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Approximation of BV space-defined functionals containing piecewise integrands with L^1 condition

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Abstract. We prove an approximation result for a class of functionals $\mathcal{G}(u) = \int_\Omega \varphi(x, Du)$ defined on $BV(\Omega)$ where $\varphi(\cdot, Du) \in L^1(\Omega)$, $\Omega \subset \mathbb{R}^N$ bounded, $\varphi(x,p)$ convex, radially symmetric and of the form

$$\varphi(x,p) = \begin{cases} g(x,p) & \text{if } |p| \le \beta \\ \psi(x)|p| + k(x) & \text{if } |p| > \beta. \end{cases}$$

We show for each $u \in BV(\Omega) \cap L^p(\Omega)$, $1 \le p < \infty$, there exist $u_k \in W^{1,1}(\Omega) \cap C^\infty(\Omega) \cap L^p(\Omega)$ so that $\mathcal{G}(u_k) \to \mathcal{G}(u)$. Approximation theorems in BV are used to prove existence results for the strong solution to the time flow $u_t = div\,(\nabla_p \varphi(x, Du))$ in $L^1((0,\infty); BV(\Omega) \cap L^p(\Omega))$, typically with additional boundary condition or penalty term in u to ensure uniqueness. The functions in this work are not covered by previous approximation theorems since for fixed p we have $\varphi(x,p) \in L^1(\Omega)$ which do not in general hold for assumptions on φ in earlier work.

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1. Introduction

In this work, we present some approximation results for functionals

$$\mathcal{G}(u) := \int_\Omega \varphi(x, Du) \tag{1}$$

defined for $u \in BV(\Omega)$ with bounded, open $\Omega \subset \mathbb{R}^N$ with the following assumptions on φ:

(1) $\varphi : \Omega \times \mathbb{R}^N \to \mathbb{R}$, where $\varphi(x,p)$ is convex in p, that is

$$\varphi(x, \lambda_1 p_1 + \lambda_2 p_2) \le \lambda_1 \varphi(x, p_1) + \lambda_2 \varphi(x, p_2)$$

for each $z \in \mathbb{R}, p_1, p_2 \in \mathbb{R}^N$, $0 \le \lambda_1, \lambda_2 \le 1$, $\lambda_1 + \lambda_2 = 1$,

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(2) $\varphi(x,p) = \varphi(x,|p|)$ for all p, and for $k \in L^1(\Omega)$ is of the form

$$\varphi(x,p) = \begin{cases} g(x,p) & \text{if } |p| \le \beta \\ \psi(x)|p| + k(x) & \text{if } |p| > \beta. \end{cases}$$

(3) φ is a Carathéodory function, with $\varphi(\cdot,p) \in L^1(\Omega)$ for each p.

From (3), φ is of linear growth in the p variable with

$$\lim_{|p|\to\infty} \frac{\varphi(x,p)}{|p|} = \psi(x).$$

We note that $\varphi(x,p)$ is continuous in p since real valued convex functions are continuous. The main result of this paper is the extension of the approximation theorems presented in, [2], [5], and [8] to include certain cases where $\varphi(\cdot,p) \in L^1(\Omega)$ for a class of integrands φ with the above assumptions (1)-(3). We note that functionals of the form (1) defined on BV have many applications to elasticity and image processing problems (see e.g. the early works of [9], [12], [14], [19]).

We recall the classic approximation theorem in [8] where it is proved that for each $u \in BV(\Omega)$, $\Omega \subset \mathbb{R}^N$ bounded, there exists a sequence $\{u_k\} \subset W^{1,1}(\Omega) \cap C^\infty(\Omega)$ so that $u_k \to u$ in $L^1(\Omega)$ and $\int_\Omega |\nabla u_k|\,dx \to \int_\Omega |Du|$. We recall $u \in BV(\Omega)$ if and only if $u \in L^1(\Omega)$ and

