EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 16, No. 4, 2023, 2247-2285
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Global existence of weak solutions to 3D compressible primitive equations of atmospheric dynamics with degenerate viscosity

Jules Ouya[1], Arouna Ouédraogo[1,*]

[1] *Département de Mathématiques, Laboratoire de Mathématiques, Informatique et Applications (L@MIA), Université Norbert Zongo, Koudougou, Burkina Faso*

Abstract. In this work, we show the existence of global weak solutions to the three-dimensional compressible primitive equations of atmospheric dynamics with degenerate viscosity density-dependent for large initial data. With a pressure law of the form ρ^2, we represent the vertical velocity as a function of the density and the horizontal one which will be important in using Faedo-Galerkin method to obtain the global existence of the approximate solutions. In analogy with the cases in [17–19], we prove that the weak solutions satisfy the basic energy inequality and the Bresch-Desjardins entropy inequality. Based on these estimates and using compactness arguments, we prove the global existence of weak solutions of (1) by vanishing the parameters in our approximate system step by step.

2020 Mathematics Subject Classifications: 35Q30, 35B40, 76D05, 34C35

Key Words and Phrases: Compressible primitive equations, global weak solutions, degenerate viscosity density-dependent

1. Introduction

We are interested in the study of equations of type Primitive Compressible Equations, CPEs. These are the equations governing the motions of the dynamics of the atmosphere. They belong to the class of geophysical fluid dynamics equations. More precisely, in the hierarchy of models, the CPEs are situated between the non-hydrostatic equations and the Saint-Venant equations. The primitive equations are derived from the said hydrostatic approximation in which, the conservation of vertical momentum is replaced by the hydrostatic equation. In general, the CPEs are obtained from the full Navier-Stokes equations for modeling the atmosphere with an anisotropic viscosity tensor. Taking advantage of the difference between the horizontal and vertical dimensions of the atmosphere (10 to 20 kilometers for altitude versus thousands of kilometers for length), we obtain the following hydrostatic model:

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v16i4.4931

Email addresses: ouyajules6@gmail.com (J. Ouya), arounaoued2002@yahoo.fr (A. Ouédraogo)

$$\begin{cases} \partial_t\rho + \operatorname{div}_x(\rho u) + \partial_y(\rho v) = 0, \\ \\ \partial_t(\rho u) + \operatorname{div}_x(\rho u \otimes u) + \partial_y(\rho u v) + \nabla_x p(\rho) + r\rho|u|u \\ \qquad\qquad\qquad\qquad = \operatorname{div}_x\big(2\nu_1 D_x(u)\big) + \partial_y(\nu_2 \partial_y u), \\ \\ \partial_y p(\rho) = -g\rho. \end{cases} \tag{1}$$

Here, $t > 0$, $x = (x_1, x_2)$ and y are respectively the temporal, horizontal and vertical variables. $(x, y) \in \Omega = \Omega_x \times (0, 1)$, with $\Omega_x = \mathbb{T}^2$ the two-dimensional torus.The functions ρ and p represent respectively the density and the pressure of the medium. They each depend on x, y and t. The vector $U = (u = (u_1, u_2), v)$ is the velocity of the flow (where u is the horizontal velocity and v the vertical velocity). $u \otimes u$ is the matrix with components $u_i u_j$, $\operatorname{div}_x = \partial_{x_1} + \partial_{x_2}$ is the divergence operator and D_x is the strain tensor with $D_x(u) = \dfrac{\nabla_x u + (\nabla_x u)^t}{2}$ along the horizontal directions. The term $r\rho|u|u$, with $r > 0$ a positive constant, comes from the quadratic friction source and is useful for the mathematical study (see [3]). ν_1 and ν_2 are the turbulence viscosities in the horizontal and vertical direction respectively. They depend on the density ρ.
In (1), The first equation expresses the conservation of mass, the second, the evolution of the momentum and the last equation, $\partial_y p(\rho) = -g\rho$, is from the hydrostatic approximation with $g > 0$ the free fall acceleration. In order to close the system, we assume that the fluid is Newtonian.
F. Wang *et al.*, in [19], investigated the global existence of weak solutions to CPE (1) in which the pressure is assumed to be $P(\rho) = c^2\rho$. In [11], Liu and Titi have studied problem (1) with $P(\rho) = \rho^\gamma, \gamma > 1$, without buoyancy (no gravity).
The main aim of this paper is to extend the result of [11, 19] for a pressure law of the form $p(\rho) = a\rho^\gamma$, with constant $\gamma > 1$ and $a > 0$ a given positive constant. This is one of the perspectives put forward by Timak Ngom in his thesis defended in 2010 [3]. But we will consider the particular case $\gamma = 2$, in which the density can not longer be written, $\rho(t, x, y) = \xi(t, x)e^{\frac{-g}{c^2}y}$ as in [19]. However, using the hydrostatic equation it can take the following form

$$\rho(t, x, y) = \xi(t, x) + \phi(y), \tag{2}$$

where $\phi(y) = \dfrac{g}{2a}(1 - y)$ and $(t, x) \in [0; +\infty[\times\mathbb{T}^2$, $\xi(t, x) \geq 0$, the new densities.
For simplicity and without lost of generality, we take $a = 1$ and

$$\nu_1(\rho) = \nu_2(\rho) = \xi(t, x) + \phi(y). \tag{3}$$

Then, for $\xi > 0$ the system (1) becomes

$$\begin{cases} \partial_t\left(\xi(1 + \frac{\phi}{\xi})\right) + \operatorname{div}_x\left(\xi(1 + \frac{\phi}{\xi})u\right) + \partial_y\left(\xi(1 + \frac{\phi}{\xi})v\right) = 0, \\ \partial_t\left(\xi(1 + \frac{\phi}{\xi})u\right) + \operatorname{div}_x\left(\xi(1 + \frac{\phi}{\xi})u \otimes u\right) + \partial_y\left(\xi(1 + \frac{\phi}{\xi})uv\right) + r\xi(1 + \frac{\phi}{\xi})|u|u \\ \qquad +\nabla_x\left(\xi(1 + \frac{\phi}{\xi})\right)^2 = \operatorname{div}_x\left(2\xi(1 + \frac{\phi}{\xi})D_x(u)\right) + \partial_y\left(\xi(1 + \frac{\phi}{\xi})\partial_y u\right), \end{cases} \tag{4}$$

where $(x, y) \in \Omega$ and $t \geq 0$.
We state the asymptotic regime

$$u_j = \sum_{i \geq 0} \varepsilon^i u_j^i, \; j = 1, 2, \quad v = \sum_{i \geq 0} \varepsilon^i v^i, \quad \rho = \sum_{i \geq 0} \varepsilon^i \rho^i = \sum_{i \geq 0} \varepsilon^i \xi^i \left(1 + \frac{\phi^i}{\xi^i}\right),$$

where $\varepsilon = \frac{\phi^0}{\xi^0}$. We introduce the asymptotic development at the main order ε^0 in (4) and omit the powers "^{0}" to obtain

$$\begin{cases} \partial_t \xi + \mathrm{div}_x(\xi u) + \partial_y (\xi v) = 0, \\ \partial_t(\xi u) + \mathrm{div}_x(\xi u \otimes u) + \partial_y(\xi u v) + \nabla_x \xi^2 + r\xi |u| u \\ \qquad\qquad\qquad\qquad = \mathrm{div}_x \left(2\xi D_x(u)\right) + \partial_y \left(\xi \partial_y u\right), \\ \partial_y \xi = 0. \end{cases} \tag{5}$$

The boundary conditions on $\partial\Omega$ are expressed as periodic conditions on $\partial\Omega_x$:

$$\begin{cases} v_{/_{y=0}} = v_{/_{y=1}} = 0, \\ \partial_y u_{/_{y=0}} = \partial_y u_{/_{y=1}} = 0. \end{cases} \tag{6}$$

The initial data are written

$$\rho|_{t=0} = \xi_0(x), \quad \rho u|_{t=0} = \xi_0 u_0 = m_0(x, y), \tag{7}$$

with $\xi_0 \geq 0$ a.e in Ω_x a bounded non-negative function, i.e., there exists a positive number M such that

$$0 \leq \xi_0 \leq M < +\infty. \tag{8}$$

Furthermore, we assume that the initial data satisfies:

$$\begin{cases} u_0 = \dfrac{m_0}{\xi_0} \quad \text{if} \quad \xi_0 \neq 0 \quad \text{and} \quad u_0 = 0 \quad \text{elsewhere}, \\ \dfrac{|m_0|^2}{\xi_0} = 0, \quad \text{a.e on} \quad \{(x, y) \in \Omega : \xi_0(x) = 0\} \end{cases} \tag{9}$$

and

$$\begin{cases} \xi_0 \in L^1(\Omega) \cap L^2(\Omega), \quad \nabla_x \sqrt{\xi_0} \in L^2(\Omega), \\ m_0 \in L^{\frac{4}{3}}(\Omega), \quad m_0 u_0 = \dfrac{m_0^2}{\xi_0} \in L^1(\Omega). \end{cases} \tag{10}$$

The factor $1 + \frac{\phi}{\xi}$ is approximately equal to 1, meaning that the difference between the vertical and horizontal components of the density is small. In general, in certain fluid dynamics problems, density can influence fluid velocity. The factor $1 + \frac{\phi}{\xi}$, close to 1, suggests that the variation in vertical density has a negligible effect on fluid velocity, indicating that other forces or factors are predominant in determining velocity. In particular, for a fluid in hydrostatic equilibrium the vertical pressure difference is balanced by the vertical density difference, so the factor $1 + \frac{\phi}{\xi}$ close to 1, indicates that the vertical density variation has a negligible effect on the pressure balance, suggesting that the fluid is mainly balanced by the horizontal density variation.

In the case of the atmosphere, air density can influence fluid velocity. In the atmosphere, vertical density variation due to temperature (and therefore pressure) variations is an important factor contributing to atmospheric circulation, including convective movements and meteorological phenomena such as updrafts and downdrafts. However, in some situations where other forces or factors are predominant, the factor $1 + \frac{\phi}{\xi}$ may be close to 1, suggesting that the variation in vertical density has a negligible effect on fluid velocity. For example, in a situation where atmospheric pressure gradients are very steep, pressure differences may be the main driver of atmospheric motions, and the vertical density variation may have a relatively small effect on fluid velocity. In such a case, the factor $1 + \frac{\phi}{\xi}$ may be close to 1.

Formally, multiplying the momentum equation $(5)_2$ by horizontal velocity u, then integrating by parts on Ω, we obtain the energy equality

$$\frac{d}{dt}\int_\Omega \left(\frac{1}{2}\xi|u|^2 + \xi^2\right) dxdy + r\int_\Omega \xi|u|^3\, dxdy + \int_\Omega \xi\left(2|D_x(u)|^2 + |\partial_y u|^2\right) dxdy = 0.$$

In this paper, motivated by [17] and especially [19], we will investigate the global existence of weak solutions to CPE (1) in which the pressure is assumed to be $P(\rho) = a\rho^\gamma$ with $\gamma > 1$ and $a > 0$ a constant ($a = 1$, $\gamma = 2$ for simplicity). The key issue in our proof is to construct the approximate solutions satisfying lower bound of the density and Bresch-Desjardins entropy. We will first face a new difficulty on how to estimate the vertical velocity v since there is no equation on it. In order to overcome this difficulty, we represent the vertical velocity v as a function of the density ξ and the horizontal velocity u and use the Faedo-Galerkin method to prove the existence of the approximate solutions. What's more, similar to [10, 17–19], we construct the approximate solutions by adding viscosity term in the continuity equation, adding drag, cold pressure, quantum and higher derivative terms in the momentum equation (see (17) for details). Using compactness arguments, we prove the global existence of weak solutions of CPE by vanishing the parameters in our approximate system step by step.

The rest of the paper is organized as follows. In the next section , we present some elementary inequality and compactness theorems which will be used frequently in the whole proof. In section 3, we show the existence of global solutions to the approximate system by using the Faedo-Galerkin method. In section 4, we deduce the Bresch-Desjardins entropy estimates. In section 5, using the standard compactness arguments, we pass to the limits as the parameters tend to zero, step by step.

2. Preliminary and main results

We give here the basic inequalities which are useful for the next, the definition of solution in our context and state the main theorems.

Lemma 1. *(Aubin-Lions, see [16]). Let X_0, X and X_1 be three Banach spaces with $X_0 \subseteq X \subseteq X_1$. Suppose that X_0 is compactly embedded in X and X is continuously embedded in X_1. For $1 \leq p, q \leq +\infty$, let*

$$W = \Big\{ u \in L^p\big([0,T]; X_0\big) : \partial_t u \in L^q\big([0,T]; X_1\big) \Big\}.$$

(I) *If $p < +\infty$, then the embedding of W into $L^p\big([0,T]; X\big)$ is compact;*

(II) *If $p = +\infty$ and $q > 1$, then the embedding of W into $C\big([0,T]; X\big)$ is compact.*

Lemma 2. *(see [19] Lemma 2.3). Let $\Omega \subset \mathbb{R}^n$ be a bounded measurable set. Suppose that $(f_n)_{n\in\mathbb{N}} \subset L^p(\Omega)$, $\|f_n\|_{L^p(\Omega)} \leq C$ with C a positive constant and $f_n \longrightarrow f$ a.e in Ω. Then*

(I) *$f \in L^p(\Omega)$ and $\|f\|_{L^p(\Omega)} \leq C$;*

(II) *$f_n \longrightarrow f$ strongly in $L^{\bar{p}}(\Omega)$, for any $\bar{p} \in [1, p)$.*

Lemma 3. *(see [14], theorem 9.3)(Gagliardo-Nirenberg interpolation inequality)*
For a function $u : \Omega \longrightarrow \mathbb{R}$ defined on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ and for all $1 \leq q, r \leq \infty$ and an integer m, suppose that a real number θ and a natural number j are such that

$$\frac{1}{p} = \frac{j}{n} + \Big(\frac{1}{r} - \frac{m}{n}\Big)\theta + \frac{1-\theta}{q} \quad \text{and} \quad \frac{j}{m} \leq \theta \leq 1.$$

Then we have

$$\|D^j u\|_p \leq C_1 \|D^m u\|_r^\theta \|u\|_q^{1-\theta} + C_2 \|u\|_s, \tag{11}$$

where $s > 0$ is an arbitrary constant. The constants C_1 and C_2 depend upon the domain Ω as well as m, n, j, r, q et θ.

To construct the regular solutions of the approximation scheme, we represent the unknown vertical velocity v as a function of the density ξ and the horizontal velocity u. To this end, by differentiating $(5)_1$ with respect to y, we obtain

$$-\xi \partial_y^2 v = \partial_y \mathrm{div}_x(\xi u). \tag{12}$$

Solving (12) yields

$$v(y) = -\frac{\mathrm{div}_x(\xi \tilde{u}(y))}{\xi} + y \frac{\mathrm{div}_x(\xi \bar{u})}{\xi}. \tag{13}$$

Then

$$\partial_y v(y) = -\frac{\mathrm{div}_x(\xi u)}{\xi} + \frac{\mathrm{div}_x(\xi \bar{u})}{\xi}, \tag{14}$$

where

$$\tilde{u}(y) = \int_0^y u(\tau) d\tau \quad \text{and} \quad \bar{u} = \int_0^1 u(\tau) d\tau.$$

We now give the definition of weak solutions in our context.

Definition 2.1. *A weak solution of the problem* (5)-(7) *is a triplet* (ξ, u, v) *satisfying:*
(1) (7) *holds in* $\mathcal{D}'(\Omega)$;
(2) ξ, u *and* v *belong to the classes*

$$\begin{cases} \xi \in L^\infty\big([0,T]; L^1(\Omega) \cap L^2(\Omega)\big), \quad \sqrt{\xi}u \in L^\infty\big([0,T]; L^2(\Omega)\big), \\ \sqrt{\xi} \in L^\infty\big([0,T]; H^1(\Omega)\big), \quad \xi^{\frac{1}{3}}u \in L^3\big([0,T]; L^3(\Omega)\big), \\ \xi\nabla_x u \in L^2\big([0,T]; L^2(\Omega)\big), \quad \xi(\nabla_x u)^t \in L^2\big([0,T]; L^2(\Omega)\big), \\ \sqrt{\xi}v \in L^2\big([0,T]; L^2(\Omega)\big), \quad \sqrt{\xi}\partial_y v \in L^2\big([0,T]; L^2(\Omega)\big), \\ \sqrt{\xi}\partial_y u \in L^2\left([0,T]; L^2(\Omega)\right), \quad \nabla_x \xi \in L^2\left([0,T]; L^2(\Omega)\right), \\ \nabla_x\sqrt{\xi} \in L^\infty\left([0,T]; L^2(\Omega)\right). \end{cases} \tag{15}$$

(3) *The mass equation is valid in the sense of the distributions and the following equality holds:*

$$\begin{aligned} &\int_\Omega m_0\Phi(0,x,y)dxdy - \int_\Omega \xi u\Phi(T,x,y)dxdy + \int_0^T\int_\Omega (\xi u\partial_t\Phi + \xi u\otimes u : \nabla_x\Phi)\, dxdydt \\ &\quad + \int_0^T\int_\Omega \big(\xi uv\partial_y\Phi + \xi^2 div_x\Phi\big)\, dxdydt - \int_0^T\int_\Omega r\xi|u|u\Phi dxdydt \\ &\qquad - \int_0^T\int_\Omega (2\xi D_x u : \nabla_x\Phi + \xi\partial_y u\partial_y\Phi)\, dxdydt = 0, \end{aligned} \tag{16}$$

for any regular test function $\Phi(t,x,y) \in C_c^\infty([0,T]\times\Omega)$.

We now state our main results.

Theorem 1. *Suppose that the initial data satisfies* (10).
Then, $\forall\ T > 0$, (5)-(7) *has a weak solution* (ξ, u, v) *in the sense of Definition 2.1.*

Theorem 2. *Under an assumption similar to the Theorem 1,* (1)-(3) *and* $(6)-(7)$ *has a weak solution* (ρ, u, v) *in the sense of Definition 2.1, if we replace* (ξ, u, v) *by* (ρ, u, v) *accordingly.*

3. Approximation problem

In this section, we construct in a similar way to [9, 18, 19] and chapter 7 of [4] the Faedo-Galerkin scheme of the approximate system (17) below, and prove the global existence of solutions to this system. Then, we will show the global existence of weak solutions to the problem (1)-(3) $(6)-(7)$ by making the parameters of our approximate system tend to 0 step by step.

3.1. Faedo-Galerkin scheme

In order to prove the global existence of weak solutions for the compressible primitive equations, we consider the following approximate system :

$$\begin{cases} \partial_t \xi + \mathrm{div}_x(\xi u) + \partial_y(\xi v) = \epsilon \Delta_x \xi, \\ \partial_t(\xi u) + \mathrm{div}_x(\xi u \otimes u) + \partial_y(\xi u v) + \nabla_x \xi^2 + r\xi|u|u + r_1 u \\ \quad +\alpha \Delta_x^2 u = \mathrm{div}_x(2\xi D(u)) + \partial_y(\xi \partial_y u) + \epsilon \nabla_x \xi \cdot \nabla_x u \\ \qquad + r_2 \nabla_x \xi^{-\beta} + k_1 \xi \nabla_x \left(\frac{\Delta_x \sqrt{\xi}}{\sqrt{\xi}} \right) + \delta \xi \nabla_x \Delta_x^5 \xi, \\ \partial_y \xi = 0, \end{cases} \tag{17}$$

where $(x,y) \in \Omega$, $t \geq 0$, $\beta \geq 10$ and the vertical velocity v can be expressed as

$$v(y) = -\frac{\mathrm{div}_x(\xi \tilde{u}(y))}{\xi} + y \frac{\mathrm{div}_x(\xi \bar{u})}{\xi}.$$

In order to keep the density bounded, we add the extra terms $\epsilon \nabla_x \xi^{-\beta}$ and $\delta \xi \nabla_x \Delta_x^5 \xi$. This allows us to take $\nabla(\ln \xi)$ as a test function to derive the Bresch-Desjardins entropy. Moreover, the term $r_1 u$ is used to control the density near the vacuum, and $r\xi|u|u$ is used to make sure that $\sqrt{\xi}u$ is strong convergence in $L^2\left([0,T]; L^2(\Omega)\right)$. The term $\frac{\Delta_x \sqrt{\xi}}{\sqrt{\xi}}$ is called Bohm potential which can be interpreted as a quantum potential.

