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Ideals of BCK-algebras and BCI-algebras based on a new form of fuzzy set

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Abstract. Ideals in BCK/BCI algebra based on Y_J^ε-fuzzy sets are studied. The fundamental properties of the level set of Y_J^ε-fuzzy sets are investigate first. The concept of (closed) Y_J^ε-fuzzy ideals in BCK/BCI-algebras is introduces, and several properties are investigated. The relationship between Y_J^ε-fuzzy ideal and Y_J^ε-fuzzy subalgebra are discussed, and also the relationship between Y_J^ε-fuzzy ideal and fuzzy ideal is identified. The characterization of (closed) Y_J^ε-fuzzy ideal using the Y-level set is established. The necessary and sufficient conditions for Y_J^ε-fuzzy ideal to be closed is explored, and conditions for Y_J^ε-fuzzy subalgebra to be Y_J^ε-fuzzy ideal are provided.

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1. Introduction

Fuzzy sets, which are introduced by Zadeh [14], are mathematical frameworks that are very useful in expressing and manipulating uncertainty and ambiguity of data with applications such as pattern recognition, decision making, control systems, image processing, data mining, expert systems, natural language processing, risk assessment and decision analysis, etc. Various studies have been conducted since the study of fuzzy sets in BCK-algebra began in 1991 (see [1, 5, 7–10]). Jun [6] introduce the notion of the J-operator in the closed interval $[0, 1]$ and investigate several properties. He used the J-operator to create a new fuzzy set called the Y_J^ε-fuzzy set and applied it to subalgebras in BCK/BCI-algebras. He introduced the concept of the Y_J^ε-fuzzy subalgebra and investigated its properties. He provided conditions for a fuzzy set to be a Y_J^ε-fuzzy subalgebra, and discussed the relationship between the fuzzy subalgebra and the Y_J^ε-fuzzy subalgebra.

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In this paper, we study the ideals of BCK/BCI-algebras based on Y_J^ε-fuzzy sets. We first investigate the underlying properties of the level sets of Y_J^ε-fuzzy sets. We introduce the concept of Y_J^ε-fuzzy ideals in BCK/BCI-algebras, and investigate several properties. We discuss the relationship between Y_J^ε-fuzzy ideal and Y_J^ε-fuzzy subalgebra, and also identify the relationship between Y_J^ε-fuzzy ideal and fuzzy ideal. We consider the characterization of Y_J^ε-fuzzy ideal using the Y-level set. We define closed Y_J^ε-fuzzy ideal, and deal with its properties. We explore the necessary and sufficient conditions for Y_J^ε-fuzzy ideal to be closed. Finally, we provide conditions for Y_J^ε-fuzzy subalgebra to be Y_J^ε-fuzzy ideal.

2. Preliminaries

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki (see [3] and [4]) and was extensively investigated by several researchers.

We recall the definitions and basic results required in this paper. See the books [2, 11] for further information regarding BCK/BCI-algebras.

By a *BCI-algebra*, we mean a structure $(X, *, 0)$, where 0 is a special element and $*$ is a binary operation on X, that satisfies the following conditions:

(I) $((a * b) * (a * c)) * (c * b) = 0$,

(II) $(a * (a * b)) * b = 0$,

(III) $a * a = 0$,

(IV) $a * b = 0,\ b * a = 0 \ \Rightarrow\ a = b$,

for all $a, b, c \in X$. If a BCI-algebra $(X, *, 0)$ satisfies the following identity:

(V) $(\forall a \in X)\ (0 * a = 0)$,

then $(X, *, 0)$ is called a *BCK-algebra.*

The order relation "$\leq_X$" in a BCK/BCI-algebra $(X, *, 0)$ is defined as follows:

$$(\forall a, b \in X)(a \leq_X b \ \Leftrightarrow\ a * b = 0). \tag{1}$$

Every BCK/BCI-algebra $(X, *, 0)$ satisfies the following conditions:

$$a * 0 = a, \tag{2}$$

$$a \leq_X b \Rightarrow a * c \leq_X b * c,\ c * b \leq_X c * a, \tag{3}$$

$$(a * b) * c = (a * c) * b, \tag{4}$$

for all $a, b, c \in X$.

Every BCK-algebra $(X, *, 0)$ satisfies:

$$(\forall x, a \in X)(x * a \leq_X x). \tag{5}$$

Every BCI-algebra $(X, *, 0)$ satisfies:

$$(\forall a, b \in X)(0 * (a * b) = (0 * a) * (0 * b)). \tag{6}$$

A BCI-algebra $(X, *, 0)$ is said to be *p-semisimple* if $0 * (0 * x) = x$ for all $x \in X$ (see [2]).

A nonempty subset S of X is called a *subalgebra* of a BCK/BCI-algebra $(X, *, 0)$ (see [11]) if $a * y \in S$ for all $a, y \in S$. A subset A of X is called an *ideal* of a BCK/BCI-algebra $(X, *, 0)$ (see [11]) if it satisfies:

$$0 \in A, \tag{7}$$

$$(\forall a \in X)\,(\forall y \in A)\,(a * y \in A \;\Rightarrow\; a \in A)\,. \tag{8}$$

An ideal A of a BCI-algebra $(X, *, 0)$ is said to be *closed* (see [2, 11]) if it is also a subalgebra of $(X, *, 0)$.

Note that an ideal A of a BCI-algebra $(X, *, 0)$ is closed if and only if $0 * a \in A$ for all $a \in A$ (see [2, Proposition 1.4.4]).

Every ideal A of a BCK/BCI-algebra $(X, *, 0)$ satisfies the next assertion.

$$(\forall a, y \in X)\,(a \leq_X y,\; y \in A \;\Rightarrow\; a \in A)\,. \tag{9}$$

A *fuzzy set* in a set X is defined to be a function $\zeta : X \to [0, 1]$. Denote by $FS(X)$ the collection of all fuzzy sets in X. Define a relation " $\subseteq$ " on $FS(X)$ by

$$(\forall \zeta, \xi \in FS(X))(\zeta \subseteq \xi \;\Leftrightarrow\; (\forall a \in X)(\zeta(a) \leq \xi(a))).$$

The *join* $(\vee)$ and *meet* $(\wedge)$ of ζ and ξ are defined by

$$(\zeta \vee \xi)(a) = \max\{\zeta(a), \xi(a)\},$$
$$(\zeta \wedge \xi)(a) = \min\{\zeta(a), \xi(a)\},$$

respectively, for all $a \in X$. The *complement* of ζ, denoted by ζ^c, is defined by

$$(\forall a \in X)(\zeta^c(a) = 1 - \zeta(a)).$$

A fuzzy set ζ in a set X of the form

$$\zeta(b) := \begin{cases} t \in (0, 1] & \text{if } b = a, \\ 0 & \text{if } b \neq a, \end{cases}$$

is said to be a *fuzzy point* with support a and value t and is denoted by $\langle a_t \rangle$.