$$\int_\Omega |Du| := \sup_{\phi \in \left\{C_0^\infty(\Omega,\mathbb{R}^N), |\phi(x)|\le 1 \text{ all } x\in\Omega\right\}} \left\{ -\int_\Omega u\,div\phi\,dx \right\} < \infty,$$

and with $\|u\|_{BV(\Omega)} := \|u\|_{L^1(\Omega)} + \int_\Omega |Du|$. In this case we have $\int_\Omega |Du| := \int_\Omega |\nabla u|\,dx + \int_\Omega |D^s u|$ for the measures $\nabla u\,dx << \mathcal{L}^N$ and $D^s u \perp \mathcal{L}^N$, and where $D^s u = 0$ if and only if $u \in W^{1,1}(\Omega)$. As $W^{1,1}(\Omega)$ is not dense in $BV(\Omega)$ we can not have $\int_\Omega |\nabla u_k - Du| \to 0$. See [7] for a detailed discussion.

As a model for image restoration, the authors in [5] consider

$$\Phi_h(u) \; : \; = \int_\Omega \varphi(x,Du) + \frac{\lambda}{2}\int_\Omega (u-u_0)^2\,dx + \int_{\partial\Omega} |u-h|\,d\mathcal{H}^{N-1}$$

for $\lambda > 0$ constant, $u_0 \in L^\infty(\Omega)$, and where

$$\varphi(x,p) = \begin{cases} \frac{1}{q(x)}|p|^{q(x)} & |p| \le \beta \\ |p| - \frac{\beta q(x) - \beta^{q(x)}}{q(x)} & |p| > \beta \end{cases}$$

for constant $\beta > 0$, $q \in L^\infty(\Omega)$, $1 < \alpha \le q(x) \le 2$. Here u and h are defined on $\partial\Omega$ in the sense of trace ([7]). The solution to

$$\min_{u\in BV(\Omega)} \Phi_h(u) \tag{2}$$

is then the restored version of the corrupted image u_0. In order to prove the existence of the weak solution of the corresponding time flow for (2), the authors show for each $u \in BV(\Omega)$ there is a sequence $u_k \in H^1(\Omega) \cap C^\infty(\Omega)$ where

$$\begin{aligned} u_k &\to u \text{ in } L^2(\Omega) \text{ and} \\ \Phi_h(u_k) &\to \Phi_h(u). \end{aligned}$$

Other approximation results are proved in [2] (Lemma 6.2) assuming lower semicontinuity or continuity in the x variable and in [3] for integrand $g(x,p)$ with a continuity condition in x which in general will not be satisfied in our case for $\varphi(\cdot,p) \in L^1(\Omega)$. We also refer the reader to [15] for lower semicontinuity and approximation theorems of functionals $\int_\Omega f(x, Du)$, $u \in BV(\Omega)$, using the work of Reshetnyak; and, for example, in [1] for the relaxation in $BV(\Omega)$ with respect to the L^1 norm for functionals $\int_\Omega f(x,u,\nabla u)\,dx$ defined on $W^{1,1}\left(\Omega; S^{d-1}\right)$ for $\Omega \subset \mathbb{R}^N$ open and bounded and S^{d-1} the unit sphere in $\mathbb{R}^d$. However the integrands $f(x,p)$ and $f(x,z,p)$ are always assumed to be lower semicontinuous or continuous on $\Omega \times \mathbb{R}^N$ or $\Omega \times \mathbb{R} \times \mathbb{R}^N$ respectively for these cases.

Importantly, we note that the approximation Lemma 6.2 in [2] is used to prove existence results there for the solution to the time dependent problem

$$\begin{cases} \frac{\partial u}{\partial t} = div \nabla_p g(x, Du) & \text{in } (0,\infty) \times \Omega \\ u(t,x) = h(x) & \text{on } (0,\infty) \times \partial\Omega \\ u(0,x) = u_0(x) & \text{for } x \in \Omega \end{cases}$$

via the strong solution using the theory of semigroups in L^2, which corresponds to the stationary problem

$$\min_{u \in BV(\Omega) \cap L^2(\Omega)} \Phi_\varphi(u),$$

with

$$\Phi_\varphi(u) := \int_\Omega g(x, Du) + \int_{\partial\Phi} |h-u| g^0(x, \nu(x))\, d\mathcal{H}^{N-1},$$

for given boundary data h. Here g is continuous on $\overline{\Omega} \times \mathbb{R}^N$, convex and continuously differentiable in the second variable p, and

$$g^0(x,p) := \lim_{t \to 0^+} t g(x, p/t).$$

Appropriately defined solutions of the above time flow in L^1 using similar semigroup methods are also proved there.