For $T > 0$, We introduce a finite dimensional space $X_n = span\{\psi_1, \ldots, \psi_n\}$, $n \in \mathbb{N}$, where ψ_i is the eigenfunction of the Laplacian: $-\Delta \psi_i = \lambda_i \psi_i$ in Ω with λ_i the eigenvalue for ψ_i. We have the periodic conditions on $\partial\Omega_x$: $\partial_y \psi_i/_{y=0} = \partial_y \psi_i/_{y=1} = 0$.

$\{\psi_i\}$ is an orthonormal basis of $L^2(\Omega)$ which is also an orthogonal basis of $H^2(\Omega)$. Let $(\xi_0, u_0) \in C^\infty(\Omega)$ be some initial data satisfying $\xi_0 \geq \lambda$ for some $\lambda > 0$. We notice that the velocity $u \in C\big([0,T]; X_n\big)$ is given by

$$u(x,y,t) = \sum_{i=1}^{n} \lambda_i(t) \psi_i(x,y), \quad (t,x,y) \in [0,T] \times \Omega$$

for some functions $\lambda_i(t)$, and the norm of u in $C([0,T]; X_n)$ can be written as

$$\|u\|_{C([0,T];X_n(\Omega))} = \sup_{t \in [0,T]} \sum_{i=1}^{n} |\lambda_i(t)|.$$

Since X_n is a finite-dimensional space, all the norms are equivalent on X_n. Hence, u can be bounded in $C\left([0,T]; C^k(\Omega)\right)$ for all $k \in \mathbb{N}$, then there exists a constant $C > 0$ dependent on k such that

$$\|u\|_{C\left([0,T];C^k(\Omega)\right)} \leq C \|u\|_{C([0,T];L^2(\Omega))}. \tag{18}$$

Now, we approach the continuity equation by adding a viscosity term, $\epsilon \Delta_x \xi$:

$$\begin{cases} \partial_t \xi + \mathrm{div}_x(\xi u) + \partial_y(\xi v) = \partial_t \xi + \mathrm{div}_x(\xi \bar{u}) = \epsilon \Delta_x \xi, \\ \xi_0 \in C^\infty(\Omega), \quad \xi_0 \geq \lambda > 0, \end{cases} \tag{19}$$

where $(t, x, y) \in [0, T] \times \Omega$ with $T > 0$.
For any given $U = (u, v)$ in $C([0, T]; X_n)$, by the classical theory of parabolic equations, there exists a classical solution $\xi(t, x) \in C^1\left([0, T]; C^3(\Omega)\right)$ to the approximated system (19). (see [12], [Lemma 3.1, [19]]) for details and proofs.
We show that this solution is continuously dependent on u and by the comparison principle, we also show that it satisfies the following inequality

$$0 < \underline{\xi}(x) e^{-\int_0^T \|\mathrm{div}_x u\|_{L^\infty} dt} \leq \xi(t, x) \leq \overline{\xi}(x) e^{\int_0^T \|\mathrm{div}_x u\|_{L^\infty} dt}, \quad \forall x \in \Omega_x = \mathbb{T}^2, \quad t \geq 0, \tag{20}$$

where

$$\underline{\xi}(x) = \inf_{x \in \mathbb{T}^2} \xi_0(x), \quad \overline{\xi}(x) = \sup_{x \in \mathbb{T}^2} \xi_0(x).$$

Moreover, from (20) and the initial conditions of (19), there exists a positive constant η such that

$$0 < \eta \leq \xi(t, x) \leq \eta^{-1}, \quad \text{for} \quad (t, x) \in [0, T] \times \Omega_x. \tag{21}$$

Thus, using the mass equation, we construct the continuous linear operator
$\mathcal{S} : C^0([0, T]; X_n) \longrightarrow C^0\left([0, T]; C^k(\Omega)\right)$ by $\mathcal{S}(u) = \xi$, and there exists a constant $C_{n,k}$ dependent on $k \geq 1$ and on n such that

$$\|\mathcal{S}(u_1) - \mathcal{S}(u_2)\|_{C\left([0,T]; C^k(\Omega)\right)} \leq C_{n,k} \|u_1 - u_2\|_{C([0,T]; L^2(\Omega))}. \tag{22}$$

3.2. Faedo-Galerkin approximation for the weak formulation of the momentum

We now want to solve the momentum equation $(17)_2$ on the space X_n. For any test function $\Phi \in X_n$, the approximate solution $u_n \in C^0\big([0, T]; X_n\big)$ satisfies:

$$\begin{aligned}
&\int_\Omega \xi_n u_n(T) \Phi dX - \int_\Omega m_0 \Phi dX - \int_0^T \int_\Omega (\xi_n u_n \otimes u_n) : \nabla_x \Phi dX dt \\
&- \int_0^T \int_\Omega \xi_n u_n v_n \partial_y \Phi dX dt + r \int_0^T \int_\Omega \xi_n |u_n| u_n \Phi dX dt + \int_0^T \int_\Omega \nabla_x \xi_n^2 \cdot \Phi dX dt \\
&+ r_1 \int_0^T \int_\Omega u_n \Phi dX dt + \alpha \int_0^T \int_\Omega \Delta_x u_n \cdot \Delta_x \Phi dX dt + \int_0^T \int_\Omega \xi_n \partial_y u_n \partial_y \Phi dX dt \\
&\quad = -2 \int_0^T \int_\Omega \xi_n D_x(u_n) \colon \nabla_x \Phi dX dt + \epsilon \int_0^T \int_\Omega (\nabla_x \xi_n \cdot \nabla_x u_n) \Phi dX dt \\
&\qquad - r_2 \int_0^T \int_\Omega \xi_n^{-\beta} \mathrm{div}_x \Phi dX dt - 2k_1 \int_0^T \int_\Omega \Phi \Delta_x \sqrt{\xi_n} \nabla_x \sqrt{\xi_n} dX dt \\
&\qquad - k_1 \int_0^T \int_\Omega \mathrm{div}_x \Phi \left(\Delta_x \sqrt{\xi_n}\right) \sqrt{\xi_n} dX dt + \delta \int_0^T \int_\Omega \Phi \xi_n \nabla_x \Delta_x^5 \xi_n dX dt,
\end{aligned} \tag{23}$$

where $m_0 = \xi_n(0,x,y)u_n(0,x,y)$ and $dX = dxdy$.
As in [4, 5, 9, 19], to solve (23) we introduce the linear operator

$$\mathcal{M}[\xi] : X_n \longrightarrow X_n^{'}, \quad <\mathcal{M}[\xi]u, v >= \int_\Omega \xi u.v dx, \qquad u, v \in X_n.$$

Using the Lax-Milgram theorem, we show that this operator is invertible and $\mathcal{M}^{-1}$ is Lipschitz continuous. Let us reformulate equation (23) as follows:

$$u_n(t) = \mathcal{M}^{-1}[\mathcal{S}(u_n(t))]\left(\mathcal{M}[\xi_0](u_0) + \int_0^T \mathcal{N}\left(\mathcal{S}(u_n), u_n\right)(s)ds\right), \tag{24}$$

where

$$\begin{aligned} \mathcal{S}(u_n) &= \xi_n, \\ \mathcal{N}\left(\mathcal{S}(u_n), u_n\right)(s) &= \operatorname{div}_x\left(2\xi_n D_x(u_n)\right) + \partial_y\left(\xi_n \partial_y u_n\right) - \operatorname{div}_x(\xi u_n \otimes u_n) - \partial_y(\xi_n u_n v_n) \\ &\quad -\alpha \Delta_x^2 u_n - \epsilon \nabla_x \xi_n \cdot \nabla_x u_n + k_1 \xi \nabla_x\left(\frac{\Delta_x \sqrt{\xi_n}}{\sqrt{\xi_n}}\right) - \nabla_x \xi_n^2 \\ &\quad + r_2 \nabla_x \xi_n^{-\beta} - r_1 u_n - r\xi |u_n| u_n + \delta \xi_n \nabla_x \Delta_x^5 \xi_n. \end{aligned}$$

For more details, we refer the readers to [4–9, 13, 15, 19].
In view of the Lipschitz continuous estimates for $\mathcal{S}$ and $\mathcal{M}^{-1}$, the nonlinear equation (24) can be solved on a short time interval $[0, \tau]$, where $\tau \leq T$, using a fixed point theorem on the Banach space $C\left([0,T]; X_n\right)$. We thus obtain a unique local solution in time (ξ_n, u_n, v_n) to problems (19) and (24).
Next we will extend this obtained local solution to be a global one. Differentiating (23) with respect to time t, taking $\Phi = u_n$ and integrating by parts with respect to x over Ω, we get

$$\begin{aligned} &\int_\Omega \frac{d}{dt}\left(\xi_n \frac{u_n^2}{2}\right) dX + \int_\Omega u_n \cdot \nabla_x \xi^2 dX + r\int_\Omega \xi_n |u_n|^3 dX + \int_\Omega \xi_n |\partial_y u_n|^2 dX \\ &+ r_1 \int_\Omega u_n^2 dX - r_2 \int_\Omega u_n \cdot \nabla_x \xi_n^{-\beta} dX + \int_\Omega 2\xi_n |D_x(u_n)|^2 dX + \alpha \int_\Omega |\Delta u_n|^2 dX \\ &+ \int_\Omega \xi_n |\partial_y u_n|^2 dX + k_1 \int_\Omega \frac{\Delta_x \sqrt{\xi_n}}{\sqrt{\xi_n}} \operatorname{div}_x\left(\xi_n u_n\right) dX + \delta \int_\Omega \operatorname{div}_x\left(\xi_n u_n\right) \Delta_x^5 \xi_n dX = 0. \end{aligned} \tag{25}$$

Furthermore, we estimate the terms of the left hand side in (25) one by one:

$$\begin{aligned} \int_\Omega u_n \cdot \nabla_x \xi_n^2 dX &= -2\int_\Omega \xi_n \operatorname{div}_x(\xi_n u_n) dX \\ &= -2\int_\Omega \xi_n\left(\epsilon \Delta_x \xi_n - \partial_t \xi_n - \partial_y(\xi_n v_n)\right) dX \end{aligned}$$

$$= \frac{d}{dt}\int_\Omega \xi_n^2 dX + 2\epsilon \int_\Omega |\nabla_x \xi_n|^2 \, dX, \tag{26}$$

where we used the fact that $\partial_y \xi_n = 0$ and integration by parts.
Next we deal with the cold pressure and high order derivative of the density terms as follows

$$\begin{aligned} -r_2 \int_\Omega u_n \cdot \nabla_x \xi_n^{-\beta} dX &= -r_2 \frac{\beta}{\beta+1}\int_\Omega \xi_n^{-\beta-1}\left(\epsilon\Delta_x \xi_n - \partial_t \xi_n - \partial_y(\xi_n v_n)\right) dX \\ &= \frac{r_2}{\beta+1}\frac{d}{dt}\int_\Omega \xi_n^{-\beta} dX + \frac{4\epsilon r_2}{\beta}\int_\Omega |\nabla_x \xi_n^{-\frac{\beta}{2}}|^2 dX, \end{aligned} \tag{27}$$

$$\begin{aligned} \delta \int_\Omega \mathrm{div}_x(\xi_n u_n)\Delta_x^5 \xi_n dX &= \delta \int_\Omega \left(\epsilon\Delta_x \xi_n - \partial_t \xi_n - \partial_y(\xi_n v_n)\right)\Delta_x^5 \xi_n dX \\ &= \frac{\delta}{2}\frac{d}{dt}\int_\Omega |\nabla_x \Delta_x^2 \xi_n|^2 dX + \delta\epsilon \int_\Omega |\Delta_x^3 \xi_n|^2 dX. \end{aligned} \tag{28}$$

Finally, we estimate the quantum term

$$\begin{aligned} k_1 \int_\Omega \frac{\Delta_x \sqrt{\xi_n}}{\sqrt{\xi_n}} \mathrm{div}_x(\xi_n u_n) dX &= k_1 \int_\Omega \frac{\Delta_x \sqrt{\xi_n}}{\sqrt{\xi_n}}\left(\epsilon\Delta_x \xi_n - \partial_t \xi_n - \partial_y(\xi_n v_n)\right) dX \\ &= k_1 \frac{d}{dt}\int_\Omega |\nabla_x \sqrt{\xi_n}|^2 dX + \frac{k_1 \epsilon}{2}\int_\Omega \xi_n |\nabla_x^2 \ln \xi_n|^2 dX, \end{aligned} \tag{29}$$

where we used

$$\begin{aligned} 2\xi_n \nabla_x\left(\frac{\Delta_x\sqrt{\xi_n}}{\sqrt{\xi_n}}\right) &= 2\xi_n \nabla_x\left(\mathrm{div}_x\left(\frac{\nabla_x\sqrt{\xi_n}}{\sqrt{\xi_n}}\right) - \nabla_x\sqrt{\xi_n}\cdot\nabla_x\frac{1}{\sqrt{\xi_n}}\right) \\ &= \xi_n \mathrm{div}_x\left(\nabla_x^2 \ln \xi_n\right) + \frac{1}{2}\xi_n \nabla_x\left(\nabla_x \ln \xi_n\right)^2 \\ &= \mathrm{div}_x\left(\xi_n \nabla_x^2 \ln \xi_n\right). \end{aligned}$$

Substituting (26)-(29) in (25), we obtain the following energy equality:

$$\begin{aligned} &\frac{d}{dt}E(\xi_n, u_n) + 2\epsilon\int_\Omega |\nabla_x \xi_n|^2 dX + r_1 \int_\Omega u_n^2 dX + r\int_\Omega \xi_n |u_n|^3 dX + \alpha\int_\Omega |\Delta u_n|^2 dX \\ &+ \int_\Omega 2\xi_n |D_x(u_n)|^2 dX + \frac{4\epsilon r_2}{\beta}\int_\Omega |\nabla_x \xi_n^{\frac{-\beta}{2}}|^2 dX + \int_\Omega \xi_n |\partial_y u_n|^2 dX \\ &+ \frac{k_1\epsilon}{2}\int_\Omega \xi_n |\nabla_x^2 \ln \xi_n|^2 dX + \delta\epsilon\int_\Omega |\Delta_x^3 \xi_n|^2 dX = 0 \end{aligned} \tag{30}$$

on $[0, \tau]$ where,

$$E(\xi_n, u_n) = \int_\Omega \left(\frac{1}{2}\xi_n u_n^2 + \xi_n^2 + \frac{r_2}{\beta+1}\xi_n^{-\beta} + k_1 |\nabla_x \sqrt{\xi_n}|^2 + \frac{\delta}{2}|\nabla_x \Delta_x^2 \xi_n|^2\right) dX,$$

and

$$E_0(\xi_n, u_n) = \int_\Omega \left(\frac{1}{2}\xi_0 u_0^2 + \xi_0^2 + \frac{r_2}{\beta+1}\xi_0^{-\beta} + k_1|\nabla_x \sqrt{\xi_0}|^2 + \frac{\delta}{2}|\nabla_x \Delta_x^2 \xi_0|^2 \right) dX.$$

Thus the energy equality (30) gives

$$\int_0^\tau \|\Delta_x u_n\|^2_{L^2(\Omega)} dt \leq E_0(\xi_n, u_n) < +\infty. \tag{31}$$

From $\dim X_n < \infty$ and (20), the density is bounded with a positive constant, which means that there exists a constant $\eta > 0$ such that

$$0 < \eta \leq \xi_n(t,x) \leq \frac{1}{\eta}, \tag{32}$$

for any $t \in [0,\tau]$ and $(x,y) \in \Omega$. Furthermore, from the basic energy equality (30) and using (32), we also obtain

$$\sup_{t\in[0,\tau]} \int_\Omega \xi_n u_n^2 dX \leq E_0(\xi_n, u_n) \leq C < \infty. \tag{33}$$

From (30), we get

$$\sup_{t\in[0,\tau]} \int_\Omega \xi_n |D_x(u_n)|^2 dX \leq E_0(\xi_n, u_n). \tag{34}$$

As all norms are equivalent on X_n, from (31) -(33), we obtain

$$\sup_{t\in[0,\tau]} \|u_n\|_{L^\infty(\Omega)} \leq C < +\infty, \tag{35}$$

$$\sup_{t\in[0,\tau]} \|\nabla_x u_n\|_{L^\infty(\Omega)} \leq C \quad \text{and} \quad \sup_{t\in[0,\tau]} \|\Delta_x u_n\|_{L^\infty(\Omega)} \leq C. \tag{36}$$

Then, we can extend τ to T by repeating the above argument several times and obtain $u_n \in C([0,T]; X_n)$. In other words, we obtain a global solution (ξ_n, u_n, v_n) of (19) and (24) for any $T > 0$. Moreover, from (30), we have

$$\sup_{t\in[0,T]} \int_\Omega \sqrt{\xi_n} u_n^2 dX \leq E_0(\xi_n, u_n). \tag{37}$$

Also,

$$E(\xi_n, u_n) \leq E_0(\xi_n, u_n), \tag{38}$$

which gives

$$\|\xi_n\|_{L^\infty(0,T;H^5(\Omega))} \leq C(E_0(\xi_n, u_n), \delta) \quad \text{and} \quad \|\xi_n^2\|_{L^\infty(0,T;L^2(\Omega))} \leq C. \tag{39}$$

Using Hölder's inequality we have:

$$\int_\Omega \xi_n \bar{u}_n^2 dxdy = \int_\Omega \xi_n \Big(\int_0^1 u_n(\tau) d\tau \Big)^2 dxdy$$

$$
\begin{aligned}
&\le \int_\Omega \xi_n\Big(\int_0^1 |u_n(\tau)|^2 d\tau\Big)dxdy\\
&\le \int_0^1 \big(\sup_{t\in[0,T]}\int_\Omega \xi_n u_n^2 dxdy\big)d\tau\\
&\le \int_0^1 E_0(\xi_n,u_n)d\tau \le E_0(\xi_n,u_n) \qquad (40)
\end{aligned}
$$

and

$$
\begin{aligned}
\int_\Omega \xi_n\tilde{u}_n^2 dxdy &= \int_\Omega \xi_n\Big(\int_0^y u_n(\tau)d\tau\Big)^2 dxdy\\
&\le \int_\Omega \xi_n\Big(\int_0^y |u_n(\tau)|^2 d\tau\Big)dxdy\\
&\le \int_\Omega \xi_n\Big(\int_0^1 |u_n(\tau)|^2 d\tau\Big)dxdy\\
&\le \int_\Omega \xi_n\Big(\int_0^1 1^2 d\tau\Big)\Big(\int_0^1 u_n^2(\tau)d\tau\Big)dxdz\\
&\le E_0(\xi_n,u_n). \qquad (41)
\end{aligned}
$$

In addition, the basic energy (30) also gives

$$
\int_0^T\int_\Omega \xi_n|\nabla_x^2\ln\xi_n|^2 dXdt \le E_0(\xi_n,u_n) \qquad (42)
$$

and

$$
\sup_{t\in[0,T]}\int_\Omega \delta|\nabla_x\Delta_x^2\xi_n|^2 dX \le E_0(\xi_n,u_n), \qquad (43)
$$

which, with (32), implies that ξ_n is a positive regular function.
We need the following Lemma from Jüngel [9].