For a fuzzy set ζ in a set X, we say that a fuzzy point $\langle a_t \rangle$ is

(i) *contained* in ζ, denoted by $\langle a_t \rangle \in \zeta$, (see [12]) if $\zeta(a) \geq t$.

(ii) *quasi-coincident* with ζ, denoted by $\langle a_t \rangle \, q \, \zeta$, (see [12]) if $\zeta(a) + t > 1$.

If a fuzzy point $\langle a_t \rangle$ is contained in ζ or is quasi-coincident with ζ, we denote it $\langle a_t \rangle \in\!\vee q\, \zeta$. If $\langle a_t \rangle\, \alpha\, \zeta$ is not established for $\alpha \in \{\in, q, \in\!\vee q\}$, it is denoted by $\langle a_t \rangle\, \overline{\alpha}\, \zeta$.

Given $t \in (0, 1]$ and a fuzzy set ζ in a set X, consider the following sets

$$(\zeta, t)_\in := \{a \in X \mid \langle a_t \rangle \in \zeta\} \text{ and } (\zeta, t)_q := \{a \in X \mid \langle a_t \rangle\, q\, \zeta\}$$

which are called the *level set* and the *q-set* of ζ related to t, respectively, in X.

Also, we consider the set

$$(\zeta, t)_{\in\vee q} := \{a \in X \mid \langle a_t \rangle \in\!\vee q\; \zeta\}$$

which is called the $\in\!\vee q$ *-set* of ζ related to t.

It is clear that $(\zeta, t)_{\in\vee q} = (\zeta, t)_\in \cup (\zeta, t)_q$ and $(\zeta, t)_q \subseteq (\zeta, s)_q$ for all $t, s \in (0, 1]$ with $t \leq s$.

A fuzzy set ζ in X is called a *fuzzy subalgebra* of a BCK/BCI-algebra $(X, *, 0)$ (see [13]) if it satisfies:

$$(\forall x, a \in X)(\zeta(x * a) \geq \zeta(x) \wedge \zeta(a)). \tag{10}$$

A fuzzy set ζ in X is called a *fuzzy ideal* of a BCK/BCI-algebra $(X, *, 0)$ (see [13]) if it satisfies:

$$(\forall x \in X)(\zeta(0) \geq \zeta(x)), \tag{11}$$

$$(\forall x, a \in X)(\zeta(x) \geq \zeta(x * a) \wedge \zeta(a)). \tag{12}$$

In [6], Jun introduced the notion of Y_J^ε-fuzzy sets based on the J-operator in the closed interval $[0, 1]$. We display the basic notions about the Y_J^ε-fuzzy sets.

We use the notation I instead of the closed interval $[0, 1]$. Let "$\ll$" be the order relation in I^2 defined as follows:

$$(\forall (m, n), (j, i) \in I^2)((m, n) \ll (j, i) \;\Leftrightarrow\; m \leq j,\, n \leq i)$$

For every $m, \varepsilon \in I$, we define $m \wedge \varepsilon := \min\{m, \varepsilon\}$ and $m \vee \varepsilon := \max\{m, \varepsilon\}$.

Consider a binary operation Y_J in I given as follows:

$$Y_J : I^2 \to I,\; (m, \varepsilon) \mapsto (1 - m) \wedge (1 - \varepsilon).$$

We will call this binary operation Y_J the *J-operator* in I (see [6]).

Let X be a set. Given a fuzzy set ζ in X and $\varepsilon \in I$, let $\varepsilon(\zeta)$ be a mapping defined by

$$\varepsilon(\zeta) : X \to I,\; x \mapsto Y_J(\varepsilon, \zeta(x)).$$

It is clear that $\varepsilon(\zeta)$ is a fuzzy set in X determined by the J-operator and ε. So we can say that $\varepsilon(\zeta)$ is a Y_J^ε*-fuzzy set* of ζ in X (see [6]).

Given a fuzzy set ζ in X and $\varepsilon \in (0, 1)$, if the Y_J^ε-fuzzy set $\varepsilon(\zeta)$ of ζ is not constant on X, then ε is said to be a *nonconstant factor* in $(0, 1)$ (see [6]).

A fuzzy set ζ in X is called a Y_J^ε*-fuzzy subalgebra* of $(X, *, 0)$ (see [6]) if it satisfies:

$$(\forall x, a \in X)(Y_J(\varepsilon, \zeta(x * a)) \leq Y_J(\varepsilon, \zeta(x)) \vee Y_J(\varepsilon, \zeta(a))). \tag{13}$$

3. Level sets of the Y_J^ε-fuzzy set

Let ζ be a fuzzy set in X, $\varepsilon \in I$ and $t \in I \setminus \{0,1\}$. Given a Y_J^ε-fuzzy set $\varepsilon(\zeta)$, we consider the sets:

$$\begin{aligned} \varepsilon(\zeta)^t &:= \{x \in X \mid Y_J(\varepsilon, \zeta(x)) \leq t\}, \\ \varepsilon(\zeta)_q^t &:= \{x \in X \mid Y_J(\varepsilon, \zeta(x)) < 1 - t\}, \end{aligned}$$

which is called the *Y-level set* and *Yq-set* of $\varepsilon(\zeta)$, respectively, related to t. We call t the *level degree* of $\varepsilon(\zeta)$.