2. Main Results

As stated in the Introduction, we prove an approximation result for a class of functionals $\int_\Omega \varphi(x, Du)$ by $\int_\Omega \varphi(x, \nabla u_k)$, $u_k \in W^{1,1}(\Omega) \cap C^\infty(\Omega)$ where $\varphi(x,p)$ satisfies (1)-(3)

and with an additional structure condition on g. Here we will use, from [6], the conjugate function g^* for given g:

$$g^*(x,q) := \sup_{p\in\mathbb{R}^N}\{q\cdot p - g(x,p)\}.$$

If g is convex in p, then it is easy to show that g^* is convex in q. Also if g is additionally continuous in p, then for a.e. x, there holds $g(x,p) = g^{**}(x,p)$ for all $p\in\mathbb{R}^N$ (see [6],[4]).

We will need the following lemma which is Proposition 1, from [17], which for the convenience of the reader we restate here.

In the sequel we define

$$\mathcal{V} := \left\{\phi\in C_0^1(\Omega,\mathbb{R}^N) : |\phi(x)|\le\psi(x) \text{ for all } x\in\Omega\right\}.$$

Lemma 1. *Assume φ satisfies the conditions (1)-(3) above:*

$$\varphi(x,p) = \begin{cases} g(x,p) & \text{if } |p|\le\beta \\ \psi(x)|p| + k(x) & \text{if } |p|>\beta, \end{cases}$$

with $\psi\in C(\Omega)\cap L^\infty(\Omega)$, $\psi\ge 0$, $k(x,u)\in L^1(\Omega)$ for each $u\in L^1(\Omega)$. Also assume for some G

$$\varphi(x,p) = G(r_1(x),...,r_K(x),p) \text{ for all } p$$

where

$$G(z_1,...,z_K,p) = \begin{cases} g_1(z_1,...,z_K,p) \text{ if } |p|\le\beta \\ z_K|p| + g_2(z_1,...,z_K) \text{ if } |p|>\beta \end{cases}$$

and where for each $|p|\le\beta$, g_1 is C^1 in the variable $\mathbf{z} = (z_1...,z_K)\in U\subset\mathbb{R}^K$, U open, $r_i\in L^1(\Omega)$ each i, $(r_1(x),...,r_K(x))\in U$ a.e. x, and $|(\nabla_{\mathbf{z}}g_1)(\mathbf{z},p)|\le C$, C independent of $(\mathbf{z},p)$. Note that $r_K(x) = \psi(x)$ and hence $z_k\ge 0$.

Then for all $u\in BV(\Omega)$ we have

$$\begin{aligned} \mathcal{G}(u) &= \int_\Omega \varphi(x,\nabla u)\,dx + \int_\Omega \psi(x)|D^s u| \qquad (3)\\ &= \sup_{\phi\in\mathcal{V}}\left\{-\int_\Omega u\,div\phi + \varphi^*(x,\phi(x))\,dx\right\}. \end{aligned}$$