Lemma 4. *For any positive regular function $\xi(x,y)$, we have*

$$
\int_\Omega |\nabla^2\sqrt{\xi}|^2 dxdy \le 7\int_\Omega \xi|\nabla^2\ln\xi|^2 dxdy \qquad (44)
$$

and

$$
\int_\Omega |\nabla\xi^{\frac14}|^4 dxdy \le 8\int_\Omega \xi|\nabla^2\ln\xi|^2 dxdy. \qquad (45)
$$

Therefore, the energy equality (30) and the Lemma (4) allow us to state the following lemma.

Lemma 5. *For any smooth positive function $\xi(t,x,y)$, we have the following estimate:*

$$
(k\epsilon)^{\frac12}\|\sqrt{\xi_n}\|_{L^2([0,T];H^2(\Omega))}+(k\epsilon)^{\frac14}\|\nabla_x\xi_n^{\frac14}\|_{L^4([0,T];L^4(\Omega))}\le C, \qquad (46)
$$

for a constant $C>0$ independent on n.

To close this section, we give a summary of the approximate solution (ξ_n, u_n, v_n) as follows.

Proposition 3.1. *Assume that (ξ_n, u_n, v_n) is the solution of (19) and (24) on $[0,T]\times\Omega$ constructed above. Then the solution satisfies the energy inequality*

$$
\begin{aligned}
&E(\xi_n,u_n) + 2\epsilon\int_0^T\int_\Omega |\nabla_x\xi_n|^2 dXdt + r_1\int_0^T\int_\Omega u_n^2 dXdt + r\int_0^T\int_\Omega \xi_n|u_n|^3 dXdt\\
&+\int_0^T\int_\Omega 2\xi_n|D_x(u_n)|^2dX + \alpha\int_0^T\int_\Omega|\Delta_x u_n|^2dXdt + \frac{4\epsilon r_2}{\beta}\int_0^T\int_\Omega|\nabla_x\xi_n^{\frac{-\beta}{2}}|^2dXdt\\
&+\delta\epsilon\int_0^T\int_\Omega|\Delta_x^3\xi_n|^2dXdt + \int_0^T\int_\Omega\xi_n|\partial_y u_n|^2dXdt + \frac{\epsilon k_1}{2}\int_0^T\int_\Omega\xi_n|\nabla_x^2\ln\xi_n|^2dXdt\\
&\quad\le E(\xi_0,u_0),
\end{aligned}
\tag{47}
$$

with

$$
E(\xi_n,u_n) = \int_\Omega\left(\frac{1}{2}\xi_n u_n^2 + \xi_n^2 + \frac{r_2}{\beta+1}\xi_n^{-\beta} + k_1|\nabla_x\sqrt{\xi_n}|^2 + \frac{\delta}{2}|\nabla_x\Delta_x^2\xi_n|^2\right)dX.
$$

In particular, we have the following estimates

$$
\begin{cases}
\sqrt{\xi_n}u_n\in L^\infty\left([0,T];L^2(\Omega)\right),\ \xi_n\in L^2\left([0,T];L^2(\Omega)\right),\ r_2\xi_n^{-\beta}\in L^\infty\left([0,T];L^1(\Omega)\right),\\
\sqrt{\xi_n}D_xu_n\in L^2\left([0,T];L^2(\Omega)\right),\ \sqrt{\alpha}\Delta_xu_n\in L^2\left([0,T];L^2(\Omega)\right),\ \sqrt{k_1}\sqrt{\xi_n}\in L^\infty\big([0,T];H^1(\Omega)\big),\\
\sqrt{\delta}\xi_n\in L^\infty\big([0,T];H^5(\Omega)\big),\ \sqrt{\epsilon}\nabla_x\sqrt{\xi_n}\in L^2\big([0,T];L^2(\Omega)\big);\sqrt{\xi_n}\partial_yu_n\in L^2\big([0,T];L^2(\Omega)\big),\\
\sqrt{\epsilon r_2}\nabla_x\xi_n^{-\frac{\beta}{2}}\in L^2\big([0,T];L^2(\Omega)\big),\ \sqrt{\delta\epsilon}\xi_n\in L^2\big([0,T];H^6(\Omega)\big),\ \sqrt{r_1}u_n\in L^2\big([0,T];L^2(\Omega)\big),\\
\xi_n^{\frac{1}{3}}u_n\in L^3\big([0,T];L^3(\Omega)\big),\ \sqrt{\epsilon k_1}\sqrt{\xi_n}\in L^2\big([0,T];H^2(\Omega)\big),\ \sqrt[4]{k_1\epsilon}\nabla_x\xi_n^{\frac{1}{4}}\in L^4\big([0,T];L^4(\Omega)\big),\\
\xi^2\in L^\infty\big([0,T];L^1(\Omega)\big),\sqrt{\xi_n}\bar{u}_n\in L^\infty\big([0,T];L^2(\Omega)\big),\ \sqrt{\xi_n}\tilde{u}_n\in L^\infty\big([0,T];L^2(\Omega)\big).
\end{cases}
\tag{48}
$$

Now, we want to pass to the limit in (47) when $n\longrightarrow+\infty$. We then state a series of convergence lemmas.

3.3. Convergence results

We fix ϵ, r, r_1, k_1, α, δ, $r_2>0$ and let us first take the limits when $n\longrightarrow+\infty$.

Lemma 6. *(Convergence of ξ_n). For any fixed positive constants ϵ, r_1, k_1, α, δ and r,* *the following estimates hold:*

$$\begin{cases} \|\partial_t\sqrt{\xi_n}\|_{L^\infty\left([0,T];W^{-1,\frac{3}{2}}(\Omega)\right)}+\|\sqrt{\xi_n}\|_{L^2\left([0,T];H^2(\Omega)\right)}\leq K, \\ \|\xi_n\|_{L^\infty\left([0,T];H^5(\Omega)\right)}+\|\partial_t\xi_n\|_{L^\infty\left([0,T];W^{-1,\frac{3}{2}}(\Omega)\right)}\leq K, \\ \|\xi_n^{-\beta}\|_{L^{\frac{5}{3}}\left([0,T];L^{\frac{5}{3}}(\Omega)\right)}\leq K, \\ \|\xi_n^2\|_{L^{\frac{5}{3}}\left([0,T];L^{\frac{5}{3}}(\Omega)\right)}\leq K, \end{cases} \tag{49}$$

where K is independent of n, depends on ϵ, r_2, δ, r_1, initial data and T. Moreover, up to an extracted subsequence ξ_n, when $n \longrightarrow +\infty$ we have:

$$\begin{cases} \sqrt{\xi_n} \longrightarrow \sqrt{\xi} \quad \textit{strongly in} \quad L^2\big([0,T];H^1(\Omega)\big) \quad \textit{and} \quad \sqrt{\xi_n} \longrightarrow \sqrt{\xi} \quad a.e, \\ \xi_n \longrightarrow \xi \quad \textit{strongly in} \quad C\big([0,T];H^5(\Omega)\big) \quad \textit{and} \quad \xi_n \longrightarrow \xi \quad a.e, \\ \xi_n^{-\beta} \longrightarrow \xi^{-\beta} \quad \textit{strongly in} \quad L^1\big([0,T];L^1(\Omega)\big) \quad \textit{and} \quad \xi_n^{-\beta} \longrightarrow \xi^{-\beta} \quad a.e. \end{cases} \tag{50}$$

Proof. The proof of (49) is done in Lemma 2.2 of [18].
It remains to prove (50). Since $\sqrt{\xi_n} \in L^\infty\big([0,T];H^1(\Omega)\big)$, using Sobolev embedding theorem (since $k=1$, $d=3$, $p=2<d$, then $W^{1,2}(\Omega)=H^1(\Omega)\hookrightarrow L^{p^*}(\Omega)$ with $p^*=\frac{dp}{d-p}=6$) we deduce that $\sqrt{\xi_n}\in L^\infty\big([0,T];L^6(\Omega)\big)$. Then, since $\sqrt{\xi_n}\in L^\infty\big([0,T];L^6(\Omega)\big)$ and $\sqrt{\xi_n}\bar{u}_n \in L^\infty\big([0,T];L^2(\Omega)\big)$, the Hölder inequality allows us to conclude that $\xi_n\bar{u}_n = \sqrt{\xi_n}\sqrt{\xi_n}\bar{u}_n \in L^\infty\big([0,T];L^{\frac{3}{2}}(\Omega)\big)$. By (19), we have

$$\begin{aligned} \partial_t\xi_n &= \epsilon\Delta_x\xi_n - div_x(\xi_n u_n) - \partial_y(\xi_n v_n) \\ &= \epsilon\Delta_x\xi_n - \mathrm{div}_x(\sqrt{\xi_n}\sqrt{\xi_n}\bar{u}_n) \in L^\infty\big([0,T];W^{-1,\frac{3}{2}}(\Omega)\big). \end{aligned} \tag{51}$$

Using (51), we get $\xi_n \in L^\infty\big([0,T];H^5(\Omega)\big)\cap L^2\big([0,T];H^6(\Omega)\big)$. From Aubin-Lions Lemma 1, we deduce that $\xi_n \in C\big([0,T];H^5(\Omega)\big)$. So, up to a subsequence, we have

$$\xi_n \longrightarrow \xi \quad \textit{strongly} \quad \textit{in} \quad C\big([0,T];H^5(\Omega)\big) \quad \text{and} \quad \xi_n \longrightarrow \xi \quad \text{a.e.}$$

Next, we claim that $\xi_n^{-\beta}$ is bounded in $L^{\frac{5}{3}}\big([0,T];L^{\frac{5}{3}}(\Omega)\big)$. Indeed, using the fact that $\nabla_x\xi_n^{-\beta/2}$ is bounded in $L^2\big([0,T];L^2(\Omega)\big)$ and the Sobolev embedding theorem, we deduce that $\xi_n^{-\beta}$ is bounded in $L^1\big([0,T];L^3(\Omega)\big)$. Then we apply Hölder inequality to get

$$\|\xi_n^{-\beta}\|_{L^{\frac{5}{3}}\left([0,T];L^{\frac{5}{3}}(\Omega)\right)}\leq \|\xi_n^{-\beta}\|^{\frac{2}{5}}_{L^\infty\left([0,T];L^1(\Omega)\right)}\|\xi_n^{-\beta}\|^{\frac{3}{5}}_{L^1\left([0,T];L^3(\Omega)\right)}\leq K. \tag{52}$$

Now, using the following Sobolev inequality, $\|\xi_n^{-1}\|_{L^\infty(\Omega)}\leq C\big(1+\|\xi_n^{-1}\|_{L^3(\Omega)}\big)^3\big(1+\|\xi_n\|_{H^{k+2}(\Omega)}\big)^2$ for $k\geq\frac{3}{2}$ (see [[1], Lemma 2.1]), and the estimates of density in (48), we get

$$\|\xi_n^{-\beta}\|_{L^\infty(\Omega)}\leq C(r_2,\delta) \quad \text{a.e} \quad \text{on} \quad [0,T]. \tag{53}$$

Thus, we have $\xi_n^{-\beta}$ converges a.e to $\xi^{-\beta}$. Thanks to (52) and Lemma 2, we have $\xi_n^{-\beta} \longrightarrow \xi^{-\beta}$ strongly in $L^1\big([0,T]; L^1(\Omega)\big)$.
Using again the mass equation (19), we have

$$\partial_t\sqrt{\xi_n} = \frac{1}{2}\frac{1}{\sqrt{\xi_n}}\partial_t\xi_n = \frac{1}{2}\frac{1}{\sqrt{\xi_n}}\big(\epsilon\Delta_x\xi_n - \mathrm{div}_x(\sqrt{\xi_n}\sqrt{\xi_n}\bar{u}_n)\big).$$

By using (53) and the fact that $\partial_t\xi_n \in L^\infty\big([0,T]; W^{-1,\frac{3}{2}}\big)$, we have $\partial_t\sqrt{\xi_n} \in L^\infty\big([0,T]; W^{-1,\frac{3}{2}}\big)$. Note that $\sqrt{\xi_n} \in L^2\big([0,T]; H^2(\Omega)\big)$ which is reflexive, so using the Aubin-Lions Lemma, we obtain
$\sqrt{\xi_n} \longrightarrow \sqrt{\xi}$ strongly in $L^2\big([0,T]; H^1(\Omega)\big)$ and $\sqrt{\xi_n} \longrightarrow \sqrt{\xi}$ a.e in $[0,T]\times\Omega$.

Furthermore, as in [5] we show that

$$\begin{cases} \|\nabla_x\xi_n\|_{L^2\left([0,T]\times\Omega\right)} \leq K, \\ \|\xi_n^2\|_{L^{\frac{5}{3}}\left([0,T];L^{\frac{5}{3}}(\Omega)\right)} \leq \|\xi_n^2\|^{\frac{2}{5}}_{L^\infty\left([0,T];L^1(\Omega)\right)}\|\xi_n^2\|^{\frac{3}{5}}_{L^1\left([0,T];L^3(\Omega)\right)} \leq K. \end{cases} \tag{54}$$

Using (49) and the fact that ξ_n^2 converges almost everywhere to ξ^2, we obtain

$$\xi_n^2 \longrightarrow \xi^2 \quad \text{strongly in} \quad L^1\big([0,T]; L^1(\Omega)\big).$$

Lemma 7. *(Convergence of momentum $\xi_n u_n$). Up to an extracted subsequence, we have $\xi_n u_n \longrightarrow \xi u$ strongly in $L^2\big([0,T]; L^2(\Omega)\big)$ and $\xi_n u_n \longrightarrow \xi u$ a.e in $[0,T]\times\Omega$.*

Proof. According to estimates (48), we know that u_n is bounded in $L^2\big([0,T]; L^2(\Omega)\big)$ which is reflexive. So, up to a subsequence, we have

$$u_n \rightharpoonup u \quad \text{weakly in} \quad L^2\big([0,T]; L^2(\Omega)\big).$$

Recalling that $\xi_n \longrightarrow \xi$ strongly in $C\big([0,T]; H^5(\Omega)\big)$, we have

$$\xi_n u_n \longrightarrow \xi u \quad \text{strongly in} \quad L^1\big([0,T]; L^1(\Omega)\big).$$

Moreover, since $\xi_n \in L^\infty\big([0,T]; H^5(\Omega)\big)$, $u_n \in L^2\big([0,T]; H^2(\Omega)\big)$, $\sqrt{\xi_n}\partial_y u_n \in L^2\big([0,T]; L^2(\Omega)\big)$, we deduce

$$\nabla(\xi_n u_n) = u_n.\nabla_x\xi_n + \xi_n\nabla_x u_n + \xi_n\partial_y u_n \in L^2\big([0,T]; L^2(\Omega)\big).$$

This last identity and the fact that $\xi_n u_n \in L^2\big([0,T]; L^2(\Omega)\big)$ give $\xi_n u_n \in L^2\big([0,T]; H^1(\Omega)\big)$. Next, we claim that $\partial_y(\xi_n u_n) \in L^2\big([0,T]; H^{-s}(\Omega)\big)$, for some $s > 0$.
Indeed,

$$\begin{aligned} \partial_t(\xi_n u_n) = &- div_x(\xi_n u_n \otimes u_n) - \partial_y(\xi_n u_n v_n) - \nabla_x\xi_n^2 - r_1 u_n \\ &- r\xi|u_n|u_n - \alpha\Delta^2 u_n + 2\mathrm{div}_x\big(\xi_n D_x(u_n)\big) + \partial_y(\xi_n\partial_y u_n) \end{aligned}$$

$$+\varepsilon\nabla_x\xi_n\cdot\nabla_x u_n + r_2\nabla_x\xi_n^{-\beta} + k_1\xi_n\nabla_x\big(\frac{\Delta_x\sqrt{\xi_n}}{\sqrt{\xi_n}}\big) + \delta\xi_n\nabla_x\Delta_x^5\xi_n. \tag{55}$$

where

$$\begin{aligned}\partial_y(\xi_n u_n v_n) &= \partial_y\Big(\xi_n u_n\big(-\frac{\mathrm{div}_x(\xi_n\tilde{u}_n)}{\xi_n} + y\frac{\mathrm{div}_x(\xi_n\bar{u}_n)}{\xi_n}\big)\Big)\\ &= \partial_y\Big(-\mathrm{div}_x(\xi_n\tilde{u}_n\otimes u_n) + \xi_n\tilde{u}_n.\nabla_x u_n + y\mathrm{div}_x(\xi_n\bar{u}_n\otimes u_n) - y\xi_n\bar{u}_n.\nabla_x u_n\Big).\end{aligned} \tag{56}$$

Based on the energy estimates (48), we get $\partial_y(\xi_n u_n)\in L^2\big([0,T];H^{-5}(\Omega)\big)$. Then, thanks to the Aubin-Lions Lemma 1, $\xi_n u_n$ converges strongly in $L^2\big([0,T];L^2(\Omega)\big)$ to a function $f\in L^2\big([0,T];L^2(\Omega)\big)$.
Also, since $\xi_n u_n \longrightarrow \xi u$ strongly in $L^1\big([0,T];L^1(\Omega)\big)$, we have $\xi_n u_n \longrightarrow \xi u$ strongly in $L^2\big([0,T];L^2(\Omega)\big)$.

We have the following result.

Lemma 8. *(see [19], Lemma 3.5). Up to an extracted subsequence, we have*

$$\begin{aligned}&\sqrt{\xi_n}u_n \longrightarrow \sqrt{\xi}u \text{ strongly in } L^2\big([0,T];L^2(\Omega)\big),\\ &\sqrt{\xi_n}\tilde{u}_n \longrightarrow \sqrt{\xi}\tilde{u} \text{ strongly in } L^2\big([0,T];L^2(\Omega)\big),\\ &\sqrt{\xi_n}\bar{u}_n \longrightarrow \sqrt{\xi}\bar{u} \text{ strongly in } L^2\big([0,T];L^2(\Omega)\big).\end{aligned}$$

By Lemma 8, we conclude that $\sqrt{\xi_n}u_n \longrightarrow \sqrt{\xi}u$, $\sqrt{\xi_n}\tilde{u}_n \longrightarrow \sqrt{\xi}\tilde{u}$ and $\sqrt{\xi_n}\bar{u}_n \longrightarrow \sqrt{\xi}\bar{u}$ almost everywhere in $[0,T]\times\Omega$.