The Y-level set and the Yq-set of $\varepsilon(\zeta)$ related to t are calculated as follows:

$$\begin{aligned} \varepsilon(\zeta)^t &= \{x \in X \mid Y_J(\varepsilon, \zeta(x)) \leq t\} \\ &= \{x \in X \mid (1-\varepsilon) \wedge (1-\zeta(x)) \leq t\} \\ &= \{x \in X \mid 1 - (\varepsilon \vee \zeta(x)) \leq t\} \\ &= \{x \in X \mid \varepsilon \vee \zeta(x) \geq 1 - t\} \end{aligned}$$

and

$$\begin{aligned} \varepsilon(\zeta)_q^t &= \{x \in X \mid Y_J(\varepsilon, \zeta(x)) < 1 - t\} \\ &= \{x \in X \mid (1-\varepsilon) \wedge (1-\zeta(x)) < 1 - t\} \\ &= \{x \in X \mid \varepsilon \vee \zeta(x) > t\}, \end{aligned}$$

respectively. The set

$$\varepsilon(\zeta)_{\in\vee q}^t := \{x \in X \mid Y_J(\varepsilon, \zeta(x)) \leq t \text{ or } Y_J(\varepsilon, \zeta(x)) < 1 - t\}$$

is called the the *Y*$\in\vee q$ *-set* of $\varepsilon(\zeta)$ related to t. It is clear that

$$\varepsilon(\zeta)_{\in\vee q}^t = \varepsilon(\zeta)^t \cup \varepsilon(\zeta)_q^t.$$

Proposition 1. *Let ζ be a fuzzy set in X and $\varepsilon \in I$ that satisfies $\varepsilon \leq \zeta(x)$ for all $x \in X$. Then $\varepsilon(\zeta)^t = (\zeta, t)_q \cup \zeta_t^1$ where $\zeta_t^1 := \{x \in X \mid \zeta(x) + t = 1\}$, and $\varepsilon(\zeta)_q^t \subseteq (\zeta, t)_\in$.*

Proof. Straightforwad.

Proposition 2. *Let ζ be a fuzzy set in X and $\varepsilon \in I$. If $s \geq t$ in $I \setminus \{0,1\}$, then $\varepsilon(\zeta)^t \subseteq \varepsilon(\zeta)^s$ and $\varepsilon(\zeta)_q^t \supseteq \varepsilon(\zeta)_q^s$.*

Proof. Straightforward.

4. Y_J^ε-fuzzy ideals

We begin this section by looking at a characterization of Y_J^ε-fuzzy subalgebra by Y-level set. In what follows, let $(X, *, 0)$ be a BCK-algebra or a BCI-algebra, and $\varepsilon \in (0,1)$ unless otherwise specified.

Theorem 1. *A fuzzy set ζ in X is a Y_J^ε-fuzzy subalgebra of $(X, *, 0)$ if and only if the nonempty Y-level set $\varepsilon(\zeta)^t$ of $\varepsilon(\zeta)$ is a subalgebra of $(X, *, 0)$ for all $t \in I \setminus \{0,1\}$.*

Proof. Assume that ζ is a Y_J^ε-fuzzy subalgebra of $(X, *, 0)$ and let $t \in I \setminus \{0,1\}$ be such that $\varepsilon(\zeta)^t \neq \emptyset$. Let $x, y \in \varepsilon(\zeta)^t$. Then $Y_J(\varepsilon, \zeta(x)) \leq t$ and $Y_J(\varepsilon, \zeta(y)) \leq t$, which imply from (13) that

$$Y_J(\varepsilon, \zeta(x * y)) \leq Y_J(\varepsilon, \zeta(x)) \vee Y_J(\varepsilon, \zeta(y)) \leq t.$$

Hence $x * y \in \varepsilon(\zeta)^t$, and therefore $\varepsilon(\zeta)^t$ is a subalgebra of $(X, *, 0)$.

Conversely, suppose that the nonempty Y-level set $\varepsilon(\zeta)^t$ is a subalgebra of $(X, *, 0)$ for all $t \in I \setminus \{0,1\}$. If (13) is not valid, then

$$Y_J(\varepsilon, \zeta(b * c)) > t \geq Y_J(\varepsilon, \zeta(b)) \vee Y_J(\varepsilon, \zeta(c))$$

for some $b, c \in X$ and $t \in I \setminus \{0,1\}$. Hence $b, c \in \varepsilon(\zeta)^t$ and $b * c \notin \varepsilon(\zeta)^t$, which is a contradiction. Therefore $Y_J(\varepsilon, \zeta(x * a)) \leq Y_J(\varepsilon, \zeta(x)) \vee Y_J(\varepsilon, \zeta(a))$ for all $x, a \in X$, which shows that ζ is a Y_J^ε-fuzzy subalgebra of $(X, *, 0)$.

Definition 1. *A fuzzy set ζ in X is called a Y_J^ε-fuzzy ideal if it satisfies:*

$$(\forall x \in X)(Y_J(\varepsilon, \zeta(0)) \leq Y_J(\varepsilon, \zeta(x))), \tag{14}$$

$$(\forall x, a \in X)(Y_J(\varepsilon, \zeta(x)) \leq Y_J(\varepsilon, \zeta(x * a)) \vee Y_J(\varepsilon, \zeta(a))). \tag{15}$$

Example 1. *Let $X = \{0, 1, 2, a, b\}$ be a set with the binary operation "$*$" given by Table 1.*

Table 1: Cayley table for the binary operation "$*$"

$*$	0	1	2	a	b
0	0	0	0	a	a
1	1	0	0	a	a
2	2	2	0	b	a
a	a	a	a	0	0
b	b	b	a	2	0

Then $(X, *, 0)$ is a BCI-algebra (see [2]). Define a fuzzy set ζ in X as follows:

$$\zeta : X \to [0,1],\ y \mapsto \begin{cases} 0.68 & \text{if } y = 0, \\ 0.61 & \text{if } y = 1, \\ 0.46 & \text{if } y = 2, \\ 0.54 & \text{if } y = a, \\ 0.46 & \text{if } y = b. \end{cases}$$

It is routine to verify that ζ is a Y_J^ε-fuzzy ideal of $(X, *, 0)$ for all $\varepsilon \in (0,1)$.