If in addition $\partial\Omega$ is Lipschitz, $u\in BV(\Omega)$, then we have the continuous trace operator $T: BV(\Omega)\to L^1(\partial\Omega,\mathcal{H}^{N-1})$ ([7]). Thus if $h\in BV(\Omega)$,

$$\begin{aligned} \mathcal{G}_h(u) &= \int_\Omega \varphi(x,\nabla u)\,dx + \int_\Omega \psi(x)|D^s u| + \int_{\partial\Omega}|u-h|\,d\mathcal{H}^{N-1} \qquad (4)\\ &= \sup_{\left\{\phi\in C^1\left(\overline{\Omega},\mathbb{R}^N\right):|\phi|\le\psi(x)\right\}}\left\{-\int_\Omega u\,div\phi + \varphi^*(x,\phi(x))\,dx + \int_{\partial\Omega}\phi\widehat{n}h\,d\mathcal{H}^{N-1}\right\}. \end{aligned}$$

Furthermore, both $\mathcal{G}$ and $\mathcal{G}_h$ are lower semicontinuous in L^1.

Before we state the proof, we note that the lower semicontinuity of $\mathcal{G}$ and $\mathcal{G}_h$ in L^1 is not covered the results in [10], [11], [13] since we only assume $\varphi(\cdot,p) \in L^1(\Omega)$ for each p and hence the condition that

$$\lim_{\widetilde{x}\to x, t\to\infty} t\varphi(\widetilde{x}, p/t) \text{ exists}$$

as stated there may not hold if $\varphi(\cdot,p)$ is only assumed to be in $L^1(\Omega)$. Also see [18] for more general results for lower semicontinuity.

For an example of an integrand φ satisfying the conditions of Lemma 1, consider the following $\mathcal{G}$ with $\alpha \in L^1(\Omega)$, $\delta > 0$:

$$\mathcal{G}(u) := \int_\Omega \varphi(x, Du)$$

with

$$\varphi(x,p) = \begin{cases} \psi(x)\sqrt{\alpha^2(x) + \delta + |p|^2} & \text{if } |p| \leq \beta \\ \psi(x)|p| + \psi(x)\frac{\alpha(x)+\delta}{\sqrt{\alpha^2(x)+\delta+\beta^2}+\beta} & \text{if } |p| > \beta. \end{cases}$$

We now state the approximation theorem.

Theorem 1. *Let $\mathcal{G}$ and $\mathcal{G}_h$ be as defined in Lemma 1 with φ satisfying the same conditions. Then for each $u \in BV(\Omega) \cap L^r(\Omega)$, $1 \leq r < \infty$ there exist a sequence $u_k \in W^{1,1}(\Omega) \cap C^\infty(\Omega) \cap L^r(\Omega)$ with*

$$\begin{aligned} \mathcal{G}(u_k) &\to \mathcal{G}(u) \text{ and} \\ u_k &\to u \text{ in } L^r(\Omega). \end{aligned}$$

In addition, if $\partial\Omega$ is Lipschitz and $h \in L^1(\partial\Omega)$ we have for each $u \in BV(\Omega)$ a sequence $u_k \in W^{1,1}(\Omega) \cap C^\infty(\Omega) \cap L^r(\Omega)$ with

$$\begin{aligned} \mathcal{G}_h(u_k) &\to \mathcal{G}_h(u), \\ u_k &\to u \text{ in } L^r(\Omega), \text{ and} \\ Tu_k &= Tu \end{aligned}$$

where Tw is the trace operator for $w \in BV(\Omega)$.

Proof. We follow [8] taking into account the extra φ^* term.

Fix $\varepsilon > 0$ and construct an open covering $\{A_i\}$ of Ω where $A_i = \Omega_{i+1} - \overline{\Omega}_{i-1}$, $A_1 = \Omega_2$ where

$$\Omega_k = \{x \in \Omega : dist(x, \partial\Omega) > 1/(m+k)\}, \, k = 0, 1, 2, ...$$

and with m so large that

$$\int_{\Omega-\Omega_0} \psi(x)|Du| < \varepsilon \text{ and} \tag{5}$$

$$|\Omega - \Omega_1| \leq \varepsilon \tag{6}$$

Now construct a sequence $\{u_\varepsilon\}$ so that

$$u_\varepsilon = \sum_{i=1}^{\infty} \eta_{\varepsilon_i} * (u\phi_i)$$

where η is the usual mollifier on $\mathbb{R}^N$, $\{\phi_i\}$ is a partition of unity subordinate to $\{A_i\}$, and the ε_i are chosen to that the four conditions all hold:

