = \lambda$$

Now, we have the following fractional ordinary differential equations

$$kX^{2\beta}(x) + X^{\beta}(x) - \lambda\, X(x) = 0 \tag{3}$$

$$T^{\alpha}(t) - \lambda\, T(t) = 0 \tag{4}$$

First, we deal with equation (3). Let the characteristic equation $\rho = \dfrac{-1 \pm \sqrt{1+4k\lambda}}{2k}$, then, using the results in [1] there are three cases:

Case(i): $1 + 4k\lambda = 0$.
So $\rho = \dfrac{-1}{2k}$, then $X(x) = c_1 e^{\frac{-x^{\beta}}{2k\beta}} + c_2 \dfrac{x^{beta}}{\beta}\, e^{\frac{-x^{\beta}}{2k\beta}}$.
the conditions $u(0,t) = u(L,t) = 0$ imply that $c_1 = c_2 = 0$. Thus, $X(x) = 0$, the trivial solution.

Case(ii): $1 + 4k\lambda = \mu^2 > 0$.
Then $\rho = \dfrac{-1 \pm \mu}{2k}$, and hence $X(x) = c_1 e^{\frac{-1+\mu}{2k}\,\frac{-x^{\beta}}{2k\beta}} + c_2 e^{\frac{-1-\mu}{2k}\,\frac{-x^{\beta}}{2k\beta}}$.
Using the condition $u(0,t) = 0$, we get$c_2 = -c_1$. So,

$$X(x) = c_1\Big(e^{\frac{-1+\mu}{2k}\,\frac{-x^{\beta}}{2k\beta}} - e^{\frac{-1-\mu}{2k}\,\frac{-x^{\beta}}{2k\beta}}\Big)$$

And using the condition $u(L,t) = 0$, we get $c_1 = 0$, thus $X(x) = 0$, therefore $1 + 4k\lambda = \mu^2 > 0$ also gives the trivial solution.

Case(iii): $1 + 4k\lambda = -\mu^2 < 0$.
Then $\rho = \dfrac{-1 \pm \mu\, i}{2k}$, and hence $X(x) = c_1 e^{\frac{-x^{\beta}}{2k\beta}} \cos\left(\cos\left(\frac{\mu\, x^{\beta}}{2k\,\beta}\right)\right) + c_2 e^{\frac{-x^{\beta}}{2k\beta}} \sin\left(\sin\left(\frac{\mu\, x^{\beta}}{2k\,\beta}\right)\right)$.
By using the condition $u(0,t) = 0$, we get $c_1 = 0$. So, $X(x) = c_2 e^{\frac{-x^{\beta}}{2k\beta}} \sin\left(\frac{\mu\, x^{\beta}}{2k\,\beta}\right)$. But

the condition $u(L,t)=0$ implies that $X(x)=c_2 e^{\frac{-x^\beta}{2k\beta}}\sin\left(\frac{\mu x^\beta}{2k\beta}\right)=0$. Since $c_2\neq 0$, then $\sin\left(\frac{\mu x^\beta}{2k\beta}\right)=0$, and then we have

$$\mu = 2kn\pi\frac{\beta}{L^\beta} \tag{5}$$

and

$$X(x)=c_2 e^{\frac{-x^\beta}{2k\beta}}\sin\Big(\sin\Big(\frac{n\pi x^\beta}{L^\beta}\Big)\Big) \tag{6}$$

Now,to solve equation (4), we will rewrite as follows:

$$t^{1-\alpha}T(t)-\lambda T(t)=0,$$

which is a separable differential equation, so

$$T(t)=A_n e^{\Big(\dfrac{-(2kn\pi\frac{\beta}{L^\beta})-1}{4k}\dfrac{t^\alpha}{\alpha}\Big)} \tag{7}$$

Combining equations (6) and (7), we get

$$u(x,t)=\sum_{n=1}^{\infty} b_n\, e^{\Big(\dfrac{-(2kn\pi\frac{\beta}{L^\beta})-1}{4k}\dfrac{t^\alpha}{\alpha}\Big)}\, e^{\frac{-x^\beta}{2k\beta}}\sin\Big(\sin\Big(\frac{n\pi x^\beta}{L^\beta}\Big)\Big) \tag{8}$$