Proposition 3. *Every Y_J^ε-fuzzy ideal ζ of $(X, *, 0)$ satisfies:*

$$(\forall x, a \in X)(x \leq_X a \ \Rightarrow\ Y_J(\varepsilon, \zeta(x)) \leq Y_J(\varepsilon, \zeta(a))). \tag{16}$$

$$(\forall x, a, y \in X)(x * a \leq_X y \ \Rightarrow\ Y_J(\varepsilon, \zeta(x)) \leq Y_J(\varepsilon, \zeta(a)) \vee Y_J(\varepsilon, \zeta(y))). \tag{17}$$

Proof. Let ζ be a Y_J^ε-fuzzy ideal of $(X, *, 0)$ and let $x, a \in X$ be such that $x \leq_X a$. Then $x * a = 0$, and so

$$\begin{aligned} Y_J(\varepsilon, \zeta(x)) &\leq Y_J(\varepsilon, \zeta(x * a)) \vee Y_J(\varepsilon, \zeta(a)) \\ &= Y_J(\varepsilon, \zeta(0)) \vee Y_J(\varepsilon, \zeta(a)) \\ &= Y_J(\varepsilon, \zeta(a)) \end{aligned}$$

by (14) and (15). Thus (16) is valid. Let $x, a, y \in X$ be such that $x * a \leq_X y$. Then

$$\begin{aligned} Y_J(\varepsilon, \zeta(x * a)) &\leq Y_J(\varepsilon, \zeta((x * a) * y)) \vee Y_J(\varepsilon, \zeta(y)) \\ &= Y_J(\varepsilon, \zeta(0)) \vee Y_J(\varepsilon, \zeta(y)) \\ &= Y_J(\varepsilon, \zeta(y)), \end{aligned}$$

and thus $Y_J(\varepsilon, \zeta(x)) \leq Y_J(\varepsilon, \zeta(x * a)) \vee Y_J(\varepsilon, \zeta(a)) \leq Y_J(\varepsilon, \zeta(y)) \vee Y_J(\varepsilon, \zeta(a))$. This completes the proof.

Corollary 1. *If ζ is a fuzzy ideal of $(X, *, 0)$, then its Y_J^ε-fuzzy set $\varepsilon(\zeta)$ satisfies:*

$$(\forall x, a \in X)(x \leq_X a \ \Rightarrow\ \varepsilon(\zeta)(x) \leq \varepsilon(\zeta)(a)).$$
$$(\forall x, a, y \in X)(x * a \leq_X y \ \Rightarrow\ \varepsilon(\zeta)(x) \leq \varepsilon(\zeta)(a) \vee \varepsilon(\zeta)(y)).$$

Theorem 2. *In a BCK-algebra $(X, *, 0)$, every Y_J^ε-fuzzy ideal is a Y_J^ε-fuzzy subalgebra for all $\varepsilon \in (0,1)$.*

Proof. Let ζ be a Y_J^ε-fuzzy ideal of a BCK-algebra $(X, *, 0)$ for all $\varepsilon \in (0,1)$. The combination of (5) and (16) induces $Y_J(\varepsilon, \zeta(x * a)) \leq Y_J(\varepsilon, \zeta(x))$, and so

$$\begin{aligned} Y_J(\varepsilon, \zeta(x * a)) \leq Y_J(\varepsilon, \zeta(x)) &\leq Y_J(\varepsilon, \zeta(x * a)) \vee Y_J(\varepsilon, \zeta(a)) \\ &\leq Y_J(\varepsilon, \zeta(x)) \vee Y_J(\varepsilon, \zeta(a)). \end{aligned}$$

Therefore ζ is a Y_J^ε-fuzzy subalgebra of $(X, *, 0)$.

In a BCI-algebra, Theorem 2 may not be true as seen in the following example.

Example 2. *Let* $(X, *, 0)$ *be a BCI-algebra and* $(\mathbb{Z}, -, 0)$ *the adjoint BCI-algebra of the additive group* $(\mathbb{Z}, +, 0)$ *of integers. Then* $(Y, \circledast, (0,0))$ *is a BCI-algebra (see [2]) where* $Y = X \times \mathbb{Z}$ *and* $\circledast$ *is a binary operation in* Y *given as follows:*

$$(\forall (x,a),(y,b) \in Y)((x,a) \circledast (y,b) = (x * y, a - b)).$$

Define a fuzzy set ζ *in* Y *as follows:*

$$\zeta : Y \to [0,1],\ c \mapsto \begin{cases} 0.87 & \text{if } c = (0,0), \\ 0.73 & \text{if } c \in X \times \mathbb{N}_0, \\ 0.42 & \text{otherwise} \end{cases}$$

wher $\mathbb{N}_0$ *is the set of all nonnegative integes. It is routine to verify that* ζ *is a* Y_J^{ε}*-fuzzy ideal of* $(Y, \circledast, (0,0))$ *for* $\varepsilon = 0.61$*. We can observe that*

$$\begin{aligned} &Y_J(\varepsilon, \zeta((0,3) \circledast (0,7))) = Y_J(0.61, \zeta(0,-4)) \\ &= (1 - 0.61) \wedge (1 - 0.42) = 0.39 \end{aligned}$$

and

$$\begin{aligned} &Y_J(\varepsilon, \zeta(0,3)) \vee Y_J(\varepsilon, \zeta(0,7)) \\ &= ((1 - 0.61) \wedge (1 - 0.73)) \vee ((1 - 0.61) \wedge (1 - 0.73)) = 0.27. \end{aligned}$$

Hence $Y_J(\varepsilon, \zeta((0,3) \circledast (0,7))) \not\leq Y_J(\varepsilon, \zeta(0,3)) \vee Y_J(\varepsilon, \zeta(0,7))$ *for* $\varepsilon = 0.61$*, which shows that* ζ *is not a* Y_J^{ε}*-fuzzy subalgebra of* $(Y, \circledast, (0,0))$*.*

The following example shows that there exists $\varepsilon \in (0,1)$ such that a Y_J^{ε}-fuzzy subalgebra may not be a Y_J^{ε}-fuzzy ideal.