1. each $\varepsilon_i < \varepsilon$, $i \geq 1$
2. $\int_\Omega |\eta_{\varepsilon_i} * (u\phi_i) - u\phi_i|^r \; dx \leq \varepsilon 2^{-i}$
3. $\int_\Omega |\eta_{\varepsilon_i} * (u\nabla\phi_i) - u\nabla\phi_i| \; dx \leq \varepsilon 2^{-i}$
4. support $\eta_{\varepsilon_i} * (u\phi_i) \subset \Omega_{i+2} - \overline{\Omega}_{i-2}$.

Summing over all i gives

$$\int_\Omega |u_\varepsilon - u| \; dx \leq \sum_{i=1}^{\infty} \int_\Omega |\eta_{\varepsilon_i} * (u\phi_i) - u\phi_i| \; dx \leq \varepsilon$$

giving $u_\varepsilon \to u$ in $L^1(\Omega)$. Hence by L^1 lower semicontinuity in Lemma 1

$$\int_\Omega \varphi(x, Du) \leq \liminf_{\varepsilon \to 0} \int_\Omega \varphi(x, Du_\varepsilon). \tag{7}$$

First we note that $|(\phi_1\eta_{\varepsilon_1} * \phi)(x)| \leq \psi(x) + \omega(\varepsilon_1)$ where the modulus of continuity ω of ψ satisfies $\omega(\varepsilon_1) \to 0$ as $\varepsilon_1 \to 0$, and that for $\varphi_c(x,p) := \varphi(x,p) + c|p|$, for each $c > 0$, satisfies the same assumptions on φ. Hence for each $u \in BV(\Omega)$

$$\sup_{|\phi(x)| \leq \psi(x)+c} \left\{ -\int_\Omega u div\phi + \varphi_c^*(x, \phi(x)) \, dx \right\} \tag{8}$$
$$= \int_\Omega \varphi(x, \nabla u) \; + c|\nabla u| \, dx + \int_\Omega (\psi(x) + c\,) d|D^s u|.$$

Now let $\phi \in C_0^1(\Omega; \mathbb{R}^N)$ with $|\phi(x)| \leq \psi(x)$ each x, then

$$-\int_\Omega u_\varepsilon div\phi + \varphi^*_{\omega(\varepsilon_1)}(x, \phi(x)) \, dx = \left(\sum_{i=1}^{\infty} -\int_\Omega (\eta_{\varepsilon_i} * (u\phi_i)) div\phi \, dx \right) \tag{9}$$
$$-\int_\Omega \varphi^*_{\omega(\varepsilon_1)}(x, \phi(x)) \, dx \tag{10}$$
$$= -\int_\Omega u div(\phi_1\eta_{\varepsilon_1} * \phi) \, dx - \int_\Omega \varphi^*_{\omega(\varepsilon_1)}(x, \phi(x)) \, dx - \sum_{i=2}^{\infty} \int_\Omega u div(\phi_i\eta_{\varepsilon_i} * \phi) \, dx$$
$$+ \sum_{i=1}^{\infty} \int_\Omega \phi(\eta_{\varepsilon_i} * (u\nabla\phi_i) - u\nabla\phi_i) \, dx$$
$$= -\int_\Omega u div(\phi_1\eta_{\varepsilon_1} * \phi) + \varphi^*_{\omega(\varepsilon_1)}(x, \eta_{\varepsilon_1} * \phi) \, dx - \sum_{i=2}^{\infty} \int_\Omega u div(\phi_i\eta_{\varepsilon_i} * \phi) \, dx$$

$$+\sum_{i=1}^{\infty}\int_{\Omega}\phi(\eta_{\varepsilon_i}*(u\nabla\phi_i)-u\nabla\phi_i)\,dx$$

$$+\int_{\Omega}\varphi^*_{\omega(\varepsilon_1)}(x,\eta_{\varepsilon_1}*\phi)-\varphi^*_{\omega(\varepsilon_1)}(x,\phi(x))\,dx := I+II+III+IV.$$