Lemma 9. *(Convergence of $(\partial_y(\xi_n u_n v_n))_n$). Let $\Phi\in C_c^\infty\big([0,T]\times\Omega\big)$ be a regular test function, then*

$$\int_0^T\int_\Omega \partial_y(\xi_n u_n v_n).\Phi dxdydt \longrightarrow \int_0^T\int_\Omega \partial_y(\xi u v).\Phi dxdydt \quad as \quad n\longrightarrow +\infty.$$

Proof. Let $\Phi\in C_c^\infty\big([0,T]\times\Omega\big)$ be a smooth function. From (56), we have

$$\begin{aligned}&\int_0^T\int_\Omega \partial_y(\xi_n u_n v_n).\Phi dxdydt\\ &\quad= -\int_0^T\int_\Omega \xi_n u_n v_n\cdot\partial_y\Phi dxdydt\\ &\quad= -\int_0^T\int_\Omega u_n\big(-div_x(\xi_n\tilde{u}_n) + ydiv_x(\xi_n\bar{u}_n)\big)\cdot\partial_y\Phi dxdydt\\ &\quad= -\int_0^T\int_\Omega \big(-div_x(\xi_n\tilde{u}_n\otimes u_n) + \xi_n\tilde{u}_n\cdot\nabla_x u_n + ydiv_x(\xi_n\bar{u}_n\otimes u_n) - y\xi_n\bar{u}_n\cdot\nabla_x u_n\big).\partial_y\Phi dxdydt\end{aligned}$$

$$
\begin{aligned}
&= -\int_0^T \int_\Omega \xi_n \tilde{u}_n \otimes u_n : \partial_y \nabla_x \Phi dxdydt + \int_0^T \int_\Omega \xi_n \bar{u}_n \otimes u_n : y\partial_y \nabla_x \Phi dxdydt \\
&\quad - \int_0^T \int_\Omega \xi_n \tilde{u}_n \cdot \nabla_x u_n \cdot \partial_y \Phi dxdydt + \int_0^T \int_\Omega \xi_n \bar{u}_n \cdot \nabla_x u_n \cdot y\partial_y \Phi dxdydt. \qquad (57)
\end{aligned}
$$

By direct computation we have $\nabla_x(\sqrt{\xi_n} u_n) = \sqrt{\xi_n}\nabla_x u_n + \nabla_x \sqrt{\xi_n} \otimes u_n$, thus

$$
\begin{aligned}
\int_0^T \int_\Omega \sqrt{\xi_n} \nabla_x u_n : \Phi dxdydt &= \int_0^T \int_\Omega \big(\nabla_x(\sqrt{\xi_n} u_n) - \nabla_x \sqrt{\xi_n} \otimes u_n\big) : \Phi dxdydt \\
&= -\int_0^T \int_\Omega (\sqrt{\xi_n} u_n) \cdot \mathrm{div}_x \Phi dxdydt - \int_0^T \int_\Omega \nabla_x \sqrt{\xi_n} \otimes u_n : \Phi dxdydt \\
&\underset{n\to+\infty}{\longrightarrow} -\int_0^T \int_\Omega (\sqrt{\xi} u) \cdot \mathrm{div}_x \Phi dxdydt - \int_0^T \int_\Omega \nabla_x \sqrt{\xi} \otimes u : \Phi dxdydt \\
&= \int_0^T \int_\Omega \sqrt{\xi} \nabla_x u : \Phi dxdydt.
\end{aligned}
$$

Hence,

$$
\sqrt{\xi_n}\nabla_x u_n \rightharpoonup \sqrt{\xi}\nabla_x u \quad weakly \quad in \quad L^2\big([0,T]; L^2(\Omega)\big).
$$

Combining this last weak convergence with (57), we get, after replacing ξ_n by $\sqrt{\xi_n}\sqrt{\xi_n}$ and using the previous Lemmas,

$$
\int_0^T \int_\Omega \partial_y(\xi_n u_n v_n).\Phi dxdydt \longrightarrow \int_0^T \int_\Omega \partial_y(\xi u v).\Phi dxdydt \quad \text{as} \quad n \longrightarrow +\infty,
$$

where $\xi v = -\mathrm{div}_x(\xi \tilde{u}) + y\mathrm{div}_x(\xi \bar{u})$.

Lemma 10. *(see Subsection 3.3.5 of [19]) (Convergence of nonlinear diffusion terms). For any smooth function $\Phi \in C_c^\infty\big([0,T] \times \Omega\big)$, we have*

$$
\begin{aligned}
\int_0^T \int_\Omega div_x(\xi_n D_x(u_n)) \cdot \Phi dxdydt \quad &\longrightarrow \quad \int_0^T \int_\Omega div_x(\xi D_x(u)) \cdot \Phi dxdydt \quad as \quad n \longrightarrow +\infty, \\
\int_0^T \int_\Omega \xi_n \nabla_x \Delta_x^5 \xi_n \cdot \Phi dxdydt \quad &\longrightarrow \quad \int_0^T \int_\Omega \xi \nabla_x \Delta_x^5 \xi \cdot \Phi dxdydt \quad as \quad n \longrightarrow +\infty, \\
\int_0^T \int_\Omega \xi_n \nabla_x \Big(\frac{\Delta_x \sqrt{\xi_n}}{\sqrt{\xi_n}}\Big) \Phi dxdydt \quad &\longrightarrow \quad \int_0^T \int_\Omega \xi \nabla_x \Big(\frac{\Delta_x \sqrt{\xi}}{\sqrt{\xi}}\Big) \Phi dxdydt \quad as \quad n \longrightarrow +\infty.
\end{aligned}
$$

Due to the compactness above, taking the limits in the approximate system of (19) and (24), then (ξ, u, v) satisfies

$$
\partial_t \xi + \mathrm{div}_x(\xi u) + \partial_y(\xi v) = \epsilon \Delta_x \xi \quad \text{on} \quad [0,T] \times \Omega
$$

and the following identity holds:

$$
\int_\Omega \xi u(T) \Phi dX - \int_\Omega m_0 \Phi dX dt - \int_0^T \int_\Omega (\xi u \otimes u) : \nabla_x \Phi dX + \int_0^T \int_\Omega \xi \partial_y u \partial_y \Phi dX dt
$$

$$
\begin{aligned}
&-\int_0^T\int_\Omega \xi uv\partial_y\Phi dXdt+\int_0^T\int_\Omega \nabla_x\xi^2\cdot\Phi dXdt+r\int_0^T\int_\Omega \xi|u|u\Phi dXdt\\
&+r_1\int_0^T\int_\Omega u\Phi dXdt+\alpha\int_0^T\int_\Omega \Delta_x u\cdot\Delta_x\Phi dXdt\\
&=-\int_0^T\int_\Omega 2\xi D_x(u)\colon\nabla_x\Phi dXdt+\epsilon\int_0^T\int_\Omega\left(\nabla_x\xi\cdot\nabla_x u\right)\Phi dX-r_2\int_0^T\int_\Omega \xi^{-\beta}\mathrm{div}_x\Phi dXdt\\
&-2k_1\int_0^T\int_\Omega \Phi\Delta_x\sqrt{\xi}\nabla_x\sqrt{\xi}dXdt-k_1\int_0^T\int_\Omega \mathrm{div}_x\Phi\left(\Delta_x\sqrt{\xi}\right)\sqrt{\xi}dXdt+\delta\int_0^T\int_\Omega \Phi\xi\nabla_x\Delta_x^5\xi dXdt,
\end{aligned}
\tag{58}
$$

for any smooth function $\Phi\in C_c^\infty\big([0,T]\times\Omega\big)$, where $m_0=\xi(0,x,y)u(0,x,y)$ and $dX=dxdy$.
Using the lower semi-continuity of convex functions, we take the limits in the energy estimate (47) to obtain the following energy inequality:

$$
\begin{aligned}
&\sup_{t\in[0,T]}E(\xi,u)+2\epsilon\int_0^T\int_\Omega|\nabla_x\xi|^2dXdt+r_1\int_0^T\int_\Omega u^2dXdt+r\int_0^T\int_\Omega \xi|u|^3dXdt\\
&+\int_0^T\int_\Omega 2\xi|D_x(u)|^2dXdt+\alpha\int_0^T\int_\Omega|\Delta_x u|^2dXdt+\frac{4\epsilon r_2}{\beta}\int_0^T\int_\Omega|\nabla_x\xi^{\frac{-\beta}{2}}|^2dXdt\\
&+\frac{\epsilon k_1}{2}\int_0^T\int_\Omega \xi|\nabla_x^2\ln\xi|^2dXdt+\delta\epsilon\int_0^T\int_\Omega|\Delta_x^3\xi|^2dXdt+\int_0^T\int_\Omega\xi|\partial_y u|^2dXdt\le E(\xi_0,u_0),
\end{aligned}
\tag{59}
$$

with

$$
E(\xi,u)=\int_\Omega\left(\frac12\xi u^2+\xi^2+\frac{r_2}{\beta+1}\xi^{-\beta}+k|\nabla_x\sqrt{\xi}|^2+\frac{\delta}{2}|\nabla_x\Delta_x^2\xi|^2\right)dX.
$$

Consequently, we obtain the existence of weak solutions of the approximated system (17), through the following.

Proposition 3.2. *For any $T>0$, the following system*

$$
\begin{cases}
\partial_t\xi+div_x(\xi u)+\partial_y\left(\xi v\right)=\epsilon\Delta_x\xi,\\[2mm]
\partial_t(\xi u)+div_x(\xi u\otimes u)+\partial_y(\xi uv)+\nabla_x\xi^2+r\xi|u|u\\
\qquad +r_1u+\alpha\Delta_x^2u=2div_x\left(\xi D(u)\right)+\partial_y\left(\xi\partial_y u\right)+\epsilon\nabla_x\xi\cdot\nabla_x u\\
\qquad +r_2\nabla_x\xi^{-\beta}+k_1\xi\nabla_x\left(\frac{\Delta_x\sqrt{\xi}}{\sqrt{\xi}}\right)+\delta\xi\nabla_x\Delta_x^5\xi,\\[2mm]
\partial_y\xi=0,
\end{cases}
\tag{60}
$$

admits a weak solution (ξ,u,v) with continuous initial data. In particular, the weak solution satisfies the energy inequality (59).

4. Bresch-Desjardins Entropy

In this section, we deal with the Bresch-Desjardins (B-D) estimate for the approximate system of the Proposition 3.2, which was first introduced by D. Bresch and B. Desjardins in [2]. By (48) and (53), we have

$$\begin{cases} \xi(t,x) \geq C(\delta, r_2) > 0 \\ \\ \xi \in L^\infty\big([0,T]; H^5(\Omega)\big) \cap L^2\big([0,T]; H^6(\Omega)\big). \end{cases} \tag{61}$$

As many authors have pointed out, a main difficulty in the proof in this type of model is to pass to the limit in the nonlinear term $\xi u \otimes u$ which requires a strong convergence of $\sqrt{\xi}u$. It seems necessary to obtain additional information on the density ξ. In this perspective, we use a mathematical entropy, called B-D entropy. Thanks to (61), we can take $\dfrac{\nabla \xi}{\xi} = \nabla \ln \xi$ as a test function to derive the B-D entropy from the momentum equation. To this end, we first take the gradient of the mass equation with respect to x, we obtain

$$\partial_t \nabla_x \xi + \nabla_x \left(\xi \mathrm{div}_x(u)\right) + \nabla_x (u \cdot \nabla_x \xi) + \partial_y \nabla_x (\xi v) = \epsilon \nabla_x \Delta_x \xi. \tag{62}$$

Multiplying (62) by 2 and writing the $\nabla_x \xi$ terms as $\xi \nabla_x \ln \xi$, we get

$$\partial_t \left(2\xi \nabla_x \ln \xi\right) + \mathrm{div}_x \left(2\xi \nabla_x^t u\right) + \mathrm{div}_x \left(2\xi \nabla_x \ln \xi \otimes u\right) + 2\partial_y \nabla_x (\xi v) = 2\epsilon \nabla_x \Delta_x \xi. \tag{63}$$

Then, we add (63) with that of the conservation of moments to obtain

$$\begin{aligned} &\partial_t \left(\xi(u + 2\nabla_x \ln \xi)\right) + \mathrm{div}_x \left(\xi(u + 2\nabla_x \ln \xi) \otimes u\right) + \partial_y \left(2\nabla_x(\xi v)\right) + \partial_y(\xi u v) \\ &+ \nabla_x \xi^2 + r_1 u + r\xi|u|u + \alpha \Delta_x^2 u = \mathrm{div}_x \left(2\xi A_x(u)\right) + \partial_y(\xi \partial_y u) \\ &+ r_2 \nabla_x \xi^{-\beta} - \epsilon \nabla_x \xi \cdot \nabla_x u + 2\epsilon \nabla_x \Delta_x \xi + k_1 \xi \nabla_x \left(\frac{\Delta_x \sqrt{\xi}}{\sqrt{\xi}}\right) + \delta \xi \nabla_x \Delta_x^5 \xi, \end{aligned} \tag{64}$$

where $A_x(u) = \dfrac{\nabla_x u - \nabla_x^t u}{2} = \left(\dfrac{\partial_{x_i} u_j - \partial_{x_j} u_i}{2}\right)_{1 \leq i,j \leq 2}$ is the vorticity rate tensor.

Lemma 11. *: Under the assumption* (61)*, we have the following B-D entropy:*

$$\begin{aligned} &\int_\Omega \big(\frac{1}{2}\xi|u + 2\nabla_x \ln \xi|^2 - 2r_1 \ln \xi\big) dxdy + 2\int_0^T \int_\Omega \xi |\partial_y v|^2 dxdydt + r\int_0^T \int_\Omega \xi |u|^3 dxdydt \\ &+ \frac{16 r_2}{\beta} \int_0^T \int_\Omega |\nabla_x \xi^{-\beta/2}|^2 dxdydt + \alpha \int_0^T \int_\Omega |\Delta_x u|^2 dxdydt + r_1 \int_0^T \int_\Omega u^2 dxdydt \\ &+ k_1 \int_0^T \int_\Omega \xi |\nabla_x^2 \ln \xi|^2 dxdydt + 2\delta \int_0^T \int_\Omega |\Delta_x^3 \xi|^2 dxdydt + 2\int_0^T \int_\Omega \xi |A_x(u)|^2 dxdydt \\ &+ 8\int_0^T \int_\Omega \xi |\nabla_x \sqrt{\xi}|^2 dxdydt + 2r_1 \epsilon \int_0^T \int_\Omega \frac{|\nabla_x \xi|^2}{\xi^2} dxdydt + \int_0^T \int_\Omega \xi |\partial_y u|^2 dxdy \end{aligned}$$

$$\leq \int_\Omega \Big(\xi_0 u_0^2 + 10(\nabla_x\sqrt{\xi_0})^2 - 2r_1\ln\xi_0\Big)\, dxdy + E_0 + C + \epsilon C(\delta, r_2) + \sqrt{\alpha}C(\delta, r_2), \tag{65}$$

where C is a generic positive constant depending on the initial data and other constants but independent of ϵ, δ, r_1, r_2, α, and $C(\delta, r_2)$ is a generic positive constant only depending on δ and r_2.

Proof. Multiplying (64) by $u + 2\nabla_x \ln\xi = \psi$ and integrating over Ω, we obtain:

$$\begin{aligned}
&\int_\Omega \partial_t(\xi\psi)\psi dxdy + \int_\Omega \mathrm{div}_x(\xi\psi\otimes u)\psi dxdy + \int_\Omega \partial_y(\xi uv)\psi dxdy + 2\int_\Omega \partial_y\nabla_x(\xi v)\psi dxdy\\
&+\int_\Omega \nabla_x\xi^2\psi dxdy + \int_\Omega \left(r_1 u + r\xi|u|u + \alpha\Delta^2 u\right)\psi dxdy - 2\int_\Omega \mathrm{div}_x(\xi A_x(u))\psi dxdy\\
&-\int_\Omega \partial_y(\xi\partial_y u)\psi dxdy - r_2\int_\Omega \nabla_x\xi^{-\beta}\psi dxdy + \epsilon\int_\Omega (\nabla_x\xi\cdot\nabla_x u)\psi dxdy - 2\epsilon\int_\Omega \big(\nabla_x\Delta_x\xi\big)\psi dxdy\\
&-k_1\int_\Omega \xi\nabla_x\Big(\frac{\Delta_x\sqrt{\xi}}{\sqrt{\xi}}\Big)\psi dxdy - \delta\int_\Omega (\xi\nabla_x\Delta_x^5\xi)\psi dxdy = 0.
\end{aligned} \tag{66}$$

The two first terms of (66) give:

- $$\begin{aligned}
&\int_\Omega \partial_t(\xi\psi)\psi dxdy + \int_\Omega \mathrm{div}_x(\xi\psi\otimes u)\psi dxdy\\
&= \int_\Omega \psi^2\partial_t\xi dxdy + \int_\Omega \xi\partial_t\frac{\psi^2}{2}dxdy + \int_\Omega \big(\xi\psi\ \mathrm{div}_x(u) + u\cdot\nabla_x(\xi\psi)\big)\psi dxdy\\
&= \int_\Omega \psi^2\partial_t\xi dxdy + \int_\Omega \xi\partial_t\frac{\psi^2}{2}dxdy + \int_\Omega \big(\xi\psi^2\mathrm{div}_x(u) + \frac{1}{2}\xi u\nabla_x\psi^2 + u\psi^2\nabla_x\xi\big)dxdy\\
&= \int_\Omega \psi^2\partial_t\xi dxdy + \int_\Omega \xi\partial_t\frac{\psi^2}{2}dxdy + \int_\Omega \Big[\psi^2\big(\xi\mathrm{div}_x(u) + u\nabla_x\xi\big) - \frac{1}{2}\mathrm{div}_x(\xi u)\psi^2\Big]dxdy\\
&= \int_\Omega \psi^2\partial_t\xi dxdy + \frac{1}{2}\int_\Omega \xi\partial_t\psi^2 dxdy + \frac{1}{2}\int_\Omega \psi^2\mathrm{div}_x(\xi u)dxdy.
\end{aligned} \tag{67}$$

Remark that $\partial_y(\xi uv) = \partial_y(\xi v\psi) - 2\partial_y(v\nabla_x\xi)$, then the third term of (66) becomes

- $$\begin{aligned}
\int_\Omega \partial_y(\xi uv)\psi dxdy &= \int_\Omega \partial_y(\xi v\psi)\psi dxdy - 2\int_\Omega \partial_y(v\nabla_x\xi)\psi dxdy\\
&= \int_\Omega \psi^2\partial_y(\xi v)dxdy + \int_\Omega \xi v\partial_y\frac{\psi^2}{2}dxdy - 2\int_\Omega \psi\nabla_x\xi\partial_y(v)dxdy\\
&= \int_\Omega \psi^2\partial_y(\xi v)dxdy - \frac{1}{2}\int_\Omega \psi^2\partial_y(\xi v)dxdy + 2\int_\Omega v\nabla_x\xi\partial_y(\psi)dxdy\\
&= \frac{1}{2}\int_\Omega \psi^2\partial_y(\xi v)dxdy + 2\int_\Omega v\nabla_x\xi\partial_y u dxdy.
\end{aligned} \tag{68}$$

Adding the last term of (67) and the first term in the right hand side of (68), we obtain

$$\frac{1}{2}\int_\Omega \psi^2\mathrm{div}_x(\xi u)dxdy + \frac{1}{2}\int_\Omega \psi^2\partial_y(\xi v)dxdy = \frac{1}{2}\epsilon\int_\Omega \psi^2\Delta_x\xi dxdy - \frac{1}{2}\int_\Omega \psi^2\partial_t\xi dxdy. \tag{69}$$

Thus, the first three terms of (66) give

$$\begin{aligned}&\int_\Omega \left(\partial_t(\xi\psi) + \mathrm{div}_x(\xi\psi\otimes u) + \partial_z(\xi uv)\right)\psi dxdy \\ &= \frac{1}{2}\frac{d}{dt}\int_\Omega \xi\psi^2 dxdy + 2\int_\Omega v\nabla_x\xi\partial_y u dxdy + \frac{1}{2}\epsilon\int_\Omega \psi^2\Delta_x\xi dxdy. \end{aligned} \tag{70}$$