Example 3. *(i) Let* $X = \{0, b_1, b_2, b_3\}$ *be a set with a binary operation "$*$" given by Table 2.*

Table 2: Cayley table for the binary operation " $*$ "

$*$	0	b_1	b_2	b_3
0	0	0	0	0
b_1	b_1	0	0	b_1
b_2	b_2	b_1	0	b_2
b_3	b_3	b_3	b_3	0

Then X *is a BCK-algebra (see [11]). A fuzzy set* ζ *in* X *defined by*

$$\zeta : X \to [0,1],\ x \mapsto \begin{cases} 0.63 & \text{if } x = 0, \\ 0.54 & \text{if } x = b_1, \\ 0.42 & \text{if } x = b_2, \\ 0.49 & \text{if } x = b_3 \end{cases}$$

is a Y_J^ε-fuzzy subalgebra of $(X, *, 0)$ for $\varepsilon = 0.52$. But it is not a Y_J^ε-fuzzy ideal of $(X, *, 0)$ for $\varepsilon = 0.52$ since

$$\begin{aligned} Y_J(\varepsilon, \zeta(b_2)) &= Y_J(0.52, 0.42) = (1 - 0.52) \wedge (1 - 0.42) \\ &= 0.48 \nleq 0.46 = (1 - 0.52) \wedge (1 - 0.54) \\ &= ((1 - 0.52) \wedge (1 - 0.54)) \vee ((1 - 0.52) \wedge (1 - 0.54)) \\ &= Y_J(0.52, \zeta(b_1)) \vee Y_J(0.52, \zeta(b_1)) \\ &= Y_J(\varepsilon, \zeta(b_2 * b_1)) \vee Y_J(\varepsilon, \zeta(b_1)). \end{aligned}$$

(ii) Consider the BCI-algebra $(X, *, 0)$ in Example 1 and let ζ be a fuzzy set in X given as follows:

$$\zeta : X \to [0, 1],\ y \mapsto \begin{cases} 0.78 & \text{if } y = 0, \\ 0.54 & \text{if } y = 1, \\ 0.37 & \text{if } y = 2, \\ 0.65 & \text{if } y = a, \\ 0.37 & \text{if } y = b. \end{cases}$$

Then ζ is a Y_J^ε-fuzzy subalgebra of $(X, *, 0)$ for $\varepsilon = 0.49$. We can observe that

$$Y_J(\varepsilon, \zeta(1)) = Y_J(0.49, 0.54) = (1 - 0.49) \wedge (1 - 0.54) = 0.46$$

and

$$\begin{aligned} &Y_J(\varepsilon, \zeta(1 * a)) \vee Y_J(\varepsilon, \zeta(a)) = Y_J(\varepsilon, \zeta(a)) \vee Y_J(\varepsilon, \zeta(a)) \\ &= Y_J(\varepsilon, \zeta(a)) = Y_J(0.49, 0.65) = (1 - 0.49) \wedge (1 - 0.65) = 0.35. \end{aligned}$$

Hence $Y_J(\varepsilon, \zeta(1)) \nleq Y_J(\varepsilon, \zeta(1 * a)) \vee Y_J(\varepsilon, \zeta(a))$, and therefore ζ is not a Y_J^ε-fuzzy ideal of $(X, *, 0)$ for $\varepsilon = 0.49$.

Theorem 3. *Let ζ be a fuzzy set in X. If $\zeta(x) \leq \varepsilon$ for all $x \in X$, then ζ is a Y_J^ε-fuzzy ideal of $(X, *, 0)$.*

Proof. Let ζ be a fuzzy set in X that satisfies $\zeta(x) \leq \varepsilon$ for all $x \in X$. Then $1 - \varepsilon \leq 1 - \zeta(x)$ for all $x \in X$. Hence

$$Y_J(\varepsilon, \zeta(0)) = (1 - \varepsilon) \wedge (1 - \zeta(0)) = 1 - \varepsilon = (1 - \varepsilon) \wedge (1 - \zeta(x)) = Y_J(\varepsilon, \zeta(x))$$

for all $x \in X$. Also, we have

$$Y_J(\varepsilon, \zeta(x)) = 1 - \varepsilon = Y_J(\varepsilon, \zeta(x * a)) \vee Y_J(\varepsilon, \zeta(a))$$

for all $x, a \in X$. Therefore ζ is a Y_J^ε-fuzzy ideal of $(X, *, 0)$.

Let ζ be a fuzzy set in X. If there exists $z \in X$ that satisfies $\zeta(z) > \varepsilon$, then ζ may not be a Y_J^ε-fuzzy ideal of $(X, *, 0)$ as shown in the example below.

Table 3: Cayley table for the binary operation “ $*$ ”

$*$	0	b_1	b_2	b_3	b_4
0	0	0	0	0	0
b_1	b_1	0	0	0	b_1
b_2	b_2	b_1	0	0	b_2
b_3	b_3	b_1	b_1	0	b_3
b_4	b_4	b_4	b_4	b_4	0

Example 4. *Let $X = \{0, b_1, b_2, b_3, b_4\}$ be a set with a binary operation "$*$" given by Table 3.*
Then $(X, *, 0)$ is a BCK-algebra and so a BCI-algebra (see [11]). Consider a fuzzy set ζ in X given as follows:

$$\zeta : X \to [0,1],\ y \mapsto \begin{cases} 0.93 & \text{if} \quad y = 0, \\ 0.46 & \text{if} \quad y = b_1, \\ 0.77 & \text{if} \quad y = b_2, \\ 0.58 & \text{if} \quad y = b_3, \\ 0.35 & \text{if} \quad y = b_4. \end{cases}$$

If $\varepsilon := 0.53$, then $Y_J(\varepsilon, \zeta(0)) \leq Y_J(\varepsilon, \zeta(x))$ for all $x \in X$. But

$$\begin{aligned} &Y_J(\varepsilon, \zeta(b_1 * b_3)) \vee Y_J(\varepsilon, \zeta(b_3)) = Y_J(0.53, 0.93) \vee Y_J(0.53, 0.58) \\ &= 0.07 \vee 0.42 = 0.42 < 0.47 = Y_J(\varepsilon, \zeta(b_1)). \end{aligned}$$

Hence ζ is not a Y_J^ε-fuzzy ideal of $(X, *, 0)$ for $\varepsilon = 0.53$.

Theorem 4. *Every fuzzy ideal of $(X, *, 0)$ is a Y_J^ε-fuzzy ideal of $(X, *, 0)$ for all $\varepsilon \in (0, 1)$.*

Proof. Let ζ be a fuzzy ideal of $(X, *, 0)$ and let $\varepsilon \in (0, 1)$. Then $\zeta^c(0) \leq \zeta^c(x)$ and $\zeta^c(x) \leq \zeta^c(x * a) \vee \zeta^c(a)$ for all $x, a \in X$. Hence

$$Y_J(\varepsilon, \zeta(0)) = (1 - \varepsilon) \wedge \zeta^c(0) \leq (1 - \varepsilon) \wedge \zeta^c(x) = Y_J(\varepsilon, \zeta(x))$$

and

$$\begin{aligned} Y_J(\varepsilon, \zeta(x)) &= (1 - \varepsilon) \wedge \zeta^c(x) \\ &\leq (1 - \varepsilon) \wedge (\zeta^c(x * a) \vee \zeta^c(a)) \\ &= ((1 - \varepsilon) \wedge \zeta^c(x * a)) \vee ((1 - \varepsilon) \wedge \zeta^c(a)) \\ &= Y_J(\varepsilon, \zeta(x * a)) \vee Y_J(\varepsilon, \zeta(a)) \end{aligned}$$

for all $x, a \in X$. Therefore ζ is a Y_J^ε-fuzzy ideal of $(X, *, 0)$ for all $\varepsilon \in (0, 1)$.