By Lemma 3 in [16] we have from the Lipschitz property of $\varphi^*_{\omega(\varepsilon_1)}$

$$\begin{aligned} IV &\leq \int_{\Omega}|\varphi^*_{\omega(\varepsilon_1)}(x,\eta_{\varepsilon_1}*\phi)-\varphi^*_{\omega(\varepsilon_1)}(x,\phi(x))|\,dx \\ &\leq \beta\int_{\Omega}|\eta_{\varepsilon_1}*\phi-\phi|\,dx. \end{aligned}$$

We now in addition to 1-4 choose ε_1 so that $\int_{\Omega_1}|\eta_{\varepsilon_1}*\phi-\phi|\,dx\leq\varepsilon$. The since $|\eta_{\varepsilon_1}*\phi|\leq\|\psi\|_\infty$ we then have

$$\begin{aligned} IV &\leq \beta\int_{\Omega}|\eta_{\varepsilon_1}*\phi-\phi|\,dx \\ &= \beta\int_{\Omega_1}|\eta_{\varepsilon_1}*\phi-\phi|\,dx+\beta\int_{\Omega-\Omega_1}|\eta_{\varepsilon_1}*\phi-\phi|\,dx \\ &\leq \beta\varepsilon+2\beta\|\psi\|\,\varepsilon\to 0 \text{ as } \varepsilon\to 0. \end{aligned}$$

Also, we have as in [8]

$$III, II\to 0 \text{ as } \varepsilon\to 0.$$

Now

$$\begin{aligned} I &= -\int_{\Omega}u\,div(\phi_1\eta_{\varepsilon_1}*\phi)+\varphi^*_{\omega(\varepsilon_1)}(x,\eta_{\varepsilon_1}*\phi)\,dx \\ &= -\int_{\Omega}u\,div(\phi_1\eta_{\varepsilon_1}*\phi)+\varphi^*_{\omega(\varepsilon_1)}(x,\phi_1\eta_{\varepsilon_1}*\phi)\,dx \\ &\quad+\int_{\Omega}\varphi^*_{\omega(\varepsilon_1)}(x,\phi_1\eta_{\varepsilon_1}*\phi)-\varphi^*_{\omega(\varepsilon_1)}(x,\eta_{\varepsilon_1}*\phi)\,dx. \end{aligned}$$

Again from Lemma 3 in [16] we have for the last line

$$\begin{aligned} |\eta| \;:&= \left|\int_{\Omega}\varphi^*_{\omega(\varepsilon_1)}(x,\phi_1\eta_{\varepsilon_1}*\phi)-\varphi^*_{\omega(\varepsilon_1)}(x,\eta_{\varepsilon_1}*\phi)\,dx\right| \\ &\leq \beta\int_{\Omega}|\phi_1\eta_{\varepsilon_1}*\phi-\eta_{\varepsilon_1}*\phi|\,dx \\ &= \beta\int_{\Omega-\Omega_1}|\phi_1-1||\eta_{\varepsilon_1}*\phi|\,dx \\ &\leq 2\beta\int_{\Omega-\Omega_1}|\eta_{\varepsilon_1}*\phi|\,dx \\ &\leq 2\beta\|\psi\|_\infty\,\varepsilon \end{aligned}$$

since $\phi_1 \equiv 1$ on Ω_1.

Therefore $\eta \to 0$ as $\varepsilon \to 0$.

