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Inclusion and Neighborhood Properties of Certain Subclasses of Analytic and Multivalent Functions

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Abstract. In the paper we introduce and investigate two new subclasses of multivalently analytic functions defined by Dziok-Srivastava operator. In this paper we obtain the coefficient estimates and the consequent inclusion relationships involving the neighborhoods of the analytic functions.

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1. Introduction

Let $A_p(n)$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{k=n}^{\infty} a_k z^k \quad (p, n \in N = \{1, 2, ...\},\ p < n), \tag{1}$$

which are analytic in the open unit disc $U = \{z : |z| < 1\}$. If $f(z) \in A_p(n)$ is given by (1) and $g(z) \in A_p(n)$ is given by

$$g(z) = z^p + \sum_{k=n}^{\infty} b_k z^k \quad (z \in U),$$

then the Hadamard product (or convolution) $(f * g)(z)$ of $f(z)$ and $g(z)$ is defined by

$$(f * g)(z) = z^p + \sum_{k=n}^{\infty} a_k b_k z^k.$$

For complex parameters $\alpha_1..., \alpha_r$ and $\beta_1, ..., \beta_s$ ($\beta_j \in C \backslash \{0, -1, -2, ...\}; j = 1, ..., s$), we define the generalized hypergeometric function ${}_rF_s(\alpha_1..., \alpha_r; \beta_1, ..., \beta_s; z)$ by

$${}_rF_s(\alpha_1..., \alpha_r; \beta_1, ..., \beta_s; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k......(\alpha_r)_k}{(\beta_1)_k.......(\beta_s)_k} \cdot \frac{z^k}{k!}$$

$$(r \le s + 1; r, s \in N_0 = N \cup \{0\};\ z \in U),$$

where $(\theta)_k$ is the Pochhammer symbol defined, in terms of the Gamma function Γ, by

$$(\theta)_k = \frac{\Gamma(\theta + k)}{\Gamma(\theta)} = \begin{cases} 1 & (k = 0) \\ \theta(\theta + 1)....(\theta + k - 1) & (k \in N). \end{cases}$$

Corresponding to a function $h_p(\alpha_1, ..., \alpha_r; \beta_1, ..., \beta_s; z)$ defined by

$$h_p(\alpha_1, ..., \alpha_r; \beta_1, ..., \beta_s; z) = z^p\ {}_rF_s(\alpha_1, ..., \alpha_r; \beta_1, ..., \beta_s; z),$$

we consider a linear operator $H_p(\alpha_1, ..., \alpha_r; \beta_1, ..., \beta_s) : A_p(n) \to A_p(n)$, defined by the convolution

$$H_p(\alpha_1, ..., \alpha_r; \beta_1, ..., \beta_s) f(z) = h_p(\alpha_1, ..., \alpha_r; \beta_1, ..., \beta_s; z) * f(z).$$

We observe that, for a function $f(z)$ of the form (1), we have

$$H_p(\alpha_1, ..., \alpha_r; \beta_1, ..., \beta_s)f(z) = z^p + \sum_{k=n}^{\infty} \Gamma_k a_k z^k,$$

where

$$\Gamma_k = \frac{(\alpha_1)_{k-p} ......(\alpha_r)_{k-p}}{(\beta_1)_{k-p} ......(\beta_s)_{k-p} \ (k-p)!} \ . \tag{2}$$

For convenience, we write

$$H^p_{r,s} = H_p(\alpha_1, ..., \alpha_r; \beta_1, ...\beta_s).$$

The linear operator $H^p_{r,s}$ was introduced by Dziok and Srivastava [1].