Theorem 5. *If ζ is a Y_J^ε-fuzzy ideal of $(X, *, 0)$ for some nonconstant factor $\varepsilon \in (0,1)$, then it is a fuzzy ideal of $(X, *, 0)$.*

Proof. Assume that ζ is a Y_J^ε-fuzzy ideal of $(X, *, 0)$ for some nonconstant factor $\varepsilon \in (0,1)$. Then

$$(1-\varepsilon) \wedge (1-\zeta(0)) = Y_J(\varepsilon, \zeta(0)) \leq Y_J(\varepsilon, \zeta(x)) = (1-\varepsilon) \wedge (1-\zeta(x))$$

for all $x \in X$. Hence $1-\zeta(0) \leq 1-\zeta(x)$, and so $\zeta(0) \geq \zeta(x)$ for all $x \in X$. For every $x, a \in X$, we have

$$\begin{aligned}(1-\varepsilon) \wedge (1-\zeta(x)) &= Y_J(\varepsilon, \zeta(x)) \leq Y_J(\varepsilon, \zeta(x*a)) \vee Y_J(\varepsilon, \zeta(a)) \\ &= ((1-\varepsilon) \wedge (1-\zeta(x*a))) \vee ((1-\varepsilon) \wedge (1-\zeta(a))) \\ &= (1-\varepsilon) \wedge ((1-\zeta(x*a)) \vee (1-\zeta(a))).\end{aligned}$$

It follows that $1-\zeta(x) \leq ((1-\zeta(x*a)) \vee (1-\zeta(a))) = 1-(\zeta(x*a) \wedge \zeta(a))$. Thus $\zeta(x) \geq \zeta(x*a) \wedge \zeta(a)$. Therefore ζ is a fuzzy ideal of $(X, *, 0)$.

Theorem 6. *A fuzzy set ζ in X is a Y_J^ε-fuzzy ideal of $(X, *, 0)$ if and only if the nonempty Y-level set $\varepsilon(\zeta)^t$ of $\varepsilon(\zeta)$ is an ideal of $(X, *, 0)$ for all $t \in I \setminus \{0,1\}$*

Proof. Assume that ζ is a Y_J^ε-fuzzy ideal of $(X, *, 0)$ and let $t \in I \setminus \{0,1\}$ be such that $\varepsilon(\zeta)^t \neq \emptyset$. If $0 \notin \varepsilon(\zeta)^t$, then $Y_J(\varepsilon, \zeta(0)) > t \geq Y_J(\varepsilon, \zeta(b))$ for some $b \in X$, which contradicts (14). Hence $0 \notin \varepsilon(\zeta)^t$. Let $x, y \in X$ be such that $x*y \in \varepsilon(\zeta)^t$ and $y \in \varepsilon(\zeta)^t$. Then $Y_J(\varepsilon, \zeta(x*y)) \leq t$ and $Y_J(\varepsilon, \zeta(y)) \leq t$. It follows from (15) that

$$Y_J(\varepsilon, \zeta(x)) \leq Y_J(\varepsilon, \zeta(x*y)) \vee Y_J(\varepsilon, \zeta(y)) \leq t.$$

Hence $x \in \varepsilon(\zeta)^t$, which shows that $\varepsilon(\zeta)^t$ is an ideal of $(X, *, 0)$.

Conversely, suppose that the nonempty Y-level set $\varepsilon(\zeta)^t$ of $\varepsilon(\zeta)$ is an ideal of $(X, *, 0)$ for all $t \in I \setminus \{0,1\}$. If there exists $c \in X$ such that $Y_J(\varepsilon, \zeta(0)) > Y_J(\varepsilon, \zeta(c))$, then $Y_J(\varepsilon, \zeta(0)) > t \geq Y_J(\varepsilon, \zeta(c))$ for some $t \in I \setminus \{0,1\}$. It follows that $c \in \varepsilon(\zeta)^t$, that is, $\varepsilon(\zeta)^t \neq \emptyset$. Hence $0 \in \varepsilon(\zeta)^t$, and so $Y_J(\varepsilon, \zeta(0)) \leq t$, which is a contradiction. Thus $Y_J(\varepsilon, \zeta(0)) \leq Y_J(\varepsilon, \zeta(x))$ for all $x \in X$. Suppose that (15) is not valid. Then

$$Y_J(\varepsilon, \zeta(x)) > t \geq Y_J(\varepsilon, \zeta(x*a)) \vee Y_J(\varepsilon, \zeta(a))$$

for some $x, a \in X$ and $t \in I \setminus \{0,1\}$. It follows that $x*a \in \varepsilon(\zeta)^t$ and $a \in \varepsilon(\zeta)^t$, but $x \notin \varepsilon(\zeta)^t$. This is a contradiction, and thus (15) is valid. Therefore ζ is a Y_J^ε-fuzzy ideal of $(X, *, 0)$.

We provide conditions for Y_J^ε-fuzzy subalgebra to be Y_J^ε-fuzzy ideal.

Theorem 7. *If a Y_J^ε-fuzzy subalgebra ζ of $(X, *, 0)$ satisfies the condition (17), then it is a Y_J^ε-fuzzy ideal of $(X, *, 0)$.*

Proof. Let ζ be a Y_J^ε-fuzzy subalgebra ζ of $(X, *, 0)$ that satisfies the condition (17). The combination of (III) and (13) induces $Y_J(\varepsilon, \zeta(0)) \leq Y_J(\varepsilon, \zeta(x))$ for all $x \in X$. For every $x, y \in X$, we have $x*(x*y) \leq_X y$ by (III), (1) and (4). It follows from (17) that $Y_J(\varepsilon, \zeta(x)) \leq Y_J(\varepsilon, \zeta(x*y)) \vee Y_J(\varepsilon, \zeta(y))$ for all $x, y \in X$. Therefore ζ is a Y_J^ε-fuzzy ideal of $(X, *, 0)$.