Thus from (8)

$$\begin{aligned} I &= -\int_\Omega u div(\phi_1 \eta_{\varepsilon_1} * \phi) + \varphi^*_{\omega(\varepsilon_1)}(x, \phi_1 \eta_{\varepsilon_1} * \phi)\, dx + \eta \\ &\leq \int_\Omega \varphi(x, \nabla u) + \omega(\varepsilon_1)|\nabla u|\, dx + \int_\Omega (\psi(x) + \omega(\varepsilon_1))\, d|D^s u| + \eta \\ &= \int_\Omega \varphi(x, Du) + \int_\Omega \omega(\varepsilon_1)|\nabla u|\, dx + \omega(\varepsilon_1) \int_\Omega d|D^s u| + \eta, \end{aligned}$$

keeping in mind that the last three terms approach 0 as $\varepsilon \to 0$. Thus we have from (9) and for each ϕ with $|\phi(x)| \leq \psi(x)$,

$$\begin{gathered} -\int_\Omega u_\varepsilon div\phi + \varphi^*(x, \phi(x))\, dx \\ \leq I + II + III + IV + \int_\Omega |\varphi^*(x, \phi(x))\, dx - \varphi^*_{\omega(\varepsilon_1)}(x, \phi(x))|\, dx \\ = I + II + III + IV + \int_\Omega |\varphi^*(x, \phi(x)) - (\varphi(x, \phi(x)) + \omega(\varepsilon_1)|\phi(x)|)^* dx \\ \leq I + II + III + IV + \omega(\varepsilon_1)|\psi|_\infty |\Omega| \\ \leq \int_\Omega \varphi(x, Du) + \int_\Omega \omega(\varepsilon_1)|\nabla u|\, dx + \omega(\varepsilon_1) \int_\Omega d|D^s u| + \eta \\ + II + III + IV + \omega(\varepsilon_1)|\psi|_\infty |\Omega|\,. \end{gathered}$$

The second inequality follows from the note before (8), the assumption $|\phi(x)| \leq \psi(x)$, and Lemma 2 in [16]. Thus we have

$$\begin{aligned} -\int_\Omega u_\varepsilon div\phi + \varphi^*(x, \phi(x))\, dx \quad \leq \quad & \int_\Omega \varphi(x, Du) + \int_\Omega \omega(\varepsilon_1)|\nabla u|\, dx \\ & + \omega(\varepsilon_1) \int_\Omega d|D^s u| + \eta \\ & + II + III + IV + \omega(\varepsilon_1)|\psi|_\infty |\Omega|\,. \end{aligned} \tag{11}$$

Taking the supremum over all such ϕ with $|\phi(x)| \leq \psi(x)$ in (11), and then letting $\varepsilon \to 0$ we have

$$\limsup_{\varepsilon \to 0} -\int_\Omega \varphi(x, Du_\varepsilon) dx \leq \int_\Omega \varphi(x, Du).$$

Combining with (7) gives the result. The second part of the theorem is proved as in the first case and as in [5] for the boundary term.

Combining Lemma 1 and Theorem 1 we have the following extension of Theorem 6.4 in [2].

Theorem 2. *Let φ satisfy the conditions of Lemma 1 and Theorem 1, then*

$$\begin{aligned}
\inf_{u\in BV(\Omega)} \mathcal{G}(u) &= \inf\left\{\int_\Omega \varphi(x,\nabla u)\,dx : u\in W^{1,1}(\Omega)\right\}, \text{ and} \\
\inf_{u\in BV(\Omega),\ u=h \text{ on } \partial\Omega} \mathcal{G}_h(u) &= \inf\left\{\int_\Omega \varphi(x,\nabla u)\,dx : u\in W^{1,1}(\Omega) \text{ and } u=h \text{ on } \partial\Omega\right\}.
\end{aligned}$$

In addition, $\mathcal{G}$, $\mathcal{G}_h$ is the greatest $L^1(\Omega)$-lower semicontinuous functional on $BV(\Omega)$ satisfying $\mathcal{G}(u) \le \int_\Omega \varphi(x,\nabla u)\,dx$, and $\mathcal{G}_h(u) \le \int_\Omega \varphi(x,\nabla u)\,dx$ for all $u\in W^{1,1}(\Omega)$ and $u\in W^{1,1}(\Omega)$ with $u=h$ on $\partial\Omega$ respectively.

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