By using $u(x,0)=f(x)$, we get

$$f(x)=\sum_{n=1}^{\infty} b_n\, e^{\frac{-x^\beta}{2k\beta}}\sin\Big(\sin\Big(\frac{n\pi x^\beta}{L^\beta}\Big)\Big).$$

So,

$$e^{\frac{-x^\beta}{2k\beta}}f(x)=\sum_{n=1}^{\infty} b_n\sin\Big(\sin\Big(\frac{n\pi x^\beta}{L^\beta}\Big)\Big) \tag{9}$$

Hence, (9) is the β-Fourier series [12] and then, we can get the sine β-Fourier coefficients as

$$b_n=\frac{2\beta}{L^\beta}\int_0^L f(x)\, e^{\frac{1}{2k}\frac{x^\beta}{2k\beta}}\frac{1}{x^{1-\beta}}\sin\Big(\sin\Big(\frac{n\pi x^\beta}{L^\beta}\Big)\Big) \tag{10}$$

which completes the solution of the differential equation (2).

3.2. Applications

Example 1. *Find the temperature $u(x,t)$ in a laterally insulated copper bar* 80 *cm long if the initial temperature is* $100\sin(\sin(\frac{\pi x}{80}))^\circ C$ *and the ends are kept at* $0\,^\circ C$. *Consider* $k=1.158\frac{cm}{m^2}$, $\alpha=0.5$ *and* $\beta=0.2$.

Solution. *The partial differential equation is given by*

$$D_t^{0.5}u - 1.158\, d_x^{2(0.2)}u = D_x^{0.2}u \tag{11}$$

with the boundary conditions $u(0,t) = u(80,t) = 0$ *and initial condition* $u(x,0) = 100\ \sin(\sin(\frac{\pi x}{80})$.
Therefore, from equation (8) *the solution is given by*

$$u(x,t) = \sum_{n=1}^{\infty} b_n\, e^{-\frac{\left(\frac{0.4632 n\pi}{\sqrt[5]{80}}\right)^2 - 1}{2.316t0.5}\, t^{0.5}}\, e^{\frac{-x^{0.2}}{0.4632}}\ \sin(\sin\left(\frac{n\pi x^{0.2}}{80}\right))\ \sin(\sin\left(\frac{n\pi x^{0.2}}{80}\right)),$$

where, $b_n = \frac{0.4}{\sqrt[5]{80}} \int_0^{80} 100\sin(\sin\left(\frac{\pi x}{80}\right))\, e^{\frac{-x^{0.2}}{0.4632}}\, \frac{dx}{x^{0.8}}$.
To verify our method, we take finite sum of $u(x,t)$*, and then the approximate solution is given by*

$$u_k(x,t) = \sum_{n=1}^{k} b_n\, e^{-\frac{\left(\frac{0.4632 n\pi}{\sqrt[5]{80}}\right)^2 - 1}{2.316t0.5}\, t^{0.5}}\, e^{\frac{-x^{0.2}}{0.4632}}\ \sin(\sin\left(\frac{n\pi x^{0.2}}{80}\right))\ \sin(\sin\left(\frac{n\pi x^{0.2}}{80}\right)).$$

Figure 2a plot the temperature $u_{100}(x,t)$ *in a laterally insulated copper equation* (11)*. In Figure 2b we plot* $u_{100}(x,t)$ *versus* x *at different values of* t, $t \in \{0.1, 0.3, 0.7, 1, 1.5, 2\}$, *where each curve represents a constant time distance function. In Figure 2c we plot* $u_{100}(x,t)$ *versus* t *at different values of* x, $x \in \{0.1, 0.3, 0.7, 1, 1.5, 2\}$ *where each curve represents a constant time function of time.*
In Figure 2d the Residual Error $Res_e(x,t) = \left| D_t^{0.5}u - 1.158\, d_x^{2(0.2)}u - D_x^{0.2}u \right|$ *are plotted to improve the solution of the PDE (11).*