The fourth term of (66) gives

$$\begin{aligned}\bullet \quad 2\int_\Omega \partial_y\nabla_x(\xi v)\psi dxdy &= -\int_\Omega 2\nabla_x(\xi v)\partial_y u dxdy - \int_\Omega 4\nabla_x(\xi v)\partial_y(\nabla_x \ln\xi)dxdy \\ &= \int_\Omega 2\xi v\partial_y \mathrm{div}_x(u)dxdy \\ &= \int_\Omega 2\big(v\partial_y\mathrm{div}_x(\xi u) - v\nabla_x\xi\partial_y u\big)dxdy \\ &= \int_\Omega 2\big(-\xi v\partial_y^2 v - v\nabla_x\xi\partial_y u\big)dxdy \\ &= \int_\Omega 2\big(\xi|\partial_y v|^2 - v\nabla_x\xi\partial_y u\big)dxdy. \end{aligned} \tag{71}$$

Thus, the first four terms of (66) are exactly

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \xi\psi^2 dxdy + \frac{1}{2}\epsilon\int_\Omega \psi^2\Delta_x\xi dxdy + \int_\Omega 2\xi|\partial_y v|^2 dxdy. \tag{72}$$

Moreover, by summing the second term of (72) with $\epsilon\int_\Omega(\nabla_x\xi\cdot\nabla_x u)\psi dxdz$, we obtain, by replacing ψ by $u + 2\nabla_x\ln\xi$,

$$\begin{aligned}&\epsilon\int_\Omega(\nabla_x\xi\cdot\nabla_x u)\psi dxdy + \frac{1}{2}\epsilon\int_\Omega \psi^2\Delta_x\xi dxdy \\ &= -2\epsilon\int_\Omega \nabla_x\xi\cdot\nabla_x^2\ln\xi\cdot u dxdy - 4\epsilon\int_\Omega \nabla_x\xi\cdot\nabla_x^2\ln\xi\cdot\nabla_x\ln\xi dxdy. \end{aligned} \tag{73}$$

The term $-2\int_\Omega \mathrm{div}_x(\xi A_x(u))\psi dxdy$ in (66) can be written:

$$\begin{aligned}\bullet -2\int_\Omega \mathrm{div}_x(\xi A_x(u))\psi dxdy &= -2\int_\Omega \mathrm{div}_x(\xi A_x(u))u dxdy - 2\int_\Omega \mathrm{div}_x(2\xi A_x(u))\nabla_x\ln\xi dxdy \\ &= 2\int_\Omega \xi|A_x(u)|^2 dxdy - 2\int_\Omega \mathrm{div}_x(2\xi A_x(u))\nabla_x\ln\xi dxdy. \end{aligned} \tag{74}$$

The other terms in (66) yield :

$$\bullet \int_\Omega r\xi|u|u\psi dxdy = \int_\Omega r\xi|u|^3 dxdy + 2r\int_\Omega |u|u\nabla_x\xi dxdy. \tag{75}$$

$$\bullet - \int_\Omega \partial_y(\xi\partial_y u)\psi dxdy = - \int_\Omega \partial_y(\xi\partial_y u)udxdy = \int_\Omega \xi|\partial_y u|^2 dxdy, \tag{76}$$

since $\int_\Omega \partial_y(\xi\partial_y u)\nabla_x \ln\xi dxdy = 0$.
Using (26), we get

$$\begin{aligned}\bullet \int_\Omega \nabla_x\xi^2\psi dxdy &= \int_\Omega \nabla_x\xi^2 udxdy + 2\int_\Omega \nabla_x\xi^2\nabla_x \ln\xi dxdy \\ &= \frac{d}{dt}\int_\Omega \xi^2 dxdy + 2\epsilon\int_\Omega |\nabla_x\xi|^2\, dxdy + 2\int_\Omega 2\xi\sqrt{\xi}\nabla_x\sqrt{\xi}.\frac{2\sqrt{\xi}\nabla_x\sqrt{\xi}}{\xi}dxdy \\ &= \frac{d}{dt}\int_\Omega \xi^2 dxdy + 2\epsilon\int_\Omega |\nabla_x\xi|^2\, dxdy + 8\int_\Omega \xi|\nabla_x\sqrt{\xi}|^2 dxdy \end{aligned} \tag{77}$$

On another note,

$$\begin{aligned}\bullet \int_\Omega r_1 u\psi dxdy &= r_1\int_\Omega u^2 dxdy + 2r_1\int_\Omega u\nabla_x \ln\xi dxdy \\ &= r_1\int_\Omega u^2 dxdy + 2r_1\int_\Omega u\frac{\nabla_x\xi}{\xi}dxdy\end{aligned} \tag{78}$$

and

$$\begin{aligned}\bullet \int_\Omega \alpha\Delta^2 u\psi dxdz &= \alpha\int_\Omega \Delta_x^2 u.udxdy + 2\alpha\int_\Omega \Delta_x^2 u\nabla_x \ln\xi dxdy \\ &= \alpha\int_\Omega |\Delta_x u|^2 dxdy + 2\alpha\int_\Omega \Delta_x u\nabla_x\Delta_x \ln\xi dxdy.\end{aligned} \tag{79}$$

Referring to (27), we obtain

$$\begin{aligned}\bullet - r_2\int_\Omega \nabla_x\xi^{-\beta}\psi dxdy &= -r_2\int_\Omega \nabla_x\xi^{-\beta}udxdy - 2r_2\int_\Omega \nabla_x\xi^{-\beta}\nabla_x \ln\xi dxdy \\ &= \frac{r_2}{\beta+1}\frac{d}{dt}\int_\Omega \xi_n^{-\beta}dxdy + \frac{4r_2}{\beta}(\epsilon+4)\int_\Omega |\nabla_x\xi_n^{-\frac{\beta}{2}}|^2 dxdy.\end{aligned} \tag{80}$$

As in (28), we get

$$\begin{aligned}\bullet - \delta\int_\Omega (\xi\nabla_x\Delta_x^5\xi)\psi dxdy &= \delta\int_\Omega \mathrm{div}_x(\xi u)\Delta_x^5\xi dxdy + 2\delta\int_\Omega \Delta_x\xi\Delta_x^5\xi dxdy \\ &= \frac{\delta}{2}\frac{d}{dt}\int_\Omega |\nabla_x\Delta_x^2\xi|^2 dxdy + \delta\epsilon\int_\Omega |\Delta_x^3\xi|^2 dxdy + 2\delta\int_\Omega |\Delta_x^3\xi|^2 dxdy.\end{aligned} \tag{81}$$

Similarly, by exploiting (29), we finally obtain

$$\bullet - k_1\int_\Omega \xi\nabla_x\big(\frac{\Delta_x\sqrt{\xi}}{\sqrt{\xi}}\big)\psi dxdy \quad = \quad k_1\int_\Omega \mathrm{div}_x(\xi u)\big(\frac{\Delta_x\sqrt{\xi}}{\sqrt{\xi}}\big)dxdy + 2k_1\int_\Omega \big(\frac{\Delta_x\sqrt{\xi}}{\sqrt{\xi}}\big)\Delta_x\xi dxdy$$

$$= k_1 \frac{d}{dt}\int_\Omega |\nabla_x \sqrt{\xi}|^2 + \frac{k_1\epsilon}{2}\int_\Omega \xi|\nabla_x^2 \ln\xi|^2 dxdy + k_1\int_\Omega \xi|\nabla_x^2\ln\xi|^2 dxdy. \quad (82)$$

Finally, putting the above results together, the equation (66) is written:

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \big(\frac{1}{2}\xi\psi^2 + \xi^2 + \frac{r_2}{\beta+1}\xi^{-\beta} + k_1|\nabla_x\sqrt{\xi}|^2 + \frac{\delta}{2}|\nabla_x\Delta_x^2\xi|^2\big)dxdy \\
&+ 2\int_\Omega \xi|\partial_y v|^2 dxdy + r\int_\Omega \xi|u|^3 dxdy + \int_\Omega \xi|\partial_y u|^2 dxdy + r_1\int_\Omega u^2 dxdy \\
&+ \frac{4r_2}{\beta}(\epsilon+4)\int_\Omega |\nabla_x\xi^{-\beta/2}|^2 dxdy + \alpha\int_\Omega |\Delta_x u|^2 dxdy + \int_\Omega 2\xi|A_x(u)|^2 dxdy \\
&\quad + \frac{k_1(\epsilon+2)}{2}\int_\Omega \xi|\nabla_x^2\ln\xi|^2 + \delta(\epsilon+2)\int_\Omega |\Delta_x^3\xi|^2 dxdy + \int_\Omega 2\epsilon|\nabla_x\xi|^2 dxdy + 8\int_\Omega \xi|\nabla_x\sqrt{\xi}|^2 dxdy \\
&\quad = 2\int_\Omega \mathrm{div}_x(2\xi A_x(u))\nabla_x\ln\xi dxdy - 2\alpha\int_\Omega \Delta_x u\nabla_x\Delta_x\ln\xi dxdy + 2\epsilon\int_\Omega \nabla_x\xi\cdot\nabla_x^2\ln\xi\cdot u dxdy \\
&\quad\ + 4\epsilon\int_\Omega \nabla_x\xi\cdot\nabla_x^2\ln\xi\cdot\nabla_x\ln\xi dxdy + 2\epsilon\int_\Omega \nabla_x\Delta_x\xi\psi dxdy \\
&\quad\quad - 2r\int_\Omega |u|u\nabla_x\xi dxdy - 2r_1\int_\Omega u\frac{\nabla_x\xi}{\xi}dxdy = \sum_{i=1}^{7} I_i. \quad (83)
\end{aligned}$$

We have $I_1 = 0$ due to the periodic conditions on Ω_x, that is,

$$\begin{aligned}
2\int_\Omega \mathrm{div}_x(2\xi A_x(u))\nabla_x\ln\xi dxdy &= \int_\Omega \partial_i\big(\xi(\partial_i u_j - \partial_j u_i)\big)\frac{\partial_j\xi}{\xi}dxdy \\
&= \int_\Omega \big(\frac{\partial_i\xi\partial_j\xi}{\xi}(\partial_i u_j - \partial_j) + (\partial_i\partial_i u_j - \partial_i\partial_j u_i)\partial_j\xi\big)dxdy \\
&= \int_\Omega (\partial_i\partial_i u_j\partial_j\xi - \partial_i\partial_j u_i\partial_j\xi)dxdy \\
&= \int_\Omega (\partial_i\partial_i u_j\partial_j\xi - \partial_j\partial_j u_i\partial_i\xi)dxdy = 0. \quad (84)
\end{aligned}$$

For I_7, we have,

$$\begin{aligned}
I_7 &= -2r_1\int_\Omega u\frac{\nabla_x\xi}{\xi}dxdy \\
&= -2r_1\int_\Omega \frac{\mathrm{div}_x(\xi u)}{\xi}dxdy \\
&= 2r_1\int_\Omega \frac{\partial_t\xi + \partial_y(\xi v) - \varepsilon\Delta_x\xi}{\xi}dxdy \\
&= 2r_1\int_\Omega \partial_t\ln\xi dxdy - 2r_1\epsilon\int_\Omega \frac{\Delta_x\xi}{\xi}dxdy \\
&= 2r_1\frac{d}{dt}\int_\Omega \ln\xi dxdy - 2r_1\epsilon\int_\Omega \frac{|\nabla_x\xi|^2}{\xi^2}dxdy. \quad (85)
\end{aligned}$$

Substituting (84) and (85) into (83) and integrating with respect to time t, we obtain

$$\begin{aligned}
&\int_\Omega \big(\frac{1}{2}\xi|\psi|^2 - 2r_1 \ln\xi\big)dxdy + 2\int_0^T\int_\Omega \xi|\partial_y v|^2 dxdydt + r\int_0^T\int_\Omega \xi|u|^3 dxdydt \\
&+ \frac{16r_2}{\beta}\int_0^T\int_\Omega |\nabla_x \xi^{-\beta/2}|^2 dxdydt + \alpha\int_0^T\int_\Omega |\Delta_x u|^2 dxdydt + r_1\int_0^T\int_\Omega u^2 dxdydt \\
&+ k_1\int_0^T\int_\Omega \xi|\nabla_x^2 \ln\xi|^2 + 2\delta\int_0^T\int_\Omega |\Delta_x^3\xi|^2 dxdydt + 2\int_0^T\int_\Omega \xi|A_x(u)|^2 dxdydt \\
&+ 8\int_0^T\int_\Omega \xi|\nabla_x\sqrt{\xi}|^2 dxdydt + 2r_1\epsilon\int_0^T\int_\Omega \frac{|\nabla_x\xi|^2}{\xi^2}dxdydt + \int_0^T\int_\Omega \xi|\partial_y u|^2 dxdydt \\
&\qquad \le \int_\Omega \big(\xi_0 u_0^2 + 10(\nabla_x\sqrt{\xi_0})^2 - 2r_1\ln\xi_0\big)dxdydt + \int_0^T \big(\sum_{i=2}^6 I_i\big)dt + E_0, \qquad (86)
\end{aligned}$$

where we used the energy inequality (59).
Now, we control the other terms I_i in (86).

$$\begin{aligned}
\bullet \int_0^T I_2 dt &= -2\alpha\int_0^T\int_\Omega \Delta_x u\cdot\nabla_x\Delta_x \ln\xi dxdydt \\
&= -2\alpha\int_0^T\int_\Omega \Delta_x u\cdot\Big(\frac{\nabla_x\Delta_x\xi}{\xi} - \frac{\Delta_x\xi\nabla_x\xi}{\xi^2} - 2\frac{(\nabla_x\xi.\nabla_x\xi)\nabla_x\xi}{\xi^2} + 2\frac{|\nabla_x\xi|^2\nabla_x\xi}{\xi^3}\Big)dxdydt \\
&\le C\alpha\int_0^T\int_\Omega |\Delta_x u|\Big(\xi^{-1}|\nabla_x^3\xi| + \xi^{-2}|\nabla_x\xi||\nabla_x^2\xi| + \xi^{-3}|\nabla_x\xi|^3\Big)dxdzdt \\
&\le C\sqrt{\alpha}\|\sqrt{\alpha}\Delta_x u\|_{L^2(L^2(\Omega))}\Big[\|\xi^{-1}\|_{L^\infty(L^\infty(\Omega))}\|\nabla_x^3\xi\|_{L^2(L^2(\Omega))} \\
&+ \|\xi^{-1}\|^2_{L^\infty(L^\infty(\Omega))}\|\nabla_x\xi\|_{L^\infty(L^\infty(\Omega))}\|\nabla_x^2\xi\|_{L^2(L^2(\Omega))} + \|\xi^{-1}\|^3_{L^\infty(L^\infty(\Omega))}\|\nabla_x\xi\|^3_{L^6(L^6(\Omega))}\Big] \\
&\le C\sqrt{\alpha}\|\sqrt{\alpha}\Delta_x u\|_{L^2(L^2(\Omega))}\Big[\|\xi^{-1}\|^3_{L^\infty(L^\infty(\Omega))}\|\nabla_x^3\xi\|_{L^2(L^2(\Omega))} + \|\xi^{-1}\|_{L^\infty(L^\infty(\Omega))}\Big] \\
&\le C(\delta, r_2)\sqrt{\alpha}. \qquad (87)
\end{aligned}$$

$$\begin{aligned}
\bullet \int_0^T I_3 dt &= 2\epsilon\int_0^T\int_\Omega \nabla_x\xi\cdot\nabla_x^2\ln\xi\cdot u dxdzdt \\
&= -\epsilon\int_0^T\int_\Omega |\frac{\nabla_x\xi}{\xi}|^2 div_x(\xi u)dxdzdt \\
&= -\epsilon\int_0^T\int_\Omega \big(\frac{|\nabla_x\xi|^2}{\xi}\nabla_x\cdot u + \frac{|\nabla_x\xi|^2}{\xi^2}u\nabla_x\xi\big)dxdzdt \\
&\le \epsilon\int_0^T\int_\Omega \frac{(\nabla_x\xi)^2}{\xi^{\frac{3}{2}}}\sqrt{\xi}\nabla_x\cdot u + \frac{(\nabla_x\xi)^2}{\xi^{\frac{5}{2}}}\sqrt{\xi}u\cdot\nabla_x\xi dxdzdt \\
&\le \epsilon\|\frac{1}{\xi}\|^{\frac{3}{2}}_{L^\infty(L^\infty(\Omega))}\|\nabla_x\xi\|_{L^\infty(L^\infty(\Omega))}\|\nabla_x\xi\|_{L^2(L^2(\Omega))}\|\sqrt{\xi}\nabla_x u\|_{L^2(L^2(\Omega))}
\end{aligned}$$

$$
\begin{aligned}
&+\epsilon\|\frac{1}{\xi}\|^{\frac{5}{2}}_{L^\infty(L^\infty(\Omega))}\|\nabla_x\xi\|^2_{L^\infty(L^\infty(\Omega))}\|\nabla_x\xi\|_{L^2(L^2(\Omega))}\|\sqrt{\xi}u\|_{L^2(L^2(\Omega))}\\
&\le \epsilon C(\delta,r_2)\big(\|\sqrt{\xi}\nabla_x u\|_{L^2(L^2(\Omega))}+1\big)\\
&\le \frac{1}{2}\|\sqrt{\xi}A_x(u)\|^2_{L^2(L^2(\Omega))}+\epsilon C(\delta,r_2), \qquad (88)
\end{aligned}
$$

where we used,

$$\|\sqrt{\xi}A_x(u)\|^2_{L^2(L^2(\Omega))}+\|\sqrt{\xi}D_x(u)\|^2_{L^2(L^2(\Omega))}=\|\sqrt{\xi}\nabla_x u\|^2_{L^2(L^2(\Omega))}.$$

In addition, we also have

$$
\begin{aligned}
\bullet \int_0^T I_4dt &= 4\epsilon\int_0^T\int_\Omega \nabla_x\xi\cdot\nabla_x^2\ln\xi\cdot\nabla_x\ln\xi dxdydt\\
&= -2\epsilon\int_0^T\int_\Omega \Delta_x\xi\cdot|\nabla_x\ln\xi|^2dxdydt\\
&\le 2\epsilon\|\frac{1}{\xi}\|^2_{L^\infty(L^\infty(\Omega))}\|\nabla_x\xi\|_{L^\infty(L^\infty(\Omega))}\|\nabla_x\xi\|_{L^2(L^2(\Omega))}\|\Delta_x\xi\|_{L^2(L^2(\Omega))}\\
&\le \epsilon C(\delta,r_2). \qquad (89)
\end{aligned}
$$

Furthermore, we have

$$
\begin{aligned}
\bullet \int_0^T I_5dt &= 2\epsilon\int_0^T\int_\Omega \nabla_x\Delta_x\xi(u+2\nabla_x\ln\xi)dxdydt\\
&\le 2\epsilon\|\nabla_x\Delta_x\xi\|_{L^2(L^2(\Omega))}\|\sqrt{\xi}u\|_{L^2(L^2(\Omega))}\|\sqrt{\xi}\|_{L^\infty(L^\infty(\Omega))}\\
&+4\epsilon\|\nabla_x\Delta_x\xi\|_{L^2(L^2(\Omega))}\|\nabla_x\xi\|_{L^2(L^2(\Omega))}\|\xi^{-1}\|_{L^\infty(L^\infty(\Omega))}\\
&\le \epsilon C(\delta,r_2) \qquad (90)
\end{aligned}
$$

and

$$
\begin{aligned}
\bullet \int_0^T I_6dt &= -2r\epsilon\int_0^T\int_\Omega |u|u\nabla_x\xi dxdydt\\
&= 2r\epsilon\int_0^T\int_\Omega \mathrm{div}_x(|u|u)\xi dxdydt\\
&= 2r\epsilon\int_0^T\int_\Omega \xi\big(|u|\mathrm{div}_x u+u\frac{u}{|u|}\nabla_x u\big)dxdydt\\
&\le C\|\sqrt{\xi}u\|_{L^2(L^2(\Omega))}\|\sqrt{\xi}\nabla_x u\|_{L^2(L^2(\Omega))}\\
&\le C+\frac{1}{2}\|\sqrt{\xi}A_x(u)\|^2_{L^2(L^2(\Omega))}. \qquad (91)
\end{aligned}
$$

So by replacing (87)-(91) in (86), we get the B-D entropy (65).