5. Closed Y_J^ε-fuzzy ideals in BCI-algebras

In this section, let $(X, *, 0)$ denote a BCI-algebra. We recall that any Y_J^ε-fuzzy ideal may not be a Y_J^ε-fuzzy subalgebra in BCI-algebras (cf. Example 2). This is a motivation for the definition below.

Definition 2. *A Y_J^ε-fuzzy ideal ζ of $(X, *, 0)$ is said to be closed if it is also a Y_J^ε-fuzzy subalgebra of $(X, *, 0)$.*

Example 5. *Let $X = \{0, b_1, b_2, b_3, b_4\}$ be a set with a binary operation "$*$" given by Table 4.*

Table 4: Cayley table for the binary operation "$*$"

$*$	0	b_1	b_2	b_3	b_4
0	0	0	0	b_3	b_3
b_1	b_1	0	0	b_3	b_3
b_2	b_2	b_2	0	b_4	b_3
b_3	b_3	b_3	b_3	0	0
b_4	b_4	b_4	b_3	b_2	0

Then $(X, *, 0)$ is a BCI-algebra (see [2]). Let ζ be a fuzzy set in X given by

$$\zeta : X \to [0,1],\ y \mapsto \begin{cases} 0.78 & \text{if } \ y = 0, \\ 0.63 & \text{if } \ y \in \{b_1, b_2\}, \\ 0.47 & \text{otherwise}, \end{cases}$$

It is routine to check that ζ is a closed Y_J^ε-fuzzy ideal of $(X, *, 0)$ for $\varepsilon := 0.46$.

Theorem 8. *A fuzzy set ζ in X given by*

$$\zeta : X \to [0,1],\ y \mapsto \begin{cases} s_1 & \text{if } \ y \in \{x \in X \mid 0 \leq_X x\}, \\ s_2 & \text{otherwise}, \end{cases}$$

where $s_1 > s_2$ in $(0,1)$, is a closed Y_J^ε-fuzzy ideal of $(X, *, 0)$.

Proof. The Y-level set $\varepsilon(\zeta)^t$ is calculated as follows:

$$\varepsilon(\zeta)^t = \begin{cases} \emptyset & \text{if } \ 0 < t < 1 - s_1, \\ \{x \in X \mid 0 \leq_X x\} & \text{if } \ 1 - s_1 \leq t < 1 - s_2, \\ X & \text{if } \ 1 - s_2 \leq t < 1. \end{cases}$$

Let $A := \{x \in X \mid 0 \leq_X x\}$, and let $y, z \in A$. Then $0 \leq_X y$ and $0 \leq_X z$, i.e., $0 * y = 0$ and $0 * z = 0$. Hence $0 * (y * z) = (0 * y) * (0 * z) = 0$ by (III) and (6), and so $0 \leq_X y * z$, i.e., $y * z \in A$. Thus A is a subalgebra of $(X, *, 0)$. It is clear that $0 \in A$. Let $y, z \in X$ be such that $y * z \in A$ and $z \in A$. Then

$$0 = 0 * (y * z) = (0 * y) * (0 * z) = (0 * y) * 0 = 0 * y$$

by (2) and (6). Hence $y \in A$, which shows that A is an ideal of $(X, *, 0)$. Therefore A is a closed ideal of $(X, *, 0)$. By the combination of Theorems 1 and 6, we conclude that ζ is a closed Y_J^ε-fuzzy ideal of $(X, *, 0)$.

Lemma 1. *A fuzzy set ζ in X is a closed Y_J^ε-fuzzy ideal of $(X, *, 0)$ if and only if the nonempty Y-level set $\varepsilon(\zeta)^t$ of $\varepsilon(\zeta)$ is a closed ideal of $(X, *, 0)$ for all $t \in I \setminus \{0, 1\}$.*

Proof. Assume that ζ is a closed Y_J^ε-fuzzy ideal of $(X, *, 0)$ and let $t \in I \setminus \{0, 1\}$ be such that $\varepsilon(\zeta)^t \neq \emptyset$. Then $\varepsilon(\zeta)^t$ is an ideal of $(X, *, 0)$ by Theorem 6. Let $x \in \varepsilon(\zeta)^t$. Then

$$Y_J(\varepsilon, \zeta(0 * x)) \overset{(13)}{\leq} Y_J(\varepsilon, \zeta(0)) \vee Y_J(\varepsilon, \zeta(x)) \overset{(14)}{\leq} Y_J(\varepsilon, \zeta(x)) \leq t,$$

and so $0 * x \in \varepsilon(\zeta)^t$. Hence $\varepsilon(\zeta)^t$ is a closed ideal of $(X, *, 0)$.

Conversely, suppose that the nonempty Y-level set $\varepsilon(\zeta)^t$ of $\varepsilon(\zeta)$ is a closed ideal of $(X, *, 0)$ for all $t \in I \setminus \{0, 1\}$. Then $\varepsilon(\zeta)^t$ is an ideal of $(X, *, 0)$, and thus ζ is a Y_J^ε-fuzzy ideal of $(X, *, 0)$ by Theorem 6. If ζ is not a Y_J^ε-fuzzy subalgebra of $(X, *, 0)$, then

$$Y_J(\varepsilon, \zeta(x * a)) > Y_J(\varepsilon, \zeta(x)) \vee Y_J(\varepsilon, \zeta(a))$$

for some $x, a \in X$. Selecting $t := Y_J(\varepsilon, \zeta(x)) \vee Y_J(\varepsilon, \zeta(a))$ induces $x, a \in \varepsilon(\zeta)^t$ and $x * a \notin \varepsilon(\zeta)^t$, which is a contradiction. Hence

$$Y_J(\varepsilon, \zeta(x * a)) \leq Y_J(\varepsilon, \zeta(x)) \vee Y_J(\varepsilon, \zeta(a))$$

for all $x, a \in X$, which shows that ζ is a Y_J^ε-fuzzy subalgebra of $(X, *, 0)$. Consequently, ζ is a closed Y_J^ε-fuzzy ideal of $(X, *, 0)$.