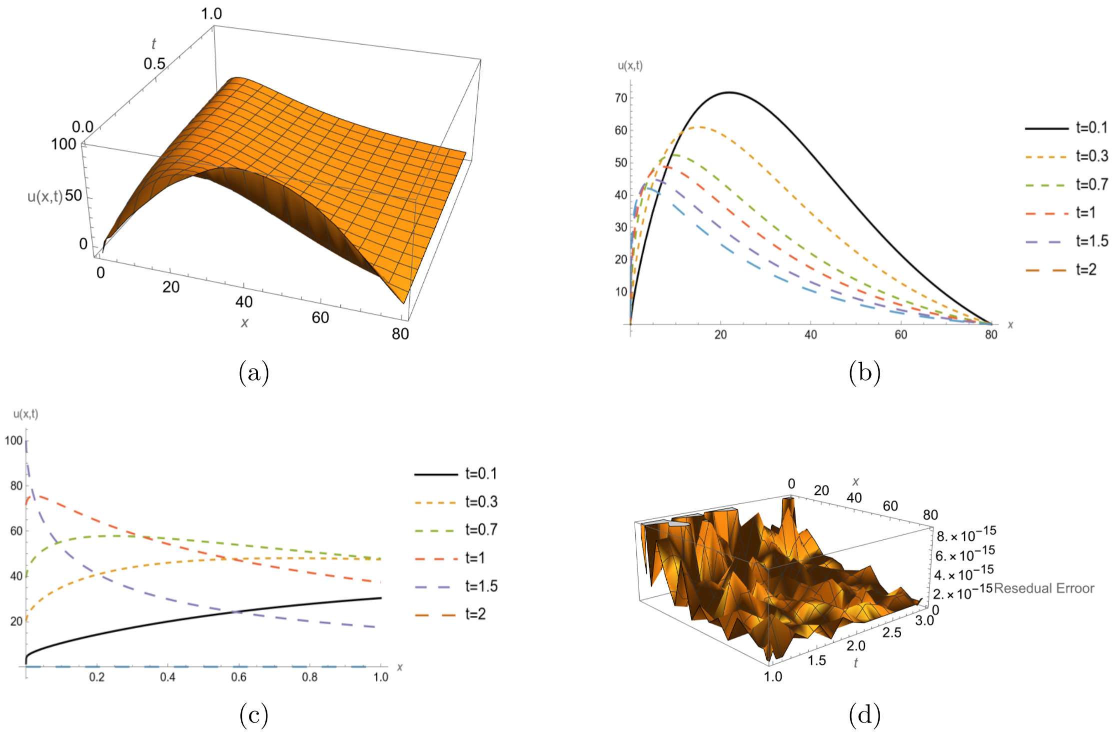


Figure 2: Graphs for equation (11). **(a)** The temperature function $u(x,t)$. **(b)** The temperature function $u(x,t)$ with fixed $t \in \{0.1, 0.3, 0.7, 1, 1.5, 2\}$. **(c)** he temperature function $u(x,t)$ with fixed $x \in \{0.1, 0.3, 0.7, 1, 1.5, 2\}$. **(d)** The Residual Error.

4. Atomic Solution

Since not every linear partial differential (fractional or not) can be solved using the separation of variables, the concept of the atomic solution is established.

Let X and Y be two Banach spaces and X^* be the dual of X. Assume $x \in X$ and $y \in Y$. The operator $T : X^* \to Y$ which is defined by $T(x^*) = x^*(x)y$ is a bounded one-rank linear operator. We write $x \otimes y$ for T. Such operators are called atoms. Atoms are among the main ingredient in the theory of tensor products. Atoms are used in the theory of best approximation in Banach spaces [16]. One of the known results [20] that we need in our paper is that: if the sum of two atoms is an atom, then either the first components are dependent, or the second ones are dependent. For more tensor products of Banach spaces, we refer to [20].