In the next section, we make the parameters of our approximate system tend to 0.

5. Proof of mains results

5.1. Proof of Theorem 1

We make the proof of Theorem 1 in several steps.

5.1.1. Passing to the limits as ϵ, $\alpha \longrightarrow 0$

We denote the solution to (60) at this level of approximation as $(\xi_{\alpha,\epsilon}, u_{\alpha,\epsilon}, v_{\alpha,\epsilon})$. From the energy inequality (59), we obtain the following uniform regularities.

$$\begin{cases} \sqrt{\xi_{\alpha,\epsilon}}u_{\alpha,\epsilon} \in L^\infty\big([0,T];L^2(\Omega)\big), \quad r_2\xi_{\alpha,\epsilon}^{-\beta} \in L^\infty\big([0,T];L^1(\Omega)\big), \\ \sqrt{\xi_{\alpha,\epsilon}}D_x(u_{\alpha,\epsilon}) \in L^2\big([0,T];L^2(\Omega)\big), \quad \sqrt{\xi_{\alpha,\epsilon}}\partial_y u_{\alpha,\epsilon} \in L^2\big([0,T];L^2(\Omega)\big), \\ \sqrt{\delta}\xi_{\alpha,\epsilon} \in L^\infty\big([0,T];H^5(\Omega)\big), \quad \sqrt{r_1}u_{\alpha,\epsilon} \in L^2\big([0,T];L^2(\Omega)\big), \\ \sqrt{\alpha}\Delta u_{\alpha,\epsilon} \in L^2\big([0,T];L^2(\Omega)\big), \quad \xi_{\alpha,\epsilon}^{\frac{1}{3}}u_{\alpha,\epsilon} \in L^3\big([0,T];L^3(\Omega)\big). \end{cases} \tag{92}$$

The B-D entropy (65) gives the following additional uniform regularities:

$$\begin{cases} \sqrt{\delta}\xi_{\alpha,\epsilon} \in L^2\big([0,T];H^6(\Omega)\big), \quad \nabla_x\xi_{\alpha,\epsilon} \in L^2\big([0,T];L^2(\Omega)\big) \\ \sqrt{r_2}\nabla_x\xi_{\alpha,\epsilon}^{-\beta/2} \in L^2\big([0,T];L^2(\Omega)\big), \quad \sqrt{\xi_{\alpha,\epsilon}}A_x(u_{\alpha,\epsilon}) \in L^2\big([0,T];L^2(\Omega)\big), \\ \sqrt{\xi_{\alpha,\epsilon}}\partial_y u_{\alpha,\epsilon} \in L^2\big([0,T];L^2(\Omega)\big), \quad \xi_{\alpha,\epsilon}^2 \in L^\infty\big([0,T];L^1(\Omega)\big), \\ \sqrt{\xi_{\alpha,\epsilon}}\nabla_x\sqrt{\xi_{\alpha,\epsilon}} \in L^\infty\big([0,T];L^2(\Omega)\big). \end{cases} \tag{93}$$

Similar to the proof of Lemma 5, we have the following uniform boundedness:

$$\sqrt{k_1}\sqrt{\xi_{\alpha,\epsilon}} \in L^2\big([0,T];H^2(\Omega)\big); \quad k_1^{\frac{1}{4}}\nabla_x\xi_{\alpha,\epsilon}^{\frac{1}{4}} \in L^4\big([0,T];L^4(\Omega)\big).$$

With the above regularities, we can show the following uniform compactness results.

Lemma 12. *Let* $(\xi_{\alpha,\epsilon}, u_{\alpha,\epsilon}, v_{\alpha,\epsilon})$ *the weak solution of* (60) *satisfying* (92) *and* (93), *then the following estimates hold:*

$$\begin{cases} \|\partial_t\sqrt{\xi_{\alpha,\epsilon}}\|_{L^\infty\left([0,T];W^{-1,\frac{3}{2}}(\Omega)\right)} + \|\sqrt{\xi_{\alpha,\epsilon}}\|_{L^2\left([0,T];H^2(\Omega)\right)} \le K \\ \|\xi_{\alpha,\epsilon}\|_{L^2\left([0,T];H^6(\Omega)\right)} + \|\partial_t\xi_{\alpha,\epsilon}\|_{L^2\left([0,T];H^{-1}(\Omega)\right)} \le K \\ \|\xi_{\alpha,\epsilon}^{-\beta}\|_{L^{\frac{5}{3}}\left([0,T];L^{\frac{5}{3}}(\Omega)\right)} \le K \\ \|\xi_{\alpha,\epsilon}u_{\alpha,\epsilon}\|_{L^2\left([0,T];W^{1,\frac{3}{2}}(\Omega)\right)} + \|\partial_t(\xi_{\alpha,\epsilon}u_{\alpha,\epsilon})\|_{L^2\left([0,T];H^{-5}(\Omega)\right)} \le K, \end{cases} \tag{94}$$

where K *is independent of* ϵ, α.

Proof. The proof of Lemma 12 follows the same lines as the proof of Lemma 6.

Thanks to Lemmas 1 and 12, when $\alpha \to 0$ and $\epsilon \to 0$, we have the following compactness results

$$\begin{cases} \sqrt{\xi_{\alpha,\epsilon}} \longrightarrow \sqrt{\xi} \quad \text{strongly in} \quad L^2\big([0,T];H^1(\Omega)\big), \\ \xi_{\alpha,\epsilon} \longrightarrow \xi \quad \text{strongly in} \quad C\big([0,T];H^5(\Omega)\big), \\ \xi_{\alpha,\epsilon} \rightharpoonup \xi \quad \text{weakly in} \quad L^2\big([0,T];H^6(\Omega)\big), \\ u_{\alpha,\epsilon} \rightharpoonup u \quad \text{weakly in} \quad L^2\big([0,T];L^2(\Omega)\big), \\ \sqrt{\xi_{\alpha,\epsilon}} \rightharpoonup \sqrt{\xi} \quad \text{weakly in} \quad L^2\big([0,T];H^2(\Omega)\big), \\ \xi_{\alpha,\epsilon}u_{\alpha,\epsilon} \longrightarrow \xi u \quad \text{strongly in} \quad L^2\big([0,T];L^p(\Omega)\big), \quad \forall 1 \le p < 3, \\ \xi_{\alpha,\epsilon}^{-\beta} \longrightarrow \xi^{-\beta} \quad \text{strongly in} \quad L^1\big([0,T];L^1(\Omega)\big), \\ \sqrt{\xi_{\alpha,\epsilon}}u_{\alpha,\epsilon} \longrightarrow \sqrt{\xi}u \quad \text{strongly in} \quad L^2\big([0,T];L^2(\Omega)\big). \end{cases} \tag{95}$$

Hence, $\sqrt{\xi_{\alpha,\epsilon}} \longrightarrow \sqrt{\xi}$, $\xi_{\alpha,\epsilon} \longrightarrow \xi$ and $\xi_{\alpha,\epsilon}^{-\beta} \longrightarrow \xi^{-\beta}$ almost everywhere (t,x).
By the results of regularities above, we can pass to the limits in (60) as $\alpha \to 0$ and $\epsilon \to 0$. For any test function $\Phi \in C_c^\infty\big([0,T]\times\Omega\big)$, we have

$$\begin{aligned} \int_0^T\int_\Omega \partial_y(\xi_{\alpha,\epsilon}u_{\alpha,\epsilon}v_{\alpha,\epsilon})\Phi dxdydt &= -\int_0^T\int_\Omega \xi_{\alpha,\epsilon}u_{\alpha,\epsilon}v_{\alpha,\epsilon}\partial_y\Phi dxdydt \\ &\longrightarrow -\int_0^T\int_\Omega \xi uv\partial_y\Phi dxdydt. \end{aligned} \tag{96}$$

Moreover, when $\epsilon,\ \alpha \longrightarrow 0$, we have

$$\begin{aligned} &\epsilon\int_0^T\int_\Omega \nabla_x\xi_{\alpha,\epsilon}\cdot\nabla_x u_{\alpha,\epsilon}\Phi dxdydt \\ &= -\epsilon\int_0^T\int_\Omega \nabla_x(\nabla_x\xi_{\alpha,\epsilon}\Phi)u_{\alpha,\epsilon}dxdydt \\ &= -\epsilon\int_0^T\int_\Omega \Delta_x\xi_{\alpha,\epsilon}\cdot u_{\alpha,\epsilon}\Phi dxdydt - \epsilon\int_0^T\int_\Omega \nabla_x\xi_{\alpha,\epsilon}u_{\alpha,\epsilon}\mathrm{div}_x\Phi dxdydt \\ &\le \epsilon\|\Delta_x\xi_{\alpha,\epsilon}\|_{L^2\left([0,T];L^2(\Omega)\right)}\|u_{\alpha,\epsilon}\|_{L^2\left([0,T];L^2(\Omega)\right)}\|\Phi\|_{L^\infty\left([0,T];L^\infty(\Omega)\right)} \\ &+ \epsilon\|\nabla_x\xi_{\alpha,\epsilon}\|_{L^2\left([0,T];L^2(\Omega)\right)}\|u_{\alpha,\epsilon}\|_{L^2\left([0,T];L^2(\Omega)\right)}\|\mathrm{div}_x\Phi\|_{L^\infty\left([0,T];L^\infty(\Omega)\right)} \longrightarrow 0 \end{aligned} \tag{97}$$

and

$$\alpha\int_0^T\int_\Omega \Delta^2 u_{\alpha,\epsilon}\cdot\Phi dxdydt = \alpha\int_0^T\int_\Omega \Delta u_{\alpha,\epsilon}\cdot\Delta\phi dxdydt$$

$$\leq \sqrt{\alpha}\|\sqrt{\alpha}\Delta u_{\alpha,\epsilon}\|_{L^2\left([0,T];L^2(\Omega)\right)}\|\Delta\Phi\|_{L^2\left([0,T];L^2(\Omega)\right)}\longrightarrow 0. \quad (98)$$

So, taking $\alpha, \epsilon \longrightarrow 0$ in (60), the system

$$\begin{cases} \partial_t\xi + \mathrm{div}_x(\xi u) + \partial_y(\xi v) = 0, \\ \partial_t(\xi u) + \mathrm{div}_x(\xi u \otimes u) + \partial_y(\xi u v) + \nabla_x \xi^2 + r_1 u + r\xi|u|u \\ = 2\mathrm{div}_x\big(\xi D_x(u)\big) + \partial_y(\xi\partial_y u) + r_2\nabla_x\xi^{-\beta} + k_1\xi\nabla_x\big(\frac{\Delta_x\sqrt{\xi}}{\sqrt{\xi}}\big) + \delta\xi\nabla_x\Delta_x^5\xi, \\ \partial_y\xi = 0 \end{cases} \quad (99)$$

holds in the sense of distribution on $[0,T]\times\Omega$.
Moreover, thanks to the lower semi-continuity of convex functions and the strong convergence of $(\xi_{\alpha,\epsilon}, u_{\alpha,\epsilon}, v_{\alpha,\epsilon})$, we can pass to the limits in the energy inequality (59) and the B-D entropy (65) as $\alpha = \epsilon \longrightarrow 0$ with δ, r_2, r_0, k_1 being fixed, to obtain

$$\begin{aligned} &E(\xi,u) + r_1\int_0^T\int_\Omega u^2 dXdt + r\int_0^T\int_\Omega \xi|u|^3 dXdt \\ &+\int_0^T\int_\Omega 2\xi|D_x(u)|^2 dXdt + \int_0^T\int_\Omega \xi|\partial_y u|^2 dXdt \leq E(\xi_0,u_0) \end{aligned} \quad (100)$$

and

$$\begin{aligned} &\int_\Omega\big(\frac{1}{2}\xi|u+2\nabla_x\ln\xi|^2 - 2r_1\ln\xi\big)dxdy + 2\int_0^T\int_\Omega \xi|\partial_y v|^2 dxdydt + r\int_0^T\int_\Omega \xi|u|^3 dxdydt \\ &+\frac{16r_2}{\beta}\int_0^T\int_\Omega |\nabla_x\xi^{-\beta/2}|^2 dxdydt + r_1\int_0^T\int_\Omega u^2 dxdydt + \int_0^T\int_\Omega \xi|\partial_y u|^2 dxdy \\ &+ k_1\int_0^T\int_\Omega \xi|\nabla_x^2\ln\xi|^2 dxdydt + 2\delta\int_0^T\int_\Omega |\Delta_x^3\xi|^2 dxdydt + 2\int_0^T\int_\Omega \xi|A_x(u)|^2 dxdydt \\ &+ 8\int_0^T\int_\Omega \xi|\nabla_x\sqrt{\xi}|^2 dxdydt \leq \int_\Omega\Big(\xi_0 u_0^2 + 10(\nabla_x\sqrt{\xi_0})^2 - 2r_1\ln\xi_0\Big)dxdy + E_0 + C. \end{aligned} \quad (101)$$

Thus, to conclude this part, we give an existence result of weak solutions to the system (99).

Proposition 5.1. *For any $T > 0$, the system*

$$\begin{cases} \partial_t\xi + div_x(\xi u) + \partial_y(\xi v) = 0, \\ \partial_t(\xi u) + div_x(\xi u \otimes u) + \partial_y(\xi u v) + \nabla_x \xi^2 + r\xi|u|u + r_1 u \\ = 2div_x\big(\xi D_x(u)\big) + \partial_y(\xi\partial_y u) + r_2\nabla_x\xi^{-\beta} + k_1\xi\nabla_x\big(\frac{\Delta_x\sqrt{\xi}}{\sqrt{\xi}}\big) + \delta\xi\nabla_x\Delta_x^5\xi, \\ \partial_y\xi = 0, \end{cases} \quad (102)$$

with $(t,x,y) \in [0,T]\times\Omega$, admits a weak solution with appropriate initial data. In particular the weak solution (ξ,u,v) satisfies the energy inequality (100) *and the B-D entropy* (101).

5.1.2. Passing to the limits as $r_2, \delta \longrightarrow 0$

In this step, we pass to the limit as $r_2 \longrightarrow 0$, $\delta \longrightarrow 0$ with k_1, r_1 fixed. We denote by $(\xi_{r_2,\delta}, u_{r_2,\delta}, v_{r_2,\delta})$ the weak solution at this level. From Proposition 5.1, we have the following regularity results:

$$\begin{cases} \sqrt{\xi_{r_2,\delta}}u_{r_2,\delta} \in L^\infty\big([0,T]; L^2(\Omega)\big), \sqrt{\xi_{r_2,\delta}}D_x(u_{r_2,,\delta}) \in L^2\big([0,T]; L^2(\Omega)\big), \sqrt{\xi_{r_2,\delta}}\partial_y u_{r_2,\delta} \in L^2\big([0,T]; L^2(\Omega)\big), \\ \sqrt{\xi_{r_2,\delta}}\nabla_x\sqrt{\xi_{r_2,\delta}} \in L^\infty\big([0,T]; L^2(\Omega)\big), \quad \sqrt{\delta}\xi_{r_2,\delta} \in L^\infty\big([0,T]; H^5(\Omega)\big), \quad \sqrt{\delta}\xi_{r_2,\delta} \in L^2\big([0,T]; H^6(\Omega)\big), \\ u_{r_2,\delta} \in L^2\big([0,T]; L^2(\Omega)\big), \quad \xi_{r_2,\delta}^{\frac{1}{3}}u_{r_2,\delta} \in L^3\big([0,T]; L^3(\Omega)\big), \quad \sqrt{\xi_{r_2,\delta}}\nabla_x u_{r_2,\delta} \in L^2\big([0,T]; L^2(\Omega)\big), \\ \sqrt{k_1}\sqrt{\xi_{r_2,\delta}} \in L^2\big([0,T]; H^2(\Omega)\big), \quad \sqrt[4]{k_1}\nabla_x\xi_{r_2,\delta}^{\frac{1}{4}} \in L^4\big([0,T]; L^4(\Omega)\big), \sqrt{\delta}\Delta_x^3\xi_{r_2,\delta} \in L^2\big([0,T]; L^2(\Omega)\big), \\ \sqrt{r_2}\nabla_x\xi_{r_2,\delta}^{-\beta/2} \in L^2\big([0,T]; L^2(\Omega)\big), \quad \sqrt{\xi_{r_2,\delta}}\partial_y v_{r_2,\delta} \in L^2\big([0,T]; L^2(\Omega)\big), \quad r_2\xi_{r_2,\delta}^{-\beta} \in L^\infty\big([0,T]; L^1(\Omega)\big), \\ r_1 \ln \xi_{r_2,\delta} \in L^\infty\big([0,T]; L^1(\Omega)\big). \end{cases} \tag{103}$$

The inequality (46) remains true for $r_2, \delta \longrightarrow 0$. Thus, we have the uniform estimates similar to the Lemma 12 with δ and r_2. Moreover, we deduce the same compactness for $(\xi_{r_2}, u_{r_2}, v_{r_2})$ and for $(\xi_\delta, u_\delta, v_\delta)$. Therefore at this level of approximation, we focus only on the convergence of $r_2\nabla\xi^{-\beta}$ and that of $\delta\xi\nabla_x\Delta_x^5\xi$. Here we state the following two Lemmas.