Theorem 9. *A Y_J^ε-fuzzy ideal ζ of $(X, *, 0)$ is closed if and only if it satisfies:*

$$(\forall x \in X)(Y_J(\varepsilon, \zeta(0 * x)) \leq Y_J(\varepsilon, \zeta(x))). \tag{18}$$

Proof. Let ζ be a closed Y_J^ε-fuzzy ideal of $(X, *, 0)$. Then the nonempty Y-level set $\varepsilon(\zeta)^t$ of $\varepsilon(\zeta)$ is a closed ideal of $(X, *, 0)$ for all $t \in I \setminus \{0, 1\}$ by Lemma 1. Then

$$Y_J(\varepsilon, \zeta(0 * x)) \overset{(13)}{\leq} Y_J(\varepsilon, \zeta(0)) \vee Y_J(\varepsilon, \zeta(x)) \overset{(14)}{\leq} Y_J(\varepsilon, \zeta(x))$$

for all $x \in X$.

Conversely, let ζ be a Y_J^ε-fuzzy ideal of $(X, *, 0)$ that satisfies the condition (18). Then

$$Y_J(\varepsilon, \zeta((x * y) * x)) \overset{(4)}{=} Y_J(\varepsilon, \zeta((x * x) * y)) \overset{(\mathrm{III})}{=} Y_J(\varepsilon, \zeta(0 * y)) \overset{(18)}{\leq} Y_J(\varepsilon, \zeta(y))$$

for all $x, y \in X$, and so

$$\begin{aligned} Y_J(\varepsilon, \zeta(x * y) &\overset{(15)}{\leq} Y_J(\varepsilon, \zeta((x * y) * x)) \vee Y_J(\varepsilon, \zeta(x)) \\ &\leq Y_J(\varepsilon, \zeta(x)) \vee Y_J(\varepsilon, \zeta(y)) \end{aligned}$$

for all $x, y \in X$. Hence ζ is a closed Y_J^ε-fuzzy ideal of $(X, *, 0)$.

Theorem 10. *Given an element $a \in X$, let ζ_a be the fuzzy set in X defined by*

$$\zeta_a : X \to [0,1],\ y \mapsto \begin{cases} s_1 & \text{if } \ y \in X_a, \\ s_2 & \text{otherwise,} \end{cases}$$

where $s_1 > s_2$ in $(0,1)$ and $X_a := \{x \in X \mid a * x = a\}$. Then ζ_a is a closed Y_J^ε-fuzzy ideal of $(X, *, 0)$.

Proof. The Y-level set $\varepsilon(\zeta_a)^t$ is calculated as follows:

$$\varepsilon(\zeta_a)^t = \begin{cases} \emptyset & \text{if } \ 0 < t < 1 - s_1, \\ X_a & \text{if } \ 1 - s_1 \leq t < 1 - s_2, \\ X & \text{if } \ 1 - s_2 \leq t < 1. \end{cases}$$

It is clear that $0 \in X_a$ by (2). For every $x \in X_a$, we have

$$0 * x \overset{\text{(III)}}{=} (a * a) * x \overset{(4)}{=} (a * x) * a \overset{x \in X_a}{=} a * a \overset{\text{(III)}}{=} 0 \in X_a \tag{19}$$

Let $x, y \in X$ be such that $x * y \in X_a$ and $y \in X_a$. Then

$$0 * (x * y) = 0 \text{ and } 0 * y = 0$$

by (19). It follows that

$$(a * x) * a \overset{(4)}{=} (a * a) * x \overset{\text{(III)}}{=} 0 * x \overset{(2)}{=} (0 * x) * (0 * y) \overset{(6)}{=} 0 * (x * y) = 0,$$

i.e., $a * x \leq_X a$. Since $x * y \in X_a$ and $y \in X_a$, we get

$$a = a * (x * y) = (a * y) * (x * y) \leq a * x.$$

Hence $a * x = a$, i.e., $x \in X_a$. This shows that X_a is a closed ideal of $(X, *, 0)$. Thus we know that the nonempty Y-level set $\varepsilon(\zeta_a)^t$ is a closed ideal of $(X, *, 0)$ for all $t \in I \setminus \{0, 1\}$. Therefore ζ_a is a closed Y_J^ε-fuzzy ideal of $(X, *, 0)$ by Lemma 1.

We explore the conditions under which a Y_J^ε-fuzzy subalgebra becomes a Y_J^ε-fuzzy ideal.

Theorem 11. *In a p-semisimple BCI-algebra $(X, *, 0)$, every Y_J^ε-fuzzy subalgebra is a Y_J^ε-fuzzy ideal.*

Proof. Let ζ be a Y_J^ε-fuzzy subalgebra of a p-semisimple BCI-algebra $(X, *, 0)$, and let $t \in I \setminus \{0,1\}$ be such that $\varepsilon(\zeta)^t \neq \emptyset$. Then $\varepsilon(\zeta)^t$ is a subalgebra of $(X, *, 0)$ by Theorem 1. It is clear that $0 \in \varepsilon(\zeta)^t$. Let $x, y \in X$ be such that $x * y \in \varepsilon(\zeta)^t$ and $y \in \varepsilon(\zeta)^t$. Then $0 * y \in \varepsilon(\zeta)^t$ and $(x * y) * (0 * y) \in \varepsilon(\zeta)^t$. On the other hand, we have

$$\begin{aligned}((x * y) * (0 * y)) * x &\overset{(4)}{=} ((x * y) * x) * (0 * y)\\ &\overset{(4)}{=} ((x * x) * y) * (0 * y)\\ &\overset{(\mathrm{III})}{=} (0 * y) * (0 * y) \overset{(\mathrm{III})}{=} 0,\end{aligned}$$

that is, $(x * y) * (0 * y) \leq_X x$. Since $(X, *, 0)$ is p-semisimple, x is a minimal element of X. It follows that $x = (x * y) * (0 * y) \in \varepsilon(\zeta)^t$. Hence $\varepsilon(\zeta)^t$ is an ideal of $(X, *, 0)$, and therefore ζ is a Y_J^ε-fuzzy ideal of $(X, *, 0)$ by Theorem 6.

Corollary 2. *If a BCI-algebra* $(X, *, 0)$ *satisfies:*

$$(\forall x, y \in X)(x * (0 * y) = y * (0 * x))$$

or

$$(\forall x \in X)(0 * x = 0 \ \Rightarrow \ x = 0),$$

then every Y_J^ε*-fuzzy subalgebra is a* Y_J^ε*-fuzzy ideal.*

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