Let us write $D_x^{\alpha} u$ to mean the partial α-derivative of u with respect to x. Further, we write $D_x^{2\alpha} u$ to mean $D_x^{\alpha} u D_x^{\alpha} u$. Similarly for derivatives with respect to y.

4.1. Application 1

In equation (12), the method of separation of variables is not possible though the equation is linear. Hence, we try to find an atomic solution to this equation

$$D_t^{\alpha} u + D_x^{\beta}\, D_x^{\beta} u = D_t^{2\alpha}\, D_x^{\beta} u,\ 0 < \alpha, \beta < 1 \tag{12}$$

$u(0,0=0)$ and $D_t^{\alpha}\, D_x^{\beta}(0,0) = 1$, where by an atomic solution we mean a solution of the form $u(x,t) = X(x)T(t)$.

Procedure

Let $u(x,t) = X(x)T(t)$. Substitute in equation (12) to get

$$X(x)T^{\alpha}(t) + X^{2\beta}(x)T(t) = X(x)T^{2\alpha}(t).$$

This can be written in tensor product form as:

$$X \otimes T^{\alpha} + X^{2\beta} \otimes T = X^{\beta} \otimes T^{2\alpha} \tag{13}$$

Let us consider the following conditions: $T(0) = 1, T^{\alpha}(0) = 1, X(0) = 0$ and $X^{\beta}(0) = 1$. In equation (12), we have the situation: the sum of two atoms is an atom. Hence, we have two cases:

Case (i): $T(t) = T^{\alpha}(t) = T^{2\alpha}(t)$. Using the result in [1], we get

$$T(t) = e^{\frac{t^{\alpha}}{\alpha}} \tag{14}$$

Now we substitute in (13) to get

$$e^{\frac{t^{\alpha}}{\alpha}} \otimes \left(X + X^{2\beta}\right) = e^{\frac{t^{\alpha}}{\alpha}} \otimes X^{\beta}.$$

Hence, $X^{2\beta} - X^{\beta} + X = 0$. Again, using the result in (Al Horani et al., 2020), we get

$$X(x) = c_1 e^{\frac{1}{2}\frac{x^{\beta}}{\beta}}\, \frac{\sqrt{3}}{2}\frac{x^{\beta}}{\beta} + c_2 e^{\frac{1}{2}\frac{x^{\beta}}{\beta}}\, \sin(\sin(\frac{\sqrt{3}}{2}\frac{x^{\beta}}{\beta})).$$

Using the conditions $X(0) = 0$ and $X^{\beta}(0) = 1$, we get

$$X(x) = \frac{2}{\sqrt{3}} e^{\frac{1}{2}\frac{x^{\beta}}{\beta}}\, \sin(\sin(\frac{\sqrt{3}}{2}\frac{x^{\beta}}{\beta})). \tag{15}$$

From (14) and (15), we obtain the atomic solution of (12) as:

$$u(x,t) = \Big(\frac{2}{\sqrt{3}} e^{\frac{1}{2}\frac{x^{\beta}}{\beta}}\, \sin(\sin(\frac{\sqrt{3}}{2}\frac{x^{\beta}}{\beta}))\Big)\, e^{\frac{t^{\alpha}}{\alpha}}. \tag{16}$$

In Figure 3, we plot $u(x,t)$ the solution of equation (12) at different values of α and β where the yellow surface at $\alpha = 1$, $\beta = 1$, the blue surface at $\alpha = 0.9$, $\beta = 0.9$, the green surface at $\alpha = 1$, $\beta = 0.5$, the red surface at $\alpha = 0.5$, $\beta = 1$ and the purple surface at $\alpha = 0.6$, $\beta = 0.6$. $(0 \le t \le 1)$, $(0 \le x \le 1)$.

Case(ii): $X(x) = X^{\beta}(x) = X^{2\beta}(x)$. In this case, equation (13) has no solution. So, there is no atomic solution.