Lemma 13. *: For any ξ_{r_2} defined as in Proposition 5.1, we have*

$$r_2\int_0^T\int_\Omega \xi_{r_2}^{-\beta}dxdydt \longrightarrow 0 \quad as \quad r_2 \longrightarrow 0.$$

Proof. The proof is motivated by the method in [18, 19]. From the entropy B-D (101), we have:

$$\sup_{t\in[0,T]}\int_\Omega \big(\ln(\frac{1}{\xi_{r_2}})\big)_+ dxdy \leq C(r_1) < +\infty. \tag{104}$$

Note that $y \in \mathbb{R}^+ \longmapsto \ln(\frac{1}{y})_+$ is a continuous convex function. Moreover, in combination with the convex function property and Fatou's Lemma, it follows that:

$$\begin{aligned} \int_\Omega \big(\ln(\frac{1}{\xi})\big)_+ dxdy &\leq \int_\Omega \liminf_{r_2\to 0}\big(\ln(\frac{1}{\xi_{r_2}})\big)_+ dxdy \\ &\leq \liminf_{r_2\to 0}\int_\Omega \big(\ln(\frac{1}{\xi_{r_2}})\big)_+ dxdy \\ &\leq C(r_1) < +\infty. \end{aligned} \tag{105}$$

This means that $\big(\ln(\frac{1}{\xi})\big)_+$ is bounded in $L^\infty\big([0,T]; L^1(\Omega)\big)$. This allows us to deduce that

$$\big|\{x/\xi(t,x) = 0\}\big| = 0 \quad \text{for almost every} \quad t \in [0,T], \tag{106}$$

where $|B|$ denotes the measure of set B.
Due to $\xi_{r_2} \longrightarrow \xi$ strongly in $C\big([0,T];H^5(\Omega)\big)$, hence $\xi_{r_2} \longrightarrow \xi \quad a.e.$ Thus the above limits and (106) deduce

$$r_2\xi_{r_2}^{-\beta} \longrightarrow 0 \quad \text{a.e} \quad \text{as} \quad r_2 \longrightarrow 0. \tag{107}$$

Moreover, using the Lemma 3 (interpolation inequality), we have

$$\|r_2\xi_{r_2}^{-\beta}\|_{L^{\frac{5}{3}}\left([0,T];L^{\frac{5}{3}}(\Omega)\right)} \leq \|r_2\xi_{r_2}^{-\beta}\|^{\frac{2}{5}}_{L^\infty\left([0,T];L^1(\Omega)\right)} \|r_2\xi_{r_2}^{-\beta}\|^{\frac{3}{5}}_{L^1\left([0,T];L^3(\Omega)\right)} \leq C, \tag{108}$$

which combines with (107) and Lemma 2, and we have

$$r_2\xi_{r_2}^{-\beta} \longrightarrow 0 \quad \text{strongly in} \quad L^1\big([0,T];L^1(\Omega)\big).$$

Lemma 14. *For any ξ_δ defined as in the proposition 5.1, we have, for any test function Φ*

$$\delta\int_0^T\int_\Omega \xi_\delta\nabla_x\Delta_x^5\xi_\delta\Phi dxdydt \longrightarrow 0 \quad as \quad \delta\longrightarrow 0. \tag{109}$$

Proof. By (103) (where δ appears) and using the Gagliardo-Nirenberg interpolation inequality, we have

$$\|\nabla_x^5\xi_\delta\|_{L^2} \leq C\|\nabla_x^6\xi_\delta\|^{\frac{9}{11}}_{L^2}\|\xi_\delta\|^{\frac{2}{11}}_{L^3}. \tag{110}$$

Thus,

$$\begin{aligned}
&\int_0^T \delta\Big(\int_\Omega |\nabla_x^5\xi_\delta|^2 dxdy\Big)^{\frac{11}{9}}dt \\
&\leq C\Big(\sup_{t\in[0,T]}\|\xi_\delta\|_{L^3}\Big)^{\frac{4}{9}}\int_0^T\delta\int_\Omega|\nabla_x^6\xi_\delta|^2dxdydt.
\end{aligned} \tag{111}$$

This implies

$$\delta^{\frac{9}{22}}\nabla_x^5\xi_\delta \in L^{\frac{22}{9}}\big([0,T];L^2(\Omega)\big).$$

For any test function $\Phi\in C_c^\infty\big([0,T];\Omega\big)$, we have

$$\begin{aligned}
&\delta\int_0^T\int_\Omega \xi_\delta\nabla_x\Delta_x^5\xi_\delta\Phi dxdydt \\
&= -\delta\int_0^T\int_\Omega \Delta_x^2\text{div}_x(\xi_\delta\Phi)\Delta_x^3\xi_\delta dxdydt \\
&= -\delta\int_0^T\int_\Omega \Delta_x^2\big(\Phi\nabla_x\xi_\delta + \xi_\delta\text{div}_x\Phi\big)\Delta_x^3\xi_\delta dxdydt.
\end{aligned} \tag{112}$$

Now we treat the term

$$\begin{aligned}&\Big|\delta\int_0^T\int_\Omega \Delta_x^2(\nabla_x\xi_\delta)\Delta_x^3\xi_\delta\Phi dxdydt\Big|\\&\leq C\delta^{\frac{1}{11}}\|\sqrt{\delta}\nabla_x^6\xi_\delta\|_{L^2(L^2)}\|\delta^{\frac{9}{22}}\nabla_x^5\xi_\delta\|_{L^{\frac{22}{9}}(L^2)}\|\Phi\|_{L^{11}(L^\infty)}\longrightarrow 0,\end{aligned} \tag{113}$$

as $\delta\longrightarrow 0$.
As before, we can control the other term of

$$\delta\int_0^T\int_\Omega \xi_\delta\nabla_x\Delta_x^5\xi_\delta\Phi dxdydt.$$

Thus, we have:

$$\delta\int_0^T\int_\Omega \xi_\delta\nabla_x\Delta_x^5\xi_\delta\Phi dxdydt\longrightarrow 0 \quad \text{when} \quad \delta\longrightarrow 0. \tag{114}$$

So, we can pass to the limit $r_2,\delta\longrightarrow 0$ in (102)). Hence

$$\begin{cases}\partial_t\xi+\operatorname{div}_x(\xi u)+\partial_y(\xi v)=0\\ \\ \partial_t(\xi u)+\operatorname{div}_x(\xi u\otimes u)+\partial_y(\xi uv)+\nabla_x\xi^2+r\xi|u|u+r_1u\\ =2\operatorname{div}_x\big(\xi D_x(u)\big)+\partial_y(\xi\partial_y u)+k_1\xi\nabla_x\big(\frac{\Delta_x\sqrt{\xi}}{\sqrt{\xi}}\big)\\ \\ \partial_y\xi=0,\end{cases} \tag{115}$$

holds in the sense of distribution on $[0,T]\times\Omega$.
Due to the lower semi-continuity of convex functions, we can obtain the following energy inequality (116) and B-D entropy (117) by passing to the limits in (100) and (101) as $r_2,\delta\longrightarrow 0$,

$$\begin{aligned}&\int_\Omega\left(\frac{1}{2}\xi u^2+\xi^2+k_1|\nabla_x\sqrt{\xi}|^2\right)dxdy+r_1\int_0^T\int_\Omega u^2dxdydt\\&+r\int_0^T\int_\Omega\xi|u|^3dxdydt+\int_0^T\int_\Omega 2\xi|D_x(u)|^2dxdydt+g\int_0^T\int_\Omega v\xi\ln\xi dxdydt\\&+\int_0^T\int_\Omega\xi|\partial_y u|^2dxdydt\leq\int_\Omega\left(\frac{1}{2}\xi_0u_0^2+\xi_0^2+k_1|\nabla_x\sqrt{\xi_0}|^2\right)ddxdy\end{aligned} \tag{116}$$

and

$$\begin{aligned}&\int_\Omega\big(\frac{1}{2}\xi|u+2\nabla_x\ln\xi|^2-2r_1\ln\xi\big)dxdy+2\int_0^T\int_\Omega\xi|\partial_y v|^2dxdydt+r\int_0^T\int_\Omega\xi|u|^3dxdydt\\&+r_1\int_0^T\int_\Omega u^2dxdydt+\int_0^T\int_\Omega\xi|\partial_y u|^2dxdy+k_1\int_0^T\int_\Omega\xi|\nabla_x^2\ln\xi|^2dxdydt\end{aligned}$$

$$
\begin{aligned}
&+2\int_0^T\int_\Omega \xi|A_x(u)|^2 dxdydt + 8\int_0^T\int_\Omega \xi|\nabla_x\sqrt{\xi}|^2 dxdydt\\
&\le \int_\Omega \Big(\xi_0 u_0^2 + 10(\nabla_x\sqrt{\xi_0})^2 - 2r_1\ln\xi_0\Big)dxdy + E_0 + C.
\end{aligned} \tag{117}
$$

We obtain the existence of the weak solution (ξ, u, v) at this level of approximation given in the following proposition.

Proposition 5.2. *For any $T > 0$, the system* (115) *admits a weak solution with appropriate initial data. In particular, the weak solution (ξ, u, w) satisfies the energy inequality* (116) *and the B-D entropy* (117).

5.1.3. Passing to the limits as $k_1, r_1 \longrightarrow 0$

In this step, we pass to the limits as $k_1, r_1 \longrightarrow 0$. We denote by $(\xi_{k_1,r_1}, u_{k_1,r_1}, v_{k_1,r_1})$ the weak solution at this level. Here, the weak solution satisfies the energy inequality (116) and the B-D entropy (117), then, we have the following regularities

$$
\begin{cases}
\sqrt{\xi_{k_1,r_1}}u_{k_1,r_1} \in L^\infty\big([0,T];L^2(\Omega)\big), \quad \sqrt{\xi_{k_1,r_1}}D_x(u_{k_1,r_1}) \in L^2\big([0,T];L^2(\Omega)\big),\\
\nabla_x\sqrt{\xi_{k_1,r_1}} \in L^\infty\big([0,T];L^2(\Omega)\big), \quad \sqrt{\xi_{k_1,r_1}}\partial_y u_{k_1,r_1} \in L^2\big([0,T];L^2(\Omega)\big),\\
\xi_{k_1,r_1}^{\frac{1}{3}}u_{k_1,r_1} \in L^3\big([0,T];L^3(\Omega)\big), \quad \sqrt{\xi_{k_1,r_1}}\nabla_x u_{k_1,r_1} \in L^2\big([0,T];L^2(\Omega)\big),\\
u_{k_1,r_1} \in L^2\big([0,T];L^2(\Omega)\big), \quad r_1\ln\xi_{k_1,r_1} \in L^\infty\big([0,T];L^1(\Omega)\big),\\
\sqrt{\xi_{k_1,r_1}}\partial_y v_{k_1,r_1} \in L^2\big([0,T];L^2(\Omega)\big), \quad \sqrt{k_1}\sqrt{\xi_{k_1,r_1}} \in L^2\big([0,T];H^2(\Omega)\big),\\
\sqrt{\xi_{k_1,r_1}}\nabla_x\sqrt{\xi_{k_1,r_1}} \in L^2\big([0,T];L^2(\Omega)\big), \quad \sqrt[4]{k_1}\nabla_x\xi_{k_1,r_1}^{\frac{1}{4}} \in L^4\big([0,T];L^4(\Omega)\big).
\end{cases} \tag{118}
$$

Lemma 15. *(Convergence of $(\sqrt{\xi_{k_1,r_1}})_{k_1,r_1}$). For $\sqrt{\xi_{k_1,r_1}}$ satisfying Proposition 5.2, we deduce that $(\sqrt{\xi_{k_1,r_1}})_{k_1,r_1}$ is bounded in $L^\infty\big([0,T];H^1(\Omega)\big)$ and $(\partial_t\sqrt{\xi_{k_1,r_1}})_{k_1,r_1}$ is bounded in $L^2\big([0,T];H^{-1}(\Omega)\big)$.*
Then, up to a subsequence, we have

$$
\sqrt{\xi_{k_1,r_1}} \longrightarrow \sqrt{\xi} \quad \textit{a.e. and strongly in} \quad L^2\big([0,T];L^2(\Omega).
$$

Furthermore, we have

$$
\xi_{k_1,r_1} \longrightarrow \xi \quad \textit{a.e. and strongly in } \ C\big([0,T];L^p(\Omega), \quad \forall \quad 1 \le p < 3.
$$

Proof. Since $\|\sqrt{\xi_{k_1,r_1}}\|^2_{L^2(\Omega)} = \|\xi_0\|_{L^1(\Omega)}$ and $\nabla_x\sqrt{\xi_{k_1,r_1}} \in L^\infty\big([0,T];L^2(\Omega)$, we have $\sqrt{\xi_{k_1,r_1}} \in L^\infty\big([0,T];H^1(\Omega)$. Then, we claim that

$$
\partial_t\sqrt{\xi_{k_1,r_1}} = -\frac{1}{2}\sqrt{\xi_{k_1,r_1}}\mathrm{div}_x(u_{k_1,r_1}) - u_{k_1,r_1}.\nabla_x\sqrt{\xi_{k_1,r_1}} - \frac{1}{2}\sqrt{\xi_{k_1,r_1}}\partial_y v_{k_1,r_1}
$$

$$= \frac{1}{2}\sqrt{\xi_{k_1,r_1}}\mathrm{div}_x(u_{k_1,r_1}) - \mathrm{div}_x(\sqrt{\xi_{k_1,r_1}}u_{k_1,r_1}) - \frac{1}{2}\sqrt{\xi_{k_1,r_1}}\partial_y v_{k_1,r_1}. \tag{119}$$

Indeed, we have

$$\partial_t\xi_{k_1,r_1} + \partial_y(\xi_{k_1,r_1}v_{k_1,r_1}) = -\mathrm{div}_x(\xi_{k_1,r_1}u_{k_1,r_1}).$$

Furthermore, $\partial_t\sqrt{\xi_{k_1,r_1}} = \frac{1}{2}\frac{1}{\sqrt{\xi_{k_1,r_1}}}\partial_t\xi_{k_1,r_1}$, hence

$$\partial_t\sqrt{\xi_{k_1,r_1}} = -\frac{1}{2}\frac{1}{\sqrt{\xi_{k_1,r_1}}}\Big(\mathrm{div}_x(\xi_{k_1,r_1}u_{k_1,r_1}) + \partial_y(\xi_{k_1,r_1}v_{k_1,r_1})\Big).$$

By developing the divergence part and replacing ξ_{k_1,r_1} by $\sqrt{\xi_{k_1,r_1}}.\sqrt{\xi_{k_1,r_1}}$, we obtain

$$\partial_t\sqrt{\xi_{k_1,r_1}} = -\frac{1}{2}\sqrt{\xi_{k_1,r_1}}\mathrm{div}_x(u_{k_1,r_1}) - u_{k_1,r_1}\nabla_x\sqrt{\xi_{k_1,r_1}} - \frac{1}{2}\sqrt{\xi_{k_1,r_1}}\partial_y v_{k_1,r_1}.$$

Adding and deducting $\sqrt{\xi_{k_1,r_1}}\mathrm{div}_x(u_{k_1,r_1})$ in the above equality, we obtain (119).

This gives $\partial_t\sqrt{\xi_{k_1,r_1}} \in L^2\big([0,T];H^{-1}(\Omega)\big)$. Using Lemma 1 we have $\sqrt{\xi_{k_1,r_1}} \longrightarrow \sqrt{\xi}$ strongly in $L^2\big([0,T];H^1(\Omega)\big)$, thus in $L^2\big([0,T];L^2(\Omega)\big)$ and this gives that $\sqrt{\xi_{k_1,r_1}} \longrightarrow \sqrt{\xi}$ a.e.
Using the Sobolev injection theorem, we deduce that $\sqrt{\xi_{k_1,r_1}}$ is bounded in $L^\infty\big([0,T];L^6(\Omega)\big)$, so $\xi_{k_1,r_1} \in L^\infty\big([0,T];L^3(\Omega)\big)$. Then, we deduce by using the Hölder inequality that

$$\xi_{k_1,r_1}u_{k_1,r_1} = \sqrt{\xi_{k_1,r_1}}\sqrt{\xi_{k_1,r_1}}u_{k_1,r_1} \in L^\infty\big([0,T];L^{\frac{3}{2}}(\Omega)\big). \tag{120}$$

This gives $\mathrm{div}_x(\xi_{k_1,r_1}u_{k_1,r_1}) \in L^\infty\big([0,T];W^{-1,\frac{3}{2}}(\Omega)\big)$. This last result, combined with $\sqrt{\xi_{k_1,r_1}}\partial_y v_{k_1,r_1} \in L^2\big([0,T];L^2(\Omega)\big)$ and the continuity equation, give $\partial_t\sqrt{\xi_{k_1,r_1}} \in L^2\big([0,T];W^{-1,\frac{3}{2}}(\Omega)\big)$. Moreover, we have

$$\nabla_x\xi_{k_1,r_1} = 2\sqrt{\xi_{k_1,r_1}}\nabla_x\sqrt{\xi_{k_1,r_1}} \in L^\infty\big([0,T];L^{\frac{3}{2}}(\Omega)\big). \tag{121}$$

Then, we conclude that ξ_{k_1,r_1} is bounded in $L^\infty\big([0,T];W^{1,\frac{3}{2}}(\Omega)\big)$. Now, we use Lemma 1 to get

$$\xi_{k_1,r_1} \longrightarrow \xi \quad \text{strongly in} \quad C\big([0,T];L^p(\Omega)\big), \quad \forall 1 \le p < 3. \tag{122}$$

Therefore, we get $\xi_{k_1,r_1} \longrightarrow \xi$ a.e.

Lemma 16. *(Convergence of the momentum).*
Up to a subsequence, the momentum $m_{k_1,r_1} = \xi_{k_1,r_1}u_{k_1,r_1}$ satisfies

$$m_{k_1,r_1} \longrightarrow m \textit{ strongly in } L^2\big([0,T];L^q(\Omega)\big),\ \forall 1 \le q < \frac{3}{2}.$$

In particular, $m_{k_1,r_1} \longrightarrow m$ for $(t,x) \in [0,T]\times\Omega$.

Proof. Since

$$\begin{aligned}\nabla_x(\xi_{k_1,r_1}u_{k_1,r_1}) &= \nabla_x\xi_{k_1,r_1}\otimes u_{k_1,r_1}+\xi_{k_1,r_1}\nabla_x u_{k_1,r_1}\\ &= 2\nabla_x\sqrt{\xi_{k_1,r_1}}\otimes\sqrt{\xi_{k_1,r_1}}u_{k_1,r_1}+\sqrt{\xi_{k_1,r_1}}\sqrt{\xi_{k_1,r_1}}\nabla_x u_{k_1,r_1}\in L^2\big([0,T];L^1(\Omega)\big)\end{aligned}$$

and

$$\partial_y(\xi_{k_1,r_1}u_{k_1,r_1})=\sqrt{\xi_{k_1,r_1}}\sqrt{\xi_{k_1,r_1}}\partial_y u_{k_1,r_1}\in L^2\big([0,T];L^{\frac{3}{2}}(\Omega)\big),$$

we use (120) to deduce that

$$\xi_{k_1,r_1}u_{k_1,r_1}\in L^2\big([0,T];W^{1,1}(\Omega)\big).$$

Now, we claim that $\partial_y(\xi_{k_1,r_1}u_{k_1,r_1})$ is bounded in $L^2\big([0,T];H^{-s}(\Omega)\big)$ for some constant $s>0$.
Indeed,

$$\begin{aligned}\partial_y(\xi_{k_1,r_1}u_{k_1,r_1}) &= 2\mathrm{div}_x\left(\xi_{k_1,r_1}D_x(u_{k_1,r_1})\right)+\partial_y(\xi_{k_1,r_1}\partial_y u_{k_1,r_1})\\ &\quad+k_1\xi_{k_1,r_1}\nabla_x\Big(\frac{\Delta_x\sqrt{\xi_{k_1,r_1}}}{\sqrt{\xi_{k_1,r_1}}}\Big)+\delta\xi_{k_1,r_1}\nabla_x\Delta_x^5\xi_{k_1,r_1}-\mathrm{div}_x(\xi_{k_1,r_1}u_{k_1,r_1}\otimes u_{k_1,r_1})\\ &\quad-\partial_y(\xi_{k_1,r_1}u_{k_1,r_1}v_{k_1,r_1})-\nabla_x\xi_{k_1,r_1}^2-r_1u_{k_1,r_1}-r\xi_{k_1,r_1}u_{k_1,r_1}|u_{k_1,r_1}|. \qquad (123)\end{aligned}$$

With the estimates of (118), we deduce that

$$\xi_{k_1,r_1}u_{k_1,r_1}\otimes u_{k_1,r_1}\in L^\infty\big([0,T];L^1(\Omega)\big),\quad \xi_{k_1,r_1}u_{k_1,r_1}v_{k_1,r_1}\in L^2\big([0,T];L^1(\Omega)\big)$$

and

$$\xi_{k_1,r_1}\partial_y u_{k_1,r_1}\in L^2\big([0,T];L^{\frac{3}{2}}(\Omega)\big),\quad \xi_{k_1,r_1}D_x(u_{k_1,r_1})\in L^2\big([0,T];L^{\frac{3}{2}}(\Omega)\big).$$

In particular, thanks to the Sobolev injection theorem, we have

$$\begin{cases}\mathrm{div}_x(\xi_{k_1,r_1}u_{k_1,r_1}\otimes u_{k_1,r_1})\in L^\infty\big([0,T];W^{-2,2}(\Omega)\big),\quad \partial_y(\xi_{k_1,r_1}u_{k_1,r_1}v_{k_1,r_1})\in L^2\big([0,T];W^{-2,2}(\Omega)\big),\\ \partial_y(\xi_{k_1,r_1}\partial_y u_{k_1,r_1})\in L^2\big([0,T];W^{-2,2}(\Omega)\big),\quad \mathrm{div}_x(\xi_{k_1,r_1}D_x(u_{k_1,r_1}))\in L^2\big([0,T];W^{-2,2}(\Omega)\big),\\ k_1\xi_{k_1,r_1}\nabla_x\Big(\frac{\Delta_x\sqrt{\xi_{k_1,r_1}}}{\sqrt{\xi_{k_1,r_1}}}\Big)=k_1\nabla_x\big(\sqrt{\xi_{k_1,r_1}}\Delta_x\sqrt{\xi_{k_1,r_1}}\big)-2k_1\Delta_x\sqrt{\xi_{k_1,r_1}}\nabla_x\sqrt{\xi_{k_1,r_1}}\in L^2\big([0,T];W^{-3,2}(\Omega)\big).\end{cases}$$

Then, we obtain the boundedness of $\partial_t(\xi_{k_1,r_1}u_{k_1,r_1})$ in $L^2\big([0,T];H^{-5}(\Omega)\big)$.
Therefore, we use Lemma 1 to conclude the proof of Lemma 16.

Remark 1. *We can define* $u(t,x,y)=\dfrac{m(t,x,y)}{\xi(t,x,y)}$ *outside the vacuum set* $\{x|\xi(t,x)=0\}$*.*
Then, we obtain $\xi_{k_1,r_1}u_{k_1,r_1}\longrightarrow\xi u$ *strongly in* $L^2\big([0,T];L^q(\Omega)\big)$, $\forall 1\leq q<1.5$*.*

With Lemmas 14 and 16, in a similar way to the proof of Lemma 8, we can deduce that when $k_1,\ r_1 \longrightarrow 0$,

$$\sqrt{\xi_{k_1,r_1}}u_{k_1,r_1} \longrightarrow \sqrt{\xi}u \quad \text{strongly in} \quad L^2\big([0,T];L^2(\Omega)\big). \tag{124}$$

Lemma 17. *(Convergence of terms $div_x(\xi_{k_1,r_1}D_x(u_{k_1,r_1})), k_1\xi_{k_1,r_1}\nabla_x\big(\frac{\Delta_x\sqrt{\xi_{k_1,r_1}}}{\sqrt{\xi_{k_1,r_1}}}\big)$ and $r_1u_{k_1,r_1}$).*
For any test function $\Phi \in C_c^\infty\big([0,T];\Omega\big)$, we have

$$r_1\int_0^T\int_\Omega u_{k_1,r_1}\Phi dxdydt \longrightarrow 0 \quad as \quad r_1 \longrightarrow 0, \tag{125}$$

$$\int_0^T\int_\Omega div_x(\xi_{k_1,r_1}D_x(u_{k_1,r_1}))\Phi dxdydt \longrightarrow \int_0^T\int_\Omega div_x(\xi D_x(u))\Phi dxdydt \ as\ r_1,k_1 \longrightarrow 0, \tag{126}$$

$$k_1\int_0^T\int_\Omega \xi_{k_1,r_1}\nabla_x\Big(\frac{\Delta_x\sqrt{\xi_{k_1,r_1}}}{\sqrt{\xi_{k_1,r_1}}}\Big)\Phi dxdydt \longrightarrow 0 \quad as \quad k_1 \longrightarrow 0. \tag{127}$$

Proof. We take $\Phi \in C_c^\infty\big([0,T];\Omega\big)$ as a test function. Firstly, we prove (125). Using Hölder's inequality, we get

$$r_1\int_0^T\int_\Omega u_{k_1,r_1}\Phi dxdydt \leq \sqrt{r_1}\|\sqrt{r_1}u_{k_1,r_1}\|_{L^2\left([0,T];L^2(\Omega)\right)}\|\Phi\|_{L^2\left([0,T];L^2(\Omega)\right)} \underset{r_1\to 0}{\longrightarrow} 0.$$

Secondly, we deal with (126). Recalling (57), we have

$$\begin{aligned}
&\int_0^T\int_\Omega \mathrm{div}_x(\xi_{k_1,r_1}D_x(u_{k_1,r_1}))\Phi dxdydt\\
&= \frac{1}{2}\int_0^T\int_\Omega \big(\xi_{k_1,r_1}u_{k_1,r_1}\cdot\Delta_x\Phi + 2\nabla_x\Phi\cdot\nabla_x\sqrt{\xi_{k_1,r_1}}\cdot\sqrt{\xi_{k_1,r_1}}u_{k_1,r_1}\big)dxdydt\\
&+ \frac{1}{2}\int_0^T\int_\Omega \big(\xi_{k_1,r_1}u_{k_1,r_1}\cdot\mathrm{div}_x(\nabla_x^t\Phi) + 2\nabla_x^t\Phi\cdot\nabla_x\sqrt{\xi_{k_1,r_1}}\cdot\sqrt{\xi_{k_1,r_1}}u_{k_1,r_1}\big)dxdydt.
\end{aligned}$$

Since $\nabla_x\sqrt{\xi_{k_1,r_1}} \in L^\infty\big([0,T];L^2(\Omega)\big)$, then the sequence $\nabla_x\sqrt{\xi_{k_1,r_1}}$ is weakly converges. Now, using Lemmas 14 and 16, and the fact that $\sqrt{\xi_{k_1,r_1}}u_{k_1,r_1} \longrightarrow \sqrt{\xi}u$ strongly in $L^2\big([0,T];L^2(\Omega)\big)$, we obtain

$$\begin{aligned}
&\frac{1}{2}\int_0^T\int_\Omega \big(\xi_{k_1,r_1}u_{k_1,r_1}\cdot\Delta_x\Phi + 2\nabla_x\Phi\cdot\nabla_x\sqrt{\xi_{k_1,r_1}}\cdot\sqrt{\xi_{k_1,r_1}}u_{k_1,r_1}\big)dxdydt\\
&+ \frac{1}{2}\int_0^T\int_\Omega \big(\xi_{k_1,r_1}u_{k_1,r_1}\cdot\mathrm{div}_x(\nabla_x^t\Phi) + 2\nabla_x^t\Phi\cdot\nabla_x\sqrt{\xi_{k_1,r_1}}\cdot\sqrt{\xi_{k_1,r_1}}u_{k_1,r_1}\big)dxdydt\\
&\underset{k_1,r_1\longrightarrow 0}{\longrightarrow} \quad \frac{1}{2}\int_0^T\int_\Omega \big(\xi u\cdot\Delta_x\Phi + 2\nabla_x\Phi\cdot\nabla_x\sqrt{\xi}\cdot\sqrt{\xi}u\big)dxdydt\\
&\qquad\qquad + \frac{1}{2}\int_0^T\int_\Omega \big(\xi u\cdot\mathrm{div}_x(\nabla_x^t\Phi) + 2\nabla_x^t\Phi\cdot\nabla_x\sqrt{\xi}\cdot\sqrt{\xi}u\big)dxdydt,
\end{aligned}$$

which implies that

$$\int_0^T\int_\Omega \mathrm{div}_x(\xi_{k_1,r_1}D_x(u_{k_1,r_1}))\Phi dxdydt \quad \underset{k_1,r_1\longrightarrow 0}{\longrightarrow} \quad \int_0^T\int_\Omega \mathrm{div}_x(\xi D_x(u))\Phi dxdydt.$$

At last, we prove (127). Recalling (118), we have

$$\begin{aligned}
&k_1\int_0^T\int_\Omega \xi_{k_1,r_1}\nabla_x\big(\frac{\Delta_x\sqrt{\xi_{k_1,r_1}}}{\sqrt{\xi_{k_1,r_1}}}\big)\Phi dxdydt\\
&= -2k_1\int_0^T\int_\Omega \Delta_x\sqrt{\xi_{k_1,r_1}}\nabla_x\sqrt{\xi_{k_1,r_1}}\Phi dxdydt - k_1\int_0^T\int_\Omega \Delta_x\sqrt{\xi_{k_1,r_1}}\sqrt{\xi_{k_1,r_1}}\mathrm{div}_x\Phi dxdydt\\
&\le 2\sqrt{k_1}\|\sqrt{k_1}\Delta_x\sqrt{\xi_{k_1,r_1}}\|_{L^2\left([0,T];L^2(\Omega)\right)}\|\nabla_x\sqrt{\xi_{k_1,r_1}}\|_{L^2\left([0,T];L^2(\Omega)\right)}\|\Phi\|_{L^\infty\left([0,T];L^\infty(\Omega)\right)}\\
&\qquad + \sqrt{k_1}\|\sqrt{k_1}\Delta_x\sqrt{\xi_{k_1,r_1}}\|_{L^2\left([0,T];L^2(\Omega)\right)}\|\sqrt{\xi_{k_1,r_1}}\|_{L^2\left([0,T];L^2(\Omega)\right)}\|\mathrm{div}_x\Phi\|_{L^\infty\left([0,T];L^\infty(\Omega)\right)}\\
&\longrightarrow 0 \quad \text{as} \quad k_1\longrightarrow 0.
\end{aligned}$$

Passing to the limits in (116) and (117) the energy inequality and the B-D entropy give respectively

$$\begin{aligned}
&\int_\Omega\left(\frac{1}{2}\xi u^2+\xi^2\right)dxdy + r\int_0^T\int_\Omega \xi|u|^3dxdydt + \int_0^T\int_\Omega \xi|\partial_y u|^2dxdydt\\
&+\int_0^T\int_\Omega 2\xi|D_x(u)|^2dxdydt \le \int_\Omega\left(\frac{1}{2}\xi_0u_0^2+\xi_0^2\right)dxdy
\end{aligned}\tag{128}$$

and

$$\begin{aligned}
&\int_\Omega\big(\frac{1}{2}\xi|u+2\nabla_x\ln\xi|^2\big)dxdy + 2\int_0^T\int_\Omega \xi|\partial_y v|^2dxdydt + r\int_0^T\int_\Omega \xi|u|^3dxdydt\\
&+\int_0^T\int_\Omega \xi|\partial_y u|^2dxdy + 2\int_0^T\int_\Omega \xi|A_x(u)|^2dxdydt + 8\int_0^T\int_\Omega \xi|\nabla_x\sqrt{\xi}|^2dxdydt\\
&\le \int_\Omega\left(\xi_0u_0^2+10(\nabla_x\sqrt{\xi_0})^2\right)dxdy + \int_\Omega\left(\frac{1}{2}\xi_0u_0^2+\xi_0^2\right)dxdy + C.
\end{aligned}\tag{129}$$

Passing to the limits in (115) when $k_1\longrightarrow 0$ and $r_1\longrightarrow 0$, the system

$$\begin{cases}
\partial_t\xi+\mathrm{div}_x(\xi u)+\partial_y(\xi v)=0\\
\partial_t(\xi u)+\mathrm{div}_x(\xi u\otimes u)+\partial_y(\xi uv)+\nabla_x\xi^2+r\xi|u|u\\
=2\mathrm{div}_x\big(\xi D_x(u)\big)+\partial_y(\xi\partial_y u)\\
\partial_y\xi=0
\end{cases}\tag{130}$$

holds in the sense of distribution on$[0,T]\times\Omega$. Therefore Theorem 1 is proved.

5.2. Proof of Theorem 2

Let us now consider the weak solution (ξ, u, v) of the system (7). From Theorem 1, we have the following regularities:

$$\begin{cases} \xi \in L^\infty\big([0,T]; H^1(\Omega)\big), \quad \sqrt{\xi}u \in L^\infty\big([0,T]; L^2(\Omega)\big), \\ \xi\nabla_x u \in L^2\big([0,T]; L^2(\Omega)\big), \qquad \xi(\nabla_x u)^t \in L^2\big([0,T]; L^2(\Omega)\big), \\ \sqrt{\xi}\partial_y v \in L^2\big([0,T]; L^2(\Omega)\big), \quad \sqrt{\xi}\partial_y u \in L^2\big([0,T]; L^2(\Omega)\big), \\ \xi^{\frac{1}{3}}u \in L^3\big([0,T]; L^3(\Omega)\big), \quad \sqrt{\xi}v \in L^2\big([0,T]; L^2(\Omega)\big). \end{cases} \tag{131}$$

Recall that $\rho(t,x,y) = \xi(t,x) + \phi(y)$, so we obtain the following properties:

$$\begin{cases} \|\sqrt{\rho}\|_{L^\infty\left([0,T];H^1(\Omega)\right)} \geq \|\sqrt{\xi}\|_{L^\infty\left([0,T];H^1(\Omega)\right)}, \\ \|\sqrt{\rho}u\|_{L^\infty\left([0,T];L^2(\Omega)\right)} \geq \|\sqrt{\xi}u\|_{L^\infty\left([0,T];L^2(\Omega)\right)}, \\ \|\sqrt[3]{\rho}u\|_{L^3\left([0,T];L^3(\Omega)\right)} \geq \|\sqrt[3]{\xi}u\|_{L^3\left([0,T];L^3(\Omega)\right)}, \\ \|\rho D_x u\|_{L^2\left([0,T];L^2(\Omega)\right)} \geq \|\xi D_x u\|_{L^2\left([0,T];L^2(\Omega)\right)}, \\ \|\sqrt{\rho}\partial_y u\|_{L^2\left([0,T];L^2(\Omega)\right)} \geq \|\sqrt{\xi}\partial_y u\|_{L^2\left([0,T];L^2(\Omega)\right)}, \\ \|\partial_y \rho\|_{L^\infty\left([0,T];L^2(\Omega)\right)} = \frac{1}{2}g|\Omega_x|. \end{cases} \tag{132}$$

In addition,

$$\begin{aligned} \|\sqrt{\rho}v\|^2_{L^2\left([0,T];L^2(\Omega)\right)} &= \int_0^T \int_0^1 \int_{\Omega_x} |\sqrt{\rho}v|^2 dxdydt \\ &= \int_0^T \int_0^1 \int_{\Omega_x} \left(\xi v^2 + \phi v^2\right) dxdydt \\ &\leq 2\int_0^T \int_0^1 \int_{\Omega_x} \xi v^2 dxdydt + g\int_0^T \int_0^1 \int_{\Omega_x} v^2 dxdydt \end{aligned} \tag{133}$$

and

$$\begin{aligned} \|\sqrt{\rho}\partial_y v\|^2_{L^2\left([0,T];L^2(\Omega)\right)} &= \int_0^T \int_0^1 \int_{\Omega_x} |\sqrt{\rho}\partial_y v|^2 dxdydt \\ &\leq 2\int_0^T \int_0^1 \int_{\Omega_x} \left(\xi(\partial_y v)^2 + \phi(\partial_y v)^2\right) dxdydt \\ &\leq C\int_0^T \int_0^1 \int_{\Omega_x} (\sqrt{\xi}\partial_y v)^2 dxdydt + g\int_0^T \int_0^1 \int_{\Omega_x} (\partial_y v)^2 dxdydt. \end{aligned} \tag{134}$$

With all of the above estimates, the Theorem 2 is proved, since (ρ, u, v) satisfies the conditions of Definition 2.1. Thus our initial system (1) admits a global weak solution.

6. Conclusion

In this paper, we discussed in dimension $d = 3$, the existence of global weak solutions to the three-dimensional compressible primitive equations of atmospheric dynamics with degenerate viscosity density-dependent for large initial data. We have proven that the weak solutions satisfy the basic energy inequality and the Bresch-Desjardins entropy inequality. We have obtained the global existence of weak solutions of (1) by vanishing the parameters in our approximate system step by step.

References

[1] Didier Bresch and Benoît Desjardins. On the construction of approximate solutions for the 2d viscous shallow water model and for compressible Navier-Stokes models. *J. Math. Pures Appl. (9)*, 86(4):362–368, 2006.

[2] Didier Bresch, Benoît Desjardins, and Chi-Kun Lin. On some compressible fluid models: Korteweg, lubrication, and shallow water systems. *Commun. Partial Differ. Equations*, 28(3-4):843–868, 2003.

[3] Mehmet Ersoy, Timack Ngom, and Mamadou Sy. Compressible primitive equations: formal derivation and stability of weak solutions. *Nonlinearity*, 24(1):79–96, 2011.

[4] Eduard Feireisl. *Dynamics of viscous compressible fluids.*, volume 26 of *Oxf. Lect. Ser. Math. Appl.* Oxford: Oxford University Press, 2004.

[5] Eduard Feireisl, Antonín Novotný, and Hana Petzeltová. On the existence of globally defined weak solutions to the Navier-Stokes equations. *J. Math. Fluid Mech.*, 3(4):358–392, 2001.

[6] Hongjun Gao, Šárka Nečasová, and Tong Tang. On the hydrostatic approximation of compressible anisotropic Navier-Stokes equations. *C. R., Math., Acad. Sci. Paris*, 359(6):639–644, 2021.

[7] M. Gisclon and I. Lacroix-Violet. About the barotropic compressible quantum Navier-Stokes equations. *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods*, 128:106–121, 2015.

[8] Quansen Jiu, Mingjie Li, and Fengchao Wang. Uniqueness of the global weak solutions to 2d compressible primitive equations. *J. Math. Anal. Appl.*, 461(2):1653–1671, 2018.

[9] Ansgar Jüngel. Global weak solutions to compressible Navier-Stokes equations for quantum fluids. *SIAM J. Math. Anal.*, 42(3):1025–1045, 2010.

[10] Jinkai Li and Edriss S. Titi. The primitive equations as the small aspect ratio limit of the Navier-Stokes equations: rigorous justification of the hydrostatic approximation. *J. Math. Pures Appl. (9)*, 124:30–58, 2019.

[11] Xin Liu and Edriss S. Titi. Global existence of weak solutions to the compressible primitive equations of atmospheric dynamics with degenerate viscosities. *SIAM J. Math. Anal.*, 51(3):1913–1964, 2019.

[12] Alessandra Lunardi. *Analytic semigroups and optimal regularity in parabolic problems*, volume 16 of *Prog. Nonlinear Differ. Equ. Appl.* Basel: Birkhäuser, 1995.

[13] A. Mellet and A. Vasseur. On the barotropic compressible Navier-Stokes equations. *Commun. Partial Differ. Equations*, 32(3):431–452, 2007.

[14] Louis Nirenberg. On elliptic partial differential equations. C.I.M.E., Principio di Minimo e sue Applicazioni alle Equazioni funzionali 1-48 (1960)., 1960.

[15] Frédéric Rousset. Weak solutions to the Navier-Stokes equation for compressible fluids [after A. Vasseur and C. Yu]. In *Séminaire Bourbaki. Volume 2016/2017. Exposés 1120–1135*, page ex. Société Mathématique de France (SMF), 2019.

[16] Jacques Simon. Compact sets in the space $L^p(0,T;B)$. *Ann. Mat. Pura Appl. (4)*, 146:65–96, 1987.

[17] Alexis F. Vasseur and Cheng Yu. Existence of global weak solutions for 3d degenerate compressible Navier-Stokes equations. *Invent. Math.*, 206(3):935–974, 2016.

[18] Alexis F. Vasseur and Cheng Yu. Global weak solutions to the compressible quantum Navier-Stokes equations with damping. *SIAM J. Math. Anal.*, 48(2):1489–1511, 2016.

[19] Fengchao Wang, Changsheng Dou, and Quansen Jiu. Global existence of weak solutions to 3d compressible primitive equations with degenerate viscosity. *J. Math. Phys.*, 61(2):021507, 33, 2020.