We denote by $T_p(n)$ the subclass of $A_p(n)$ consisting of functions $f(z)$ of the form:

$$f(z) = z^p - \sum_{k=n}^{\infty} a_k z^k \quad (a_k \geq 0) \ . \tag{3}$$

By using the linear operator $H^p_{r,s}$ we introduce a new subclass $S(p, n, q, \lambda, \beta)$ of the class $T_p(n)$, which consists of functions $f(z) \in T_p(n)$ satisfying the inequality:

$$\left| \frac{z(H^p_{r,s}f)^{(1+q)}(z) + \lambda z^2 (H^p_{r,s}f)^{(2+q)}(z)}{\lambda z(H^p_{r,s}f)^{(1+q)}(z) + (1-\lambda)(H^p_{r,s}f)^{(q)}(z)} - (p-q) \right| < \beta \tag{4}$$

$$(z \in U; p \in N; \ q \in N_0; q < k-1; k \geq n; 0 \leq \lambda \leq 1; \beta > 0).$$

Also, let $P(p, n, q, \lambda, \beta)$ denote the subclass of $T_p(n)$ consisting of functions $f(z)$ which satisfy the inequality:

$$\left| (1-\lambda)\frac{(H^p_{r,s}f)^{(q)}(z)}{z^{p-q}} + \lambda \frac{(H^p_{r,s}f)^{(1+q)}(z)}{(p-q)z^{p-q-1}} - (p-q+1)_q \right| < \beta \tag{5}$$

$$(z \in U; p \in N; q \in N_0; q < k-1; k \geq n; \lambda \geq 0; \beta > 0) \ .$$

Now we define two classes related to the classes $S(p, n, q, \lambda, \beta)$ and $P(p, n, q, \lambda, \beta)$.

A function $f(z) \in T_p(n)$ is said to be in the class $S^\gamma(p, n, q, \lambda, \beta)$ if there exists a function $g(z) \in S(p, n, q, \lambda, \beta)$ such that

$$\left| \frac{f(z)}{g(z)} - 1 \right| < \gamma \quad (z \in U; \ \gamma > 0). \tag{6}$$

Analogously, a function $f(z) \in T_p(n)$ is said to be in the class $P^{\gamma}(p,n,q,\lambda,\beta)$ if there exists a function $g(z) \in P(p,n,q,\lambda,\beta)$ such that the inequality (6) holds true.

We note that for suitable chosen parameters the classes were investigated by (among others) Srivastava *et al.* ([2] and [3]). Also, following the earlier investigation by Goodman [4], Ruscheweyh [5], and others we define the $(n,\delta)-$ neighborhood of a function $f(z)$ of the form (3) by

$$N_{n,\delta}(f) = \left\{ g(z) = z^p - \sum_{k=n}^{\infty} b_k z^k \in T_p(n) : \quad \sum_{k=n}^{\infty} k \left| a_k - b_k \right| \le \delta \right\}. \tag{7}$$

In particular, if

$$h(z) = z^p \ (p \in N),$$

we immediately have

$$N_{n,\delta}(h) = \left\{ g(z) = z^p - \sum_{k=n}^{\infty} b_k z^k \in T_p(n) : \quad \sum_{k=n}^{\infty} k \left| b_k \right| \le \delta \right\}. \tag{8}$$

The neighborhoods of function was studied among others by Altintas *et al.* ([6], [7] and [8]), Srivastava *et al.* ([2], [3], [9]and [10]) and Aouf [11] (see also Prajapart and Raina [12]). In this paper we obtain the coefficient estimates and the consequent inclusion relationships involving the neighborhoods of some analytic functions.

2. Coefficient Estimates

In our investigation of the inclusion relations involving $N_{n,\delta}(h)$, we shall require Theorems 1 and 2 below.

Theorem 1. *Let the function $f(z) \in T_p(n)$ be defined by (3). Then $f(z)$ is in the class $S(p,n,q,\lambda,\beta)$ if and only if*

$$\sum_{k=n}^{\infty} (k+\beta-p) C_k a_k \le \beta C_p, \tag{9}$$

where

$$C_k = [1+\lambda(k-q-1)]\left(k-q+1\right)_q \Gamma_k \tag{10}$$

and Γ_k *is given by (2).*

Proof. Let a function $f(z)$ of the form (3) belong to the class $S(p,n,q,\lambda,\beta)$. Then, in view of (3) and (4), we obtain the following inequality:

$$\operatorname{Re}\left\{\frac{z(H_{r,s}^{p}f)^{(1+q)}(z)+\lambda z^2(H_{r,s}^{p}f)^{(2+q)}(z)}{\lambda z(H_{r,s}^{p}f)^{(1+q)}(z)+(1-\lambda)(H_{r,s}^{p}f)^{(q)}(z)}-(p-q)\right\} > -\beta \ (z\in U),$$

or, equivalently,

$$\operatorname{Re}\left\{\frac{-\sum\limits_{k=n}^{\infty}(k-p)C_k a_k z^{k-p}}{C_p-\sum\limits_{k=n}^{\infty}C_k a_k z^{k-p}}\right\} > -\beta \quad (z\in U).$$