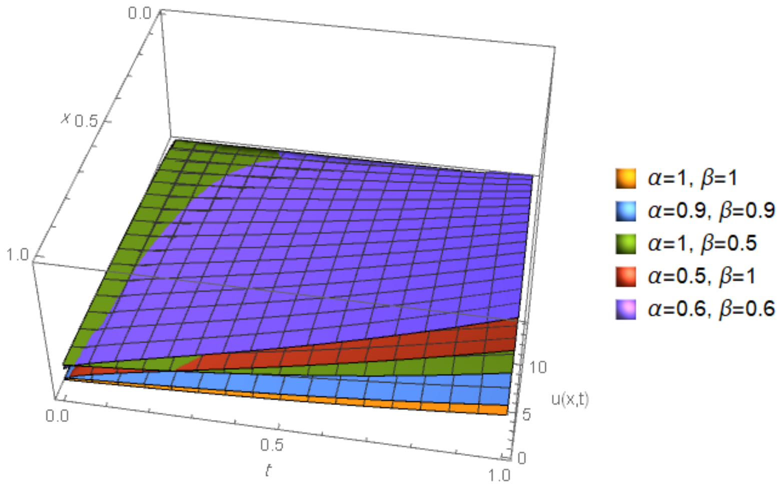


Figure 3: The solution $u(x,t)$ of equation (12).

4.2. Application 2

The solution of

$$D_x^{\frac{3}{2}}u+\sqrt{x}\,D_y^{\frac{3}{2}}u=D_x^{\frac{1}{2}}u\,D_y^{\frac{1}{2}}u \tag{17}$$

$u(0,0)=1, \dfrac{\partial u}{\partial x}(0,0)=1$ and $\dfrac{\partial u}{\partial y}(0,0)=1$, where by an atomic solution we mean a solution of the form $u(x,y)=P(x)Q(y)$.

Procedure

Let $u(x,t)=X(x)T(t)$. Substitute in equation (17) to get

$$P^{\frac{3}{2}}(x)Q(y)+\sqrt{x}P(x)Q^{\frac{3}{2}}(y)=P^{\frac{1}{2}}(x)Q^{\frac{1}{2}}(y).$$

This can be written in tensor product form as:

$$P^{\frac{3}{2}}\otimes Q+\sqrt{x}P\otimes Q^{\frac{3}{2}}=P^{\frac{1}{2}}\otimes Q^{\frac{1}{2}} \tag{18}$$

Let us use the following conditions: $P(0)=1, P'(0)=1, Q(0)=0$ and $Q'(0)=1$.
Hence, we have two cases from equation (18):

Case (i): $P^{\frac{3}{2}}(x)=\sqrt{x}P(x)=P^{\frac{1}{2}}(x)$. Then we get $\sqrt{x}P''=\sqrt{x}P(x)=P^{\frac{1}{2}}(x)$. Using the result in [1], we get

$$P(x)=e^x. \tag{19}$$

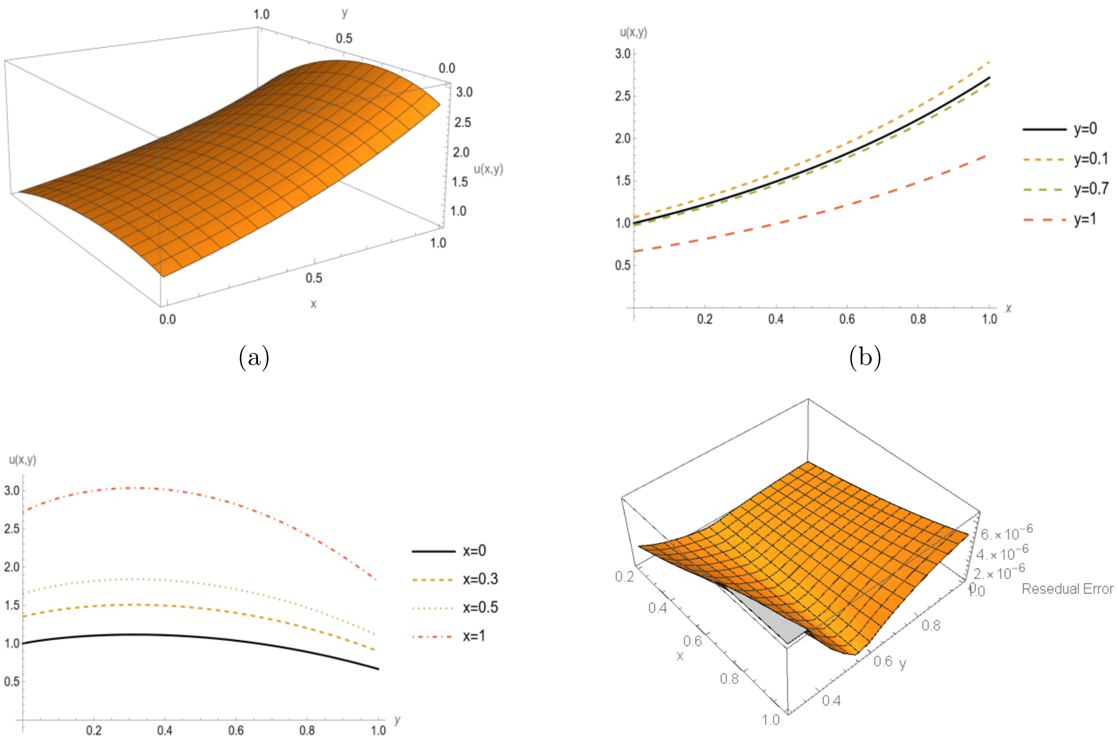
Now we substitute in (18) to get: $\sqrt{x}e^x\otimes\left(Q+Q^{\frac{3}{2}}\right)=P^{\frac{1}{2}}\otimes Q^{\frac{1}{2}}$. Hence,

$$Q+Q^{\frac{3}{2}}-Q^{\frac{1}{2}}=0.$$

Again, using the result in [1] we have

$$\sqrt{y}Q''(y)-\sqrt{y}Q'(y)+Q(y)=0 \tag{20}$$

This equation can be solved numerically by using the reproducing Kernel Hilbert space method [11], where the conditions Q0=1 and Q'(0)=1, are used. From (19) and the numerical solution of (20), we obtain the atomic solution of (17).
In Figure 4(a) we plot the solution $u(x,y)$ of equation (17) where $0 \leq x \leq 1$ and $0 \leq y \leq 1$. In Figure 4(b) we plot $u(x,y)$ versus x at different values of $y, y \in \{0, 0.1, 0.7, 1\}$, where each curve represents a constant time exponential function. In Figure 4(c) we plot $u(x,y)$ versus y at different values of $x, x \in \{0, 0.3, 0.5, 1\}$, where each curve represents a constant time $Q(y)$. In Figure 4(d) we plot the Residual Error for equation (17).
$Res_e(x,y) = \left| D_x^{\frac{3}{2}} u + \sqrt{x} D_y^{\frac{3}{2}} u + -D_x^{\frac{1}{2}} u D_y^{\frac{1}{2}} u \right|$ which proves that the obtained result is very efficient.



(a) (b) (c) (d)

Figure 4: Graphs for equation (17). **(a)** The solution $u(x,y)$. **(b)** The solution $u(x,y)$ when $y \in \{0, 0.1, 0.7, 1\}$. **(c)** The solution $u(x,y)$ when $x \in \{0, 0.3, 0.5, 1\}$. **(d)** The Residual Error.

Case(ii): $Q(y) = Q^{\frac{3}{2}}(y) = Q^{\frac{1}{2}}(Y)$. Has no solution. So, there is no atomic solution.

5. Conclusion

The main goal of this study is to find solutions for linear partial differential equations that involve fractional conformable derivatives. To achieve this, we have used two methods: separation of variables and tensor product method. These methods allow us to obtain both exact and numerical solutions. Our results show that these procedures are efficient, reliable and sufficient. In other words, we have found that the solutions we obtained using these methods are accurate and can be trusted. This is important because it means that these methods can be used to solve similar problems in the future. Overall, our study provides a valuable contribution to the field of partial differential equations, demonstrating the effectiveness of the separation of variables and tensor product method for solving these types of equations.

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