Setting $z=r\,(0\le r<1)$ we obtain

$$\frac{\sum\limits_{k=n}^{\infty}(k-p)C_k a_k r^{k-p}}{C_p-\sum\limits_{k=n}^{\infty}C_k a_k r^{k-p}} < \beta \quad (0\le r<1).$$

We observe that the expression in the denominator of the left-hand side of is positive for $r=0$ and also for $0<r<1$. Thus we have

$$\sum_{k=n}^{\infty}(k+\beta-p)C_k a_k r^{k-p} \le \beta C_p,$$

and, by letting $r\to 1^-$ through real values, we obtain the desired assertion of Theorem 1. Conversely, by applying the hypothesis (9) and letting $|z|=1$, we find from (3) that

$$\left|\frac{z(H_{r,s}^{p}f)^{(1+q)}(z)+\lambda z^2(H_{r,s}^{p}f)^{(2+q)}(z)}{\lambda z(H_{r,s}^{p}f)^{(1+q)}(z)+(1-\lambda)(H_{r,s}^{p}f)^{(q)}(z)}-(p-q)\right| = \left|\frac{\sum\limits_{k=n}^{\infty}(k-p)C_k a_k z^{k-p}}{C_p-\sum\limits_{k=n}^{\infty}C_k a_k z^{k-p}}\right|$$

$$\le \frac{\sum_{k=n}^{\infty}(k-p)C_k a_k}{C_p - \sum_{k=n}^{\infty} C_k a_k} \le \beta \frac{C_p - \sum_{k=n}^{\infty} C_k a_k}{C_p - \sum_{k=n}^{\infty} C_k a_k} = \beta.$$

Hence, by the maximum modulus theorem, we have $f(z) \in S(p,n,q,\lambda,\beta)$, which evidently completes the proof of Theorem 1.

Similarly, we can prove the following theorem.

Theorem 2. *Let the function $f(z) \in T_p(n)$ be given by (3). Then $f(z) \in P(p,n,q,\lambda,\beta)$ if and only if*

$$\sum_{k=n}^{\infty} \left[p-q+\lambda\left(k-p\right)\right](k-q+1)_q \Gamma_k a_k \le \beta\left(p-q\right). \tag{11}$$

where Γ_k is given by (2).

Using Theorems 1 and 2 we obtain following two corollaries.

Corollary 1. *If the function $f(z)$ given by (3) belongs to the class $P(p,n,q,\lambda,\beta)$, then*

$$a_k \le \frac{\beta C_p}{(k+\beta-p)C_k}, \quad (k = n, n+1, \ldots),$$

where C_k is given by (10). The result is sharp.

Corollary 2. *If the function $f(z)$ given by (3) belongs to the class $S(p,n,q,\lambda,\beta)$, then*

$$a_k \le \frac{\beta\left(p-q\right)}{\left[p-q+\lambda\left(k-p\right)\right](k-q+1)_q \Gamma_k}, \quad (k = n, n+1, \ldots),$$

where Γ_k is given by (2). The result is sharp.

3. Neighborhoods Properties

Our first inclusion relation $N_{n,\delta}(h)$ is given in the following theorem.

Theorem 3. *If*

$$C_n \leq C_k \quad (k = n, n+1, ...), \tag{12}$$

then

$$S(p,n,q,\lambda,\beta) \subset N_{n,\delta}(h), \tag{13}$$

where C_k is given by (10) and

$$\delta = \frac{n\beta C_p}{(n+\beta-p)C_n} \quad (p \geq \beta).$$

Proof. Let $f(z) \in S(p,n,q,\lambda,\beta)$. Using Theorem 1, by (12), we have

$$(n+\beta-p)C_n \sum_{k=n}^{\infty} a_k \leq \sum_{k=n}^{\infty}(k+\beta-p)C_k a_k \leq \beta C_p,$$

which readily yields

$$\sum_{k=n}^{\infty} a_k \leq \frac{\beta C_p}{(n+\beta-p)C_n}. \tag{14}$$

Making use of (9) again, in conjunction with (12) and (14), we get

$$C_n \sum_{k=n}^{\infty} k a_k \leq \beta C_p + (p-\beta)C_n \sum_{k=n}^{\infty} a_k \leq \beta C_p + \frac{(p-\beta)\beta C_p}{n+\beta-p} = \frac{n\beta C_p}{n+\beta-p}$$

Hence

$$\sum_{k=n}^{\infty} k a_k \leq \frac{n\beta C_p}{(n+\beta-p)C_n} = \delta,$$

which, by means of the definition (8), establishes the inclusion relation (13) asserted by Theorem 1.

Remark 1. *Putting $\lambda = 0, \beta = |b|$, $b \in C \setminus \{0\}$, replacing n by $n+p$ $(p,n \in N)$ and taking $r = 2; s = 1; \alpha_1 = \mu + p$ $(\mu > -p; p \in N); \alpha_2 = \beta_1 = 1$ in Theorem 3, we obtain the result obtained by Raina and Srivastava [9].*

In a similar manner, by applying the assertion (11) of Theorem 2 instead of the assertion (9) of Theorem 1 to functions in the class $P(p,n,q,\lambda,\beta)$ we can prove the following inclusion relationship.

Theorem 4. *If*

$$(n-q+1)_q\Gamma_n \le (k-q+1)_q\Gamma_k \quad (k=n,n+1,...), \tag{15}$$

then

$$P(p,n,q,\lambda,\beta) \subset N_{n,\delta}(h),$$

where

$$\delta = \frac{n\beta\,(p-q)}{\left[p-q+\lambda\,(n-p)\right](n-q+1)_q\Gamma_n} \quad (q+\lambda p \ge p).$$

Theorem 5. *Let $g(z) \in S(p,n,q,\lambda,\beta)$. If C_k given by (10) satisfies (12) and*

$$\gamma > \frac{\delta}{n}\frac{(n+\beta-p)C_n}{(n+\beta-p)C_n-\beta C_p} \quad (\delta > 0), \tag{16}$$

then

$$N_{n,\delta}(g) \subset S^{\gamma}(p,n,q,\lambda,\beta).$$

Proof. Suppose that $f(z) \in N_{n,\delta}(k)$. We find from (8) that

$$\sum_{k=n}^{\infty} k\left|a_k - b_k\right| \le \delta,$$

which readily implies that

$$\sum_{k=n}^{\infty}\left|a_k - b_k\right| \le \frac{\delta}{n}.$$

Next, since $g(z) \in S(p,n,q,\lambda,\beta)$, we have [c.f. equation (14)] that

$$\sum_{k=n}^{\infty} b_k \le \frac{\beta C_p}{(n+\beta-p)C_n},$$

so that

$$\left|\frac{f(z)}{g(z)}-1\right| \le \frac{\sum_{k=n}^{\infty}\left|a_k-b_k\right|}{1-\sum_{k=n}^{\infty} b_k} \le \frac{\delta}{n}\frac{(n+\beta-p)C_n}{(n+\beta-p)C_n-\beta C_p}$$

Thus, by (16) we have. $f(z) \in S^{\gamma}(p,n,q,\lambda,\beta)$. This evidently proves Theorem 5.

The proof of Theorem 6 below is similar to that of Theorem 5 above therefore, we omit the details involved

Theorem 6. *Let $g(z) \in P(p,n,q,\lambda,\beta)$. If the condition (15) holds true and*

$$\gamma > \frac{\delta}{n} \frac{\left[p-q+\lambda\left(n-p\right)\right](n-q+1)_q \Gamma_k}{\left[p-q+\lambda\left(n-p\right)\right](n-q+1)_q \Gamma_n - \beta\left(p-q\right)} \quad (\delta > 0),$$

then

$$N_{n,\delta}(g) \subset P^{\gamma}(p,n,q,\lambda,\beta).